Photons in general relativity

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Abstract

This paper is grounded exclusively on the GR-Hamilton's variational principle, reducible to the principle of geodesics in the case of a body in free motion in a gravitational field. The first two sections deal with the theory of gravitational light bending, available in all general relativity texts, but rewritten here using exclusively formulas between quantities, more suitable for the physical interpretation than the formulas between measurements commonly used (c = 1, G = 1).

The third section deals the main question: what properties should GR assign to a particle moving according to a null-geodesics? Without resorting to any previous theory, the general relativity itself imposes very stringent and interesting properties for "photons" (of course in agreement with those asserted by special relativity in the absence of a gravitational field).

1 Null geodesics of light

In a static gravitational field the speed of light is notoriously:

$$c_g = \sqrt{g_{00}}\, c \tag{1}$$

For simplicity's sake it is convenient to use the Schwarzschild metric, in which the static gravitational field has spherical symmetry and r_s is the Schwarschild radius:

$$g_{oo} = g_{oo}(r) = 1 - \frac{r_s}{r} \tag{2}$$

The differential equation of an one-dimensional radial motion in the case of a null-geodesics (i.e. of light) is:

$$\left(\frac{dr}{d\tau}\right)^2 = c^2\mathcal{E}^2 - \left(1 - \frac{r_s}{r}\right)\frac{\mathcal{L}^2}{r^2} \tag{3}$$

where $\mathcal{E}$ and $\mathcal{L}$ are the integration constants of the energy and of the angular momentum:

$$\left(1 - \frac{r_s}{r}\right)\frac{dt}{d\tau} = \mathcal{E} \tag{4}$$

$$r^2\frac{d\phi}{d\tau} = \mathcal{L} \tag{5}$$

It will also be useful to choose a notation for the ratio of these constants:

$$b = \frac{\mathcal{L}}{c\,\mathcal{E}} = \frac{l}{cE} \tag{6}$$

We will call effective potential for the null-geodesic the quantity:

$$V(r) = \left(1 - \frac{r_s}{r}\right)\frac{\mathcal{L}^2}{r^2} \tag{7}$$

Since $\left(\frac{dr}{d\lambda}\right)^2$ is non-negative, the trajectory of the light ray develops exclusively in regions for which it results:

$$c^2\mathcal{E}^2 - V(r) \geq 0 \tag{8}$$

It is elementary to check that the function $V(r) = \left(1 - \frac{r_s}{r}\right)\frac{\mathcal{L}^2}{r^2}$ has a maximum at the point $(\frac{3}{2}r_s, \frac{4}{27}\frac{\mathcal{L}^2}{r_s^2})$. In order for light to not be confined, so that propagation is possible for every value of r, it is necessary that:

$$c^2\mathcal{E}^2 \geq \frac{4}{27}\frac{\mathcal{L}^2}{r_s^2} \qquad \rightarrow \quad b < \sqrt{\frac{27}{4}}\,r_s \tag{9}$$

2 Gravitational light bending

By eliminating the parameter τ between (3) and the integral of energy (5) we obtain the equation of the spatial trajectory $r(\phi)$, which allows us to analyze the deflection of light in Schwarzschild geometry for a light ray that is not captured:

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 = \left(\frac{d\frac{1}{r}}{d\phi}\right)^2 = \frac{1}{b^2} - \left(1 - \frac{r_s}{r}\right)\frac{1}{r^2} \tag{10}$$

or in equivalent form (with the minus sign because ϕ is a decreasing function of r):

$$-\frac{d\phi}{dr} = \frac{\mathcal{L}}{r^2}\left[c^2\mathcal{E}^2 - \left(1 - \frac{r_s}{r}\right)\frac{\mathcal{L}^2}{r^2}\right]^{-\frac{1}{2}} = \frac{1}{r^2}\left[\frac{1}{b^2} - \left(1 - \frac{r_s}{r}\right)\frac{1}{r^2}\right]^{-\frac{1}{2}} \tag{11}$$

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$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{dr}{d\phi}\right)^2\left(\frac{d\phi}{d\tau}\right)^2 \qquad c^2\mathcal{E}^2 - \left(1 - \frac{r_s}{r}\right)\frac{\mathcal{L}^2}{r^2} = \left(\frac{dr}{d\phi}\right)^2\left(\frac{\mathcal{L}}{r^2}\right)^2 \qquad \left(\frac{d\phi}{dr}\right)^2 = \left(\frac{\mathcal{L}}{r^2}\right)^2\left(c^2\mathcal{E}^2 - \left(1 - \frac{r_s}{r}\right)\frac{\mathcal{L}^2}{r^2}\right)^{-1}$

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The impact parameter b in figure coincides with the constant (6):

$$b = \lim_{r\to+\infty} r\sin\phi(r) = \frac{\mathcal{L}}{c\,\mathcal{E}} = \frac{c\,l}{E} \tag{12}$$

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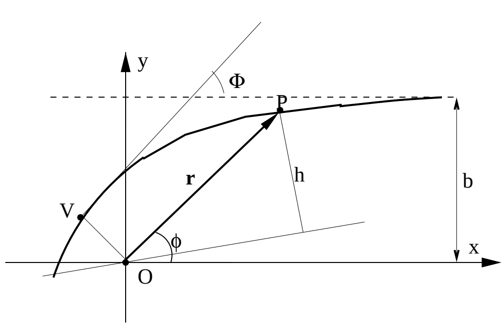
Indeed using De l'Hopital's theorem we have:



$$\lim_{r\to+\infty} r\sin\phi(r) = \lim_{r\to+\infty} r\frac{\sin\phi(r)}{\phi(r)}\,\phi(r) = \lim_{r\to+\infty} r\phi(r)$$

$$= \lim_{r\to+\infty}\frac{\phi(r)}{\frac{1}{r}} = \lim_{r\to+\infty}\frac{\frac{d\phi(r)}{dr}}{-\frac{1}{r^2}} = \lim_{r\to+\infty}(-r^2)\frac{d\phi(r)}{dr}$$

and recalling the equation of the light ray (10) we get:

$$\lim_{r\to+\infty} r\sin\phi(r) = \lim_{r\to+\infty}(-r^2)\frac{d\phi(r)}{dr} = \lim_{r\to+\infty}\left[\frac{1}{b^2} - \left(1 - \frac{r_s}{r}\right)\frac{1}{r^2}\right]^{-\frac{1}{2}} = b$$

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The calculus of the deflection angle carried out by integration of equation (11) can be found on Wald [1]. An alternative calculation method uses the change of variable $r \to u$ which allows us to write the equation of the light ray (10) in a form suitable for the evaluation of the light bending with perturbative methods.

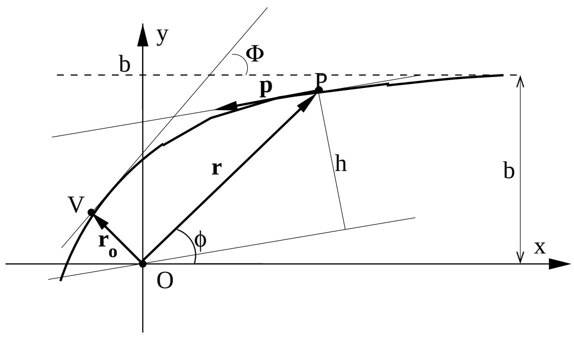
$$\frac{b}{r} = u \qquad \left(\frac{du}{d\phi}\right)^2 = 1 - \left(1 - \frac{r_s u}{b}\right) u^2 \qquad \frac{du}{d\phi} = -\sqrt{1 - \left(1 - \frac{r_s u}{b}\right) u^2} \tag{13}$$

The result obtained for the gravitational deflection of light is notoriously twice that calculated in the classical mechanical theory of central fields:

$$\Delta\Phi = 2\Phi = \frac{2r_s}{b} = \frac{4GM}{c^2 b} \tag{14}$$

3 Gravitational deflection of photons

Let us now examine the gravitational deflection assuming that the light is of corpuscular nature, that is, formed by particles that we will call photons. To determine the minimum distance r_o of the light ray as a function of the impact parameter b, it is sufficient to look for the zeros of the derivative (13) in order to find the maximum value of the variable $u = \frac{b}{r}$ (aka the minimum value of r):



$$\left(\frac{du}{d\phi}\right)^2 = 1 - \left(1 - \frac{r_s u}{b}\right) u^2 = 0 \qquad \to \qquad 1 - u^2 + \frac{r_s}{b} u^3 = 0 \qquad \to \qquad r^3 - b^2 r + r_s b^2 = 0$$

The minimum distance r_o is the only real root of this equation; recalling (2) and (6) we see that it satisfies the relation:

$$r_o = b\sqrt{1 - \frac{r_s}{r_0}} = b\sqrt{g_{oo}(r_o)} = \frac{cl}{E}\sqrt{g_{oo}(r_o)} \tag{15}$$

In the presence of a spherically symmetric gravitational field and in agreement with the two integrals found, the motion of the photon is characterized by the conservation of energy E and of the angular momentum l with respect to the central body.
It is natural to define the momentum of the photon within a gravitational field in such a way that we can interpret in the figure the angular momentum l as the product of the momentum p_g of the photon by the lever arm h with respect to the central pole O:

$$p_g = \frac{l}{h} \tag{16}$$

According to the expression (15) at the minimum distance r_o from O and in the absence of the gravitational field (i.e. at infinity), the formulas for the photon momentum are:

$$p_o = \frac{l}{r_o} \qquad\qquad p = p_\infty = \frac{l}{b} = \frac{E}{c} \qquad\qquad \frac{p_o}{p} = \frac{b}{r_o} = \frac{1}{\sqrt{g_{oo}(r_o)}} \tag{17}$$

We can see that the momentum-energy relation valid for photons at infinity, i.e. in the absence of a gravitational field coincides with the property known from special relativity, here obtained with only the concepts of GR:

$$E = p\,c \tag{18}$$

Furthermore, since in the (17) the distance r_o can take on any value (by varying the parameter b), it can be asserted that the photon's momentum p_g in the presence of a gravitational field in all generality is:

$$p_g = \frac{p}{\sqrt{g_{oo}(r)}} \tag{19}$$

Remembering that in a static gravitational field the speed of light ìs given by (1) we arrive at the following remarkable conclusion:

$$E = p_g\, c_g = p\, c \tag{20}$$

Energy is a constant of motion, coherently the momentum p_g of a photon turns out to be inversely proportional to the local speed c_g of light, at least in the case of static gravitational fields.
Now introducing a pseudo-Planck-constant h_g depending on the gravitational field:

$$h_g = h\,\sqrt{g_{00}} \tag{21}$$

and assuming the Planck-Einstein formula for the energy of photons $E = h\nu$ (in the absence of a gravitational field) one can write that:

$$E = h_g \nu_g \qquad\qquad p_g = \frac{h_g}{\lambda_g} \tag{22}$$

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$E = h\nu = h\,(\nu_g\sqrt{g_{00}}) = h_g\nu_g$
$E = h\nu = p_g c_g = p_g\lambda_g\nu_g \quad \rightarrow \; p_g = \frac{h\nu}{\lambda_g\nu_g} = \frac{h}{\lambda_g}\sqrt{g_{00}} = \frac{h_g}{\lambda_g}$
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In absence of a gravitational field these formulas reduce to the classical ones of photons ($E = h\nu,\; p = h/\lambda$) and establish an interesting connection between GR and QM.

4 Conclusions

These results coincide with those obtained in the article by Vaclav Vavricuk [5], who arrives at them by transforming the formulas of special relativity through Maxwell's equations in GR. In this paper instead we have made exclusive use of the properties of null-like geodesics, that is, we have exploited only the fundamental notions of GR, without any reference to previous theories. It is worth noting that no mention was made of mass, but only of the energy and momentum of photons.
The results (22) suggest the bold hypothesis that the Planck constant h measured in terrestrial laboratories may be in fact h_g, that is, that its value depends on the terrestrial gravitational field. This hypothesis could be experimentally checked by carrying out the measurement aboard an orbiting station, but the required precision is not currently achievable.

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