Supportive intersection (without analyticity)

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Preface

Content of the paper is divided into two parts. But it is in the reversed order for the organization.

Part I. On a manifold, we apply the analysis in Part II below to define an intersection called supportive intersection for singular cycles. It has a topological descend to the cup-product. The result is motivated by a problem in cohomology theory. The tool is the notion of currents. A current which is a functional was first introduced by de Rham in 1955. Ever since then, currents played a central role in geometry. However, the part about the support has not been in focus. For instance, the cup-product has been extensively studied in the past. Yet, there is no adequate control on the support of cohomological classes. So, we would like to introduce the supportive intersection that will catch this property. The purpose of this paper is to build the foundation for exploring further. In the end, we'll give an application in this direction.

Part II. This is the technical foundation for the geometry above, but it may have an independent interest. It consists of a functional analysis on a very specific type of convergence of currents. In terms of classical analysis, it is an extension of mollifiers. Classically, mollifier is mostly applied as a smoother for a distribution which is usually viewed as a current of degree 0. We extend the mollifier to currents where the degrees are positive.

Key words: forms, currents, de Rham's regularization, Lebesgue measure

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Abstract

Let X be a differential manifold. Let $\mathscr{D}'(X)$ be the space of currents, and $\mathcal{S}(X)$ the Abelian group freely generated by regular cells, each of which is a pair of a polyhedron Π and a differential embedding of a neighborhood of Π to X. In this paper, we define a variant that is a bilinear map

$$\begin{array}{ccc} \mathcal{S}(X) \times \mathcal{S}(X) & \rightarrow & \mathscr{D}'(X) \\ (c_1, c_2) & \rightarrow & [c_1 \wedge c_2] \end{array}$$

called the supportive intersection such that

1) the support of $[c_1 \wedge c_2]$ is contained in the intersection of the supports of c_1 and c_2;

2) if c_1, c_2 are closed, $[c_1 \wedge c_2]$ is also closed and its cohomology class is the cup-product of the cohomology classes of c_1, c_2.

Then we show a connection between the supportive intersection and the Hodge's problem.

I.1 Introduction to the supportive intersection

In transcendental geometry, support as a closed set is attached to other more notable invariants such as forms, chains and various abstract classes. Although itself has no independent interest, support has always served as a foundation for other structures to be built on. We found that it could play a role in the global structure. In this paper we start exploring it in the ring of the singular cohomology theory, and show an application. The paper is organized in a reversed order. We first go straight to its verification and consequence in

Part I, and leave the foundational analysis in Part II. Let X be a differential manifold. We would like to construct a variant that is a bilinear map

$$\begin{array}{ccc} \mathcal{S}(X)\times\mathcal{S}(X) & \to & \mathscr{D}'(X) \\ (c_1, c_2) & \to & [c_1\wedge c_2] \end{array} \tag{I.1}$$

such that

Condition I.1. *(supportivity) the support of* $[c_1\wedge c_2]$ *is contained in the intersection of the supports of* c_1 *and* c_2*;*

Condition I.2. *(cohomologicality) if* c_1, c_2 *are closed,* $[c_1\wedge c_2]$ *is also closed and its cohomology class is the cup-product of the cohomology classes of* c_1, c_2*.*

The idea goes back to de Rham's work on currents. De Rham started this direction where he constructed, for an arbitrary current T, the regularization $R_\epsilon T$ for a real number $\epsilon > 0$ such that the regularization weakly converges to T as $\epsilon \to 0$. From that, de Rham obtained an interpretation of the geometric intersection number. Based on this intersection number, he worked out his famous theory in topology. However, the properties of the regularization is far beyond the topology where he finally landed in. For instance, the supportive property we are interested in is one of them. In particular, it satisfies that

1) (homotopy) there exists another linear operator A_ϵ satisfying the homotopy formula

$$R_\epsilon T - T = bA_\epsilon T + A_\epsilon bT \tag{I.2}$$

where b is the boundary operator on currents,

2) (supportivity) the support of $R_\epsilon T$ is contained in any given neighborhood of the support of T provided ϵ is sufficiently small;

As the general theory moves on, de Rham's old method slowly becomes an echo of reminiscence that only serves as a reminder. For instance, in Thom isomorphism, any closed manifold M is cohomologous to a smooth form whose support is in any tubular neighborhood of M. This clearly follows from the supportivity, condition 2) above. In general, the development of the property 1) provided the axiomatic basis for homological algebra. But on the other hand, the implication of the property 2) lags behind, and the subtlety in his analysis is no longer in the mainstream.

In this paper, we are going to focus on the property 2). We work with chains which are known to be a particular type of currents. Let c be a regular chain. Denote the current of the integration over c by T_c.

First we generalize the notion of a mollifier in functional analysis.

Definition I.3. *(blow-up forms)*

Let $\digamma_\epsilon$ *for* $\epsilon > 0$ *be a family of smooth forms of degree* r *in an Euclidean space* $\mathbb{R}^m$*. If there are an orthogonal decomposition* $\mathbb{R}^m = \mathbb{R}^r \oplus \mathbb{R}^{m-r}$ *with coordinate*

$\mathbf{u}$ for the subspace $\mathbb{R}^r$ and a smooth form $\digamma_1(\mathbf{u})$ on $\mathbb{R}^r$ with a compact support such that

$$\digamma_\epsilon = \pi^* \digamma_1(\frac{\mathbf{u}}{\epsilon}) \tag{I.3}$$

where $\pi : \mathbb{R}^m \to \mathbb{R}^r$ is the orthogonal projection, then $\digamma_\epsilon$ is called a blow-up form from $\digamma_1(\mathbf{u})$ along $\mathbb{R}^r$ at $\mathbb{R}^{m-r}$.

Definition I.4. *(de Rham's convergence) Let $\digamma_\epsilon$ be a blow-up form and c a regular cell in $\mathbb{R}^m$. The weak convergence of*

$$T_c \wedge \digamma_\epsilon \tag{I.4}$$

to a current as $\epsilon \to 0$ is called de Rham's convergence.

Remark In [2], G. de Rham proved the convergence for a special case with a topological assumption. It is conceivable that if the supportive intersection is defined to be the limit of de Rham's convergence of the current $T_{c_1} \wedge R_\epsilon(T_{c_2})$, the conditions I.1 and I.2 simply follow from de Rham's properties 1) and 2). The precise statement is our main theorem in the following.

Theorem I.5. *(Main theorem 1) Let X be a differential manifold. For*

$$(c_1, c_2) \in \mathcal{S}(X) \times \mathcal{S}(X)$$

with $dim(c_1) + dim(c_2) \geq dim(X)$, the current

$$[c_1 \wedge c_2] \tag{I.5}$$

satisfying Condition I.1 and Condition I.2 exists if de Rham's convergence holds. We call $[c_1 \wedge c_2]$ a supportive intersection.

Remark. The theorem provides us with an intersection beyond the topology. For instance, a supportive intersection does not require the objects to be closed. So the supportive intersection is not topological.

We organize the rest of chapter as follows. In Section 2, we prove that the de Rham's regularization $R_\epsilon c$ for any regular cell c is a blow-up form. In section 3, applying the de Rham's convergence in Part II, we'll directly verify that for $(c_1, c_2) \in \mathcal{S}(X) \times \mathcal{S}(X)$ indeed satisfies Conditions I.1 and I.2.

I.2 The blow-up form of de Rham's Regularization

The technique is a particular regularization of currents. In the literature, there are many different types of regularization such as de Rham's, Heat kernel, Demailly's psh regularization, etc. But we'll use the original regularization constructed by de Rham due to its full control on the support. We'll state it below.

Theorem I.6. *(G. de Rham) Let ϵ be a small positive number. Let $\mathscr{E}(X)$ be the space of smooth forms on X. Then there exist linear operators,*

$$\begin{array}{rcl} R_\epsilon : \mathscr{D}'(X) & \to & \mathscr{E}(X) \\ A_\epsilon : \mathscr{D}'(X) & \to & \mathscr{D}'(X) \end{array} \tag{I.6}$$

such that for $T \in \mathscr{D}'(X)$

(1) a homotopy formula

$$R_\epsilon T - T = bA_\epsilon T + A_\epsilon bT, \tag{I.7}$$

holds where b is the boundary operator,

(2) supp$(R_\epsilon T)$, supp$(A_\epsilon T)$ are contained in any given neighborhood of supp(T) provided ϵ is sufficiently small,

(3) If a smooth differential form ϕ has the bounded semi-norm $||\bullet||_{q,K}$ where q is a whole number and K is a compact set and ϵ is bounded above, then $R_\epsilon T_\phi, A_\epsilon T_\phi$ are also bounded in the same semi-norm,

(4)

$$\lim_{\epsilon\to 0} R_\epsilon T = T, \quad \lim_{\epsilon\to 0} A_\epsilon T = 0$$

in the weak topology of $\mathscr{D}'(X)$. Furthermore, the convergence is uniform on the set of forms with the bounded semi-norms $||\bullet||_{q,K}$.

The operator R_ϵ is called regulator. The collection of the data used in the regularization is called de Rham data. In particular, it consists of countably many ordered, covering open sets $U_1, \cdots$ where the local regularization occur independently in each U_i. The global regularization R_ϵ is just the iteration of the local regularization.

This paper needs the properties (1) and (2) only. In addition, de Rham showed that the kernel of the operator R_ϵ is a differential form, i.e. a differential form ϱ_ϵ on $X \times X$ such that for any $\phi \in \mathscr{D}(X)$ and $T \in \mathscr{D}'(X)$ with a compact support

$$R_\epsilon(T)[\phi] = (T \otimes T_\phi)[\varrho_\epsilon]$$

where the tensor product $T \otimes T_\phi$ is a current on $X \times X$ with a compact support. In the following, we prove a local property of this smooth kernel.

Proposition I.7.
At each point of X, there is a neighborhood $U \simeq \mathbb{R}^m$, such that the smooth kernel ϱ_ϵ of de Rham's regulator R_ϵ with sufficiently small ϵ is restricted to a blow-up form on $\mathbb{R}^m \times \mathbb{R}^m$ at the diagonal Δ.

Proof. We need to analyze the local structure of the rgulator. So, we start with the reviewing of the de Rham's construction in its local charts. Let $\mathbb{R}^m$ be the Euclidean space of dimension m with a linear structure. Let $y_1, \cdots, y_m$ be its coordinates under a basis. They will be collectively denoted by the **bold** letter $\mathbf{y}$. Same **bold** fonts for various Euclidean spaces will be used throughout this paper. Let $f(\mathbf{y}) \in \mathscr{D}(\mathbb{R}^m)$ be a function (i.e. a mollifier) supported in the unit ball such that

$$\int_{\mathbf{y}\in\mathbb{R}^m} f(\mathbf{y}) d\mu_{\mathbf{y}} = 1, \tag{I.8}$$

where $d\mu_{\mathbf{y}}$ is the volume form

$$dy_1 \wedge \cdots \wedge dy_m.$$

Let

$$\vartheta_\epsilon(\mathbf{y}) = \frac{1}{\epsilon^m} f(\frac{\mathbf{y}}{\epsilon}) d\mu_{\mathbf{y}}, \epsilon > 0 \tag{I.9}$$

be the m-form on $\mathbb{R}^m$. Then the de Rham's regulator on $\mathbb{R}^m$ is the operator that sends each current T on $\mathbb{R}^m$ to the form

$$\pm T[\vartheta_\epsilon\big(\mathbf{x} - \mathbf{y}\big)|_{\mathbf{y}}] \tag{I.10}$$

where the sign $\pm$ is determined by the dimension of T and m, the current T is evaluated at the double form but in the variable $\mathbf{y}$. The operator depends on the coordinates of $\mathbb{R}^m$. We denote this regulator by $\mathcal{R}_\epsilon$. The form

$$\pm\vartheta_\epsilon\big(\mathbf{x} - \mathbf{y}\big) \tag{I.11}$$

on $\mathbb{R}^m \times \mathbb{R}^m$ is denoted by $\theta_\epsilon(\mathbf{x}-\mathbf{y})$ where $\mathbf{x}, \mathbf{y}$ are the variables for the first and second factors in $\mathbb{R}^m \times \mathbb{R}^m$. Notice that $\theta_\epsilon(\mathbf{x} - \mathbf{y})$ is the smooth kernel of $\mathcal{R}_\epsilon$ (with respect to the degree of c). * The extension to the global X is through a countable iteration of the local $\mathcal{R}_\epsilon$. The extension requires countably many local charts $U_i \simeq \mathbb{R}^m$ in de Rham data that covers X. The covering is locally finite. By the continuity, we may only consider the point q not on the boundaries of U_i. Such an extension at the point q can be described as follows. Since the de Rham's covering is locally finite, there are finitely many ordered open sets, $U_1, U_2, \cdots, U_n$ that contain q. It suffices to consider the regularization in these open sets. We denote the regulator on each U_i by $\mathcal{R}^i_\epsilon$ and its smooth kernel by $\theta^i_\epsilon(\mathbf{x}_i - \mathbf{y}_i)$. By the partition of unity, we may only consider the current T

*We should note that this kernel is supported in a neighborhood of the diagonal. The assertion is the base of de Rham's construction. For instance, the evaluation (I.11) is based on that.

compactly supported in the overlap $\underset{i}{\cap} U_i$. Then the global R_ϵ sends the T to a smooth form

$$\mathcal{R}_\epsilon^n \circ \mathcal{R}_\epsilon^{n-1} \circ \cdots \circ \mathcal{R}_\epsilon^1(T). \tag{I.12}$$

Above is the description of de Rham's construction around the point q. The following is our work to show that the kernel of (I.12) is a blow-up form. First we'll express the kernel. In each local regulator

$$\mathcal{R}_\epsilon^i : \mathscr{D}'(\mathbb{R}^m) \to \mathscr{E}(\mathbb{R}^m)$$

we denote the $\mathbb{R}^m$ in the domain space by $\mathbb{R}^m_{\mathbf{y}_i}$ with the variable $\mathbf{y}_i$, and the $\mathbb{R}^m$ in the target space by $\mathbb{R}^m_{\mathbf{x}_i}$ with variable $\mathbf{x}_i$. We identify $\mathbb{R}^m_{\mathbf{y}_i} = \mathbb{R}^m_{\mathbf{x}_{i-1}}$ and denote it by $\mathbb{R}^m_{i,(i-1)}$ (which is diffeomorphic to $\mathbb{R}^m$). Then each product $\mathbb{R}^m_{\mathbf{x}_i} \times \mathbb{R}^m_{\mathbf{y}_i}$ for $i = n, \cdots, 1$ is embedded in

$$\mathbb{R}^m_{\mathbf{x}_n} \times \mathbb{R}^m_{n,n-1} \times \mathbb{R}^m_{n-1,n-2} \times \cdots \times \mathbb{R}^m_{2,1} \times \mathbb{R}^m_{\mathbf{y}_1}$$

as the zero-section of the trivial bundle. So, we pull back each $\theta_\epsilon^i(\mathbf{x}_i - \mathbf{y}_i)$ to the the product

$$\mathbb{R}^m_{\mathbf{x}_n} \times \mathbb{R}^m_{n,n-1} \times \mathbb{R}^m_{n-1,n-2} \times \cdots \times \mathbb{R}^m_{2,1} \times \mathbb{R}^m_{\mathbf{y}_1}$$

and denote the pullbacks with the same notation $\theta_\epsilon^i(\mathbf{x}_i - \mathbf{y}_i)$. Then according to (I.12), the local expression of the global kernel $\varrho_\epsilon(\mathbf{x}_n, \mathbf{y}_1)$ is the fibre integral

$$\int_{(\mathbf{y}_n,\cdots,\mathbf{y}_2)\in\mathbb{R}^m_{n,n-1}\times\mathbb{R}^m_{n-1,n-2}\times\cdots\times\mathbb{R}^m_{2,1}} \theta_\epsilon^n(\mathbf{x}_n - \mathbf{y}_n) \wedge \theta_\epsilon^{n-1}(\mathbf{x}_{n-1} - \mathbf{y}_{n-1}) \wedge \cdots \wedge \theta_\epsilon^2(\mathbf{x}_2 - \mathbf{y}_2) \wedge \theta_\epsilon^1(\mathbf{x}_1 - \mathbf{y}_1), \tag{I.13}$$

where $\theta_\epsilon^i(\mathbf{x}_i - \mathbf{y}_i)$ is the smooth kernel of $\mathcal{R}_i$. So the global kernel $\varrho_\epsilon(\mathbf{x}_n, \mathbf{y}_1)$ is an m-form on the product

$$\mathbb{R}^m_{\mathbf{x}_n} \times \mathbb{R}^m_{\mathbf{y}_1} = \mathbb{R}^m \times \mathbb{R}^m$$

where $\mathbf{x}_n, \mathbf{y}_1$ are the coordinates for the first and second factor of the kernel. In (I.13), we define the new coordinates:

$$\mathbf{w}_i = \mathbf{x}_i - \mathbf{y}_i \tag{I.14}$$

where $i = 1, \cdots, n-1$, also

$$\mathbf{x}_n - \mathbf{y}_1 - \left(\mathbf{w}_1 + \cdots + \mathbf{w}_{n-1}\right) = \mathbf{x}_n - \mathbf{y}_n. \tag{I.15}$$

Then (I.13) is equal to

$$\int_{(\mathbf{w}_{n-1},\cdots,\mathbf{w}_1)\in\mathbb{R}^m_{n,n-1}\times\mathbb{R}^m_{n-1,n-2}\times\cdots\times\mathbb{R}^m_{2,1}} \theta_\epsilon^n\left(\mathbf{x}_n - \mathbf{y}_1 - \big(\mathbf{w}_1 + \cdots + \mathbf{w}_{n-1}\big)\right) \wedge \theta_\epsilon^{n-1}(\mathbf{w}_{n-1}) \wedge \cdots \wedge \theta_\epsilon^1(\mathbf{w}_1), \tag{I.16}$$

Divide each variable by ϵ, we obtain that $\varrho_\epsilon(\mathbf{x}_n, \mathbf{y}_1)$ is equal to

$$\int_{(\mathbf{w}_{n-1},\cdots,\mathbf{w}_1)\in\mathbb{R}^m_{n,n-1}\times\mathbb{R}^m_{n-1,n-2}\times\cdots\times\mathbb{R}^m_{2,1}} \theta_1^n\left(\frac{\mathbf{x}_n-\mathbf{y}_1}{\epsilon} - \big(\mathbf{w}_1+\cdots+\mathbf{w}_{n-1}\big)\right) \wedge \theta_1^{n-1}(\mathbf{w}_{n-1}) \wedge\cdots\wedge \theta_1^1(\mathbf{w}_1). \tag{I.17}$$

So, if we denote the m form on $\mathbb{R}^m$,

$$\int_{(\mathbf{w}_{n-1},\cdots,\mathbf{w}_1)\in\mathbb{R}^m_{n,n-1}\times\mathbb{R}^m_{n-1,n-2}\times\cdots\times\mathbb{R}^m_{2,1}} \theta_1^n\left(\frac{\mathbf{z}}{\epsilon} - \big(\mathbf{w}_1+\cdots+\mathbf{w}_{n-1}\big)\right) \wedge \theta_1^{n-1}(\mathbf{w}_{n-1}) \wedge\cdots\wedge \theta_1^1(\mathbf{w}_1) \tag{I.18}$$

by $F_\epsilon(\mathbf{z})$ for the variable $\mathbf{z}$ of $\mathbb{R}^m$, then

$$\varrho_\epsilon(\mathbf{x}_n, \mathbf{y}_1) = \kappa^* F_\epsilon \tag{I.19}$$

where κ is the map: $(\mathbf{x}_n, \mathbf{y}_1) \to \mathbf{x}_n - \mathbf{y}_1$. Since all forms $\theta_1^j(\mathbf{z}), j = n, \cdots, 1$ have compact supports, so $\varrho_\epsilon(\mathbf{x}_n, \mathbf{y}_1)$ is a blow-up form from a compactly supported form F_1. We complete the proof.

□

The following proposition proves the first part of Main theorem I.5.

Proposition I.8. *Let X be a differential manifold of dimension m. For chains c_1, c_2 in $\mathcal{S}(X)$ with $dim(c_1) + dim(c_2) \geq m$, the exterior product*

$$T_{c_1} \wedge R_\epsilon c_2 \tag{I.20}$$

converges weakly to a current as $\epsilon \to 0$.

Proof. The convergence is local. Thus we assume the following calculation occurs in a neighborhood diffeomorphic to $\mathbb{R}^m$. It suffices to assume $c_2 : \Pi_p \to \mathbb{R}^m$ is a regular cell and it lies in an open neighborhood U. We subdivide c_1 to a sum of smaller regular cells so that there are finitely many regular cells σ_j that cover the $supp(R_\epsilon c_2)$ for sufficient small ϵ and $supp(\sigma_j) \subset U$. Then

$$T_{c_1} \wedge R_\epsilon c_2 = \sum_j T_{\sigma_j} \wedge R_\epsilon c_2.$$

So, it suffices to prove the proposition for c_1 whose support lies in U. For a test form ϕ, the evaluation

$$\Big(T_{c_1} \wedge R_\epsilon c_2\Big)[\phi]$$

is equal to the integral in $X \times X$ as

$$\int_{(\mathbf{x},\mathbf{y})\in c_1\times c_2} \varrho_\epsilon(\mathbf{x}, \mathbf{y}) \wedge P^*(\phi)(\mathbf{x}) \tag{I.21}$$

where $P : X\times X \to X(1st\ copy)$ is the projection, $\varrho_\epsilon(\mathbf{x},\mathbf{y})$ is the kernel of R_ϵ. By Proposition I.7, $\varrho_\epsilon(\mathbf{x},\mathbf{y})$ is a blow-up form in the Euclidean space $U\times U \simeq \mathbb{R}^{2m}$ at the diagonal Δ_U. Thus (I.21) is the evaluation of the current

$$T_{c_1\times c_2} \wedge \varrho_\epsilon(\mathbf{x},\mathbf{y}) \tag{I.22}$$

at a particular form $P^*(\phi)$. Since $dim(c_1\times c_2) \geq m = deg(\varrho_\epsilon(\mathbf{x},\mathbf{y}))$, by Theorem II.2 (below), the limit

$$\lim_{\epsilon\to 0}\int_{(\mathbf{x},\mathbf{y})\in c_1\times c_2} \varrho_\epsilon(\mathbf{x},\mathbf{y}) \wedge P^*(\phi)(\mathbf{x}) \tag{I.23}$$

exists, and bounded by $||\phi||_\infty$. Hence

$$\lim_{\epsilon\to 0} T_{c_1} \wedge R_\epsilon c_2$$

is a current. The proof is completed.

$\square$

I.3 The supportive intersection

Definition I.9. *Let X be a differential manifold. Let c_1, c_2 be two chains in $\mathcal{S}(X)$. We define*

$$[c_1 \wedge c_2]$$

to be the weak limit

$$\lim_{\epsilon\to 0}\big(T_{c_1} \wedge R_\epsilon c_2\big).$$

It gives a rise to a well-defined bilinear map

$$\mathcal{S}(X) \times \mathcal{S}(X) \to \mathscr{D}'(X).$$

We call the map the supportive intersection.

The following properties (1) and (2) complete the proof for the second part of Main theorem I.5.

Property I.10.
Let X a differential manifold of dimension m. For chains c_1, c_2 in $\mathcal{S}(X)$, the supportive intersection $[c_1 \wedge c_2]$ satisfies:

(1) (Supportivity)

$$supp([c_1 \wedge c_2]) \subset supp(c_1) \cap supp(c_2). \tag{I.24}$$

(2) (Cohomologicity) if c_1, c_2 are closed, $[c_1 \wedge c_2]$ is closed and

$$\langle [c_1 \wedge c_2] \rangle = \langle c_1 \rangle \smile \langle c_2 \rangle \tag{I.25}$$

where $\langle \bullet \rangle$ denotes the cohomology class of a singular cycle.

(3) (Leibniz rule) If $deg(c_1) = p$, then the differential map of chains follows Leibniz rule,

$$d[c_1 \wedge c_2] = [dc_1 \wedge c_2] + (-1)^p [c_1 \wedge dc_2], \tag{I.26}$$

where the differential map d is the operator $(-1)^{p+1} b$ for the boundary operator b acting on chains of the codimension p.

Proof. (1) Suppose

$$\mathbf{a} \notin supp(c_1) \cap supp(c_2).$$

Then $\mathbf{a}$ must be either outside of $supp(c_1)$ or outside of $supp(c_2)$. Let's assume first it is not in $supp(c_2)$. Since the support of a currents is closed, we choose a small neighborhood $U_{\mathbf{a}}$ of $\mathbf{a}$ in X, but disjoint from $supp(c_2)$. Let ϕ be a C^∞-form of X with a compact support in $U_{\mathbf{a}}$. According to the part (2) of Theorem I.6, when ϵ is small enough $R_\epsilon(c_2)$ is zero in $U_{\mathbf{a}}$. Hence

$$[c_1 \wedge c_2][\phi] = 0. \tag{I.27}$$

Hence $\mathbf{a} \notin supp([c_1 \wedge c_2])$. If $\mathbf{a} \notin supp(c_1)$, $U_{\mathbf{a}}$ can be chosen disjoint with $supp(c_1)$. Then since $\phi \in \mathscr{D}(U_{\mathbf{a}})$ is a C^∞-form of X with a compact support in $U_{\mathbf{a}}$ disjoint with $supp(c_1)$, the restriction of ϕ to c_1 is zero. Hence

$$[c_1 \wedge c_2][\phi] = 0.$$

Then $\mathbf{a} \notin supp([c_1 \wedge c_2])$. Thus

$$\mathbf{a} \notin supp(c_1) \cap supp(c_2)$$

will always imply

$$\mathbf{a} \notin supp([c_1 \wedge c_2]).$$

This completes the proof.

(2) By the homotopy formula (I.7) in Theorem I.6, $R_\epsilon c_2$ is closed. Next let ϕ be a test form. By the definition

$$\begin{aligned} & b[c_1 \wedge c_2][\phi] \\ & = \lim_{\epsilon \to 0} \int_{c_1} R_\epsilon c_2 \wedge d\phi \\ & (\text{ since } c_1 \text{ is closed}) \\ & = \pm \lim_{\epsilon \to 0} \int_{c_1} dR_\epsilon c_2 \wedge \phi = 0. \end{aligned} \tag{I.28}$$

So $[c_1 \wedge c_2]$ is closed. For the closed test form ϕ, the supportive intersection number,

$$deg\Big(\big\langle [c_1 \wedge c_2] \big\rangle \smile \langle \phi \rangle\Big) \tag{I.29}$$

is a well-defined real number that is equal to

$$\lim_{\epsilon \to 0} c_1 [R_\epsilon(c_2) \wedge \phi]. \tag{I.30}$$

By the de Rham theorem

$$c_1 [R_\epsilon(c_2) \wedge \phi]$$

is the topological intersection number

$$deg(\langle c_1 \rangle \smile \langle R_\epsilon c_2 \rangle \smile \langle \phi \rangle). \tag{I.31}$$

By the homotopy formula (I.7) again, $\langle R_\epsilon c_2 \rangle = \langle c_2 \rangle$. Thus

$$\lim_{\epsilon \to 0} c_1 [R_\epsilon(c_2) \wedge \phi] = deg(\langle c_1 \rangle \smile \langle c_2 \rangle \smile \langle \phi \rangle). \tag{I.32}$$

Formulas (I.29) and (I.32) imply

$$\langle [c_1 \wedge c_2] \rangle = \langle c_1 \rangle \smile \langle c_2 \rangle \tag{I.33}$$

(3) Let ϕ be a test form. Let

$$deg(c_1) = p, deg(c_2) = q.$$

Then

$$\begin{aligned}
& b[c_1 \wedge c_2][\phi] \\
& = \lim_{\epsilon \to 0} \int_{c_1} R_\epsilon c_2 \wedge d\phi \\
& (\text{ Leibniz Rule for } C^\infty \text{ forms }) \\
& = \lim_{\epsilon \to 0} \int_{c_1} (-1)^q d(R_\epsilon c_2 \wedge \phi) + (-1)^{q+1} dR_\epsilon c_2 \wedge \phi \\
& = \lim_{\epsilon \to 0} (-1)^q \int_{bc_1} R_\epsilon c_2 \wedge \phi + \lim_{\epsilon \to 0} (-1)^{q+1} \int_{c_1} dR_\epsilon c_2 \wedge \phi \\
& (\text{By Formula (I.7), } d \text{ and } R_\epsilon \text{ commute}) \\
& = \lim_{\epsilon \to 0} (-1)^q \int_{bc_1} R_\epsilon c_2 \wedge \phi + \lim_{\epsilon \to 0} (-1)^{q+1} \int_{c_1} R_\epsilon dc_2 \wedge \phi \\
& = (-1)^q [bc_1 \wedge c_2][\phi] + (-1)^{q+1} [c_1 \wedge dc_2][\phi]
\end{aligned}$$

Hence

$$b[c_1 \wedge c_2] = (-1)^q [bc_1 \wedge c_2] + (-1)^{q+1} [c_1 \wedge dc_2]. \tag{I.34}$$

After change the sign, we found (I.34) is (I.26).

□

I.4 Application

The motivation of the supportive intersection lies in the Hodge's problem ([6]): describe the topological cycles contained in algebraic sets. Let X be a complex projective manifold. Denote the cohomology group of degree i with rational coefficients by $H^i(X;\mathbb{Q})$. Grothendieck in [4] converted Hodge's homology to sheaf cohomology by introducing an "arithmetic" filtration

$$\begin{array}{ccccccc} N^n H^i(X;\mathbb{Q}) & \subset \cdots & \subset & N^p H^i(X;\mathbb{Q}) & \subset \cdots & \subset N^0 H^i(X;\mathbb{Q}) \\ & & & & & \| \\ & & & & & H^i(X;\mathbb{Q}), \end{array} \tag{I.35}$$

formed by the subgroups,

$$N^p H^i(X) := \bigcup_{cod(V)\geq p} ker\Big(H^i(X;\mathbb{Q}) \to H^i(X-V;\mathbb{Q})\Big) \tag{I.36}$$

where V are algebraic sets and ker stands for the kernel of the restriction map. Today this filtration is named as the coniveau filtration, where the number p is called the coniveau and $i-2p$ is called the level. Grothendieck's re-formulation arose from his vision in the homological algebra ([5]). It successively extends Hodge's problem to that over other types of fields. However, we found there is more in Hodge's transcendental vision.

Corollary I.11. *Let X be a complex projective manifold of dimension n. Let $u \in H^2(X;\mathbb{Q})$ be a hyperplane section class. Let p, q, k be non-negative integers satisfying*

$$p+q+k=n, h=q-p, q\geq p.$$

Then the image of the map

$$\begin{array}{rcl} L_k^h : N^p H^{2p+k}(X) & \to & H^{2q+k}(X) \\ \alpha & \to & \alpha \cdot u^h. \end{array} \tag{I.37}$$

lies in the subgroup $N^q H^{2q+k}(X)$ and the map is injective.

Proof. Let σ be a regular cycle representing a class in $N^p H^{2p+k}(X)$. Then σ is contained in an algebraic set V of codimension p. Let W be a generic plane section of codimension h. Then the intersection $V \cap W$ is proper, i.e. it has complex codimension q. By the supportivity of the supportive intersection

$$supp([\sigma \wedge W]) \subset supp(V) \cap supp(W) = supp(V \bullet W) \tag{I.38}$$

where $V \bullet W$ is the algebo-geometric intersection in Serre's formula, and support of it is $|V \bullet W|$ (=support of the integration current).

Hence the cohomological class of $[\sigma \wedge W]$ lies in

$$ker\Big(H^{2q+k}(X;\mathbb{Q}) \to H^{2q+k}(X - supp(V \bullet W);\mathbb{Q})\Big) \subset N^q H^{2q+k}(X). \tag{I.39}$$

On the other hand the map is injective by the hard Lefschetz theorem. We complete the proof.

□

Remark: The surjectivity for $k = 0$ is the Lefschetz standard conjecture. The measure-theoretical analysis in [7] reveals that map L_k^h to $N^q H^{2q+k}(X; \mathbb{Q})$ is, indeed, surjective. Therefore there is a structural duality

$$N^p H^{2p+k}(X) \simeq N^q H^{2q+k}(X)$$

arising from the support of currents.

II Part II: The foundation

Abstract

This chapter establishes the foundation in local analysis for Chapter 1, i.e. we prove the de Rham's convergence. In terms of technique, we extend the convergence of mollifiers.

II.1 Introduction to currents of subdivision

On a differential manifold, let c be a singular chain whose current of the integration is denoted by T_c. The current $T_c \wedge \omega_\epsilon$ is called the current of subdivision. It has has been explored by G. de Rham in the case $dim(c) = deg(\omega_\epsilon)$. His study focused on topology which led to his theory in cohomology. The simplest case is in the Euclidean space where the codimension of c is zero. It coincides with mollifiers. Let $\mathbf{x} = (x_1, \cdots, x_n)$ be the coordinates of $\mathbb{R}^n$ with the volume form $dx_1 \wedge \cdots \wedge dx_n$ denoted by $d\mu$. Let ω_ϵ for $\epsilon > 0$ be the differential n-form,

$$\frac{1}{\epsilon^n} f(\frac{\mathbf{x}}{\epsilon}) d\mu \tag{II.1}$$

where $f(\mathbf{x})$ is a function of a mollifier, i.e. a smooth bump function around the origin such that

$$\int_{\mathbb{R}^n} f(\mathbf{x}) d\mu = 1.$$

Let c be an n-dimensional polyhedron in $\mathbb{R}^n$ that contains the origin as its interior point. Then the current $T_c \wedge \omega_\epsilon$, as $\epsilon \to 0$, converges weakly to the δ function at the origin (see chapter 3, [3]). In this chapter we would like to show that if the form ω_ϵ does not have the top-degree and does not meet the de Rham's topological requirement, the convergence in the sense of measures still holds. This measure theoretical convergence suggests a direction other than de Rham's more topological approach. † To state the convergence as a theorem, we first extends the mollifier to differential forms of lower degrees.

†See [2] for G. de Rham's more topological approach.

Definition II.1. *(blow-up forms)*

Let $\digamma_\epsilon$ for $\epsilon > 0$ be a family of smooth forms of degree r in an Euclidean space $\mathbb{R}^n$. If there are an orthogonal decomposition $\mathbb{R}^n = \mathbb{R}^r \oplus \mathbb{R}^{n-r}$ with coordinate $\mathbf{u}$ for the subspace $\mathbb{R}^r$ and a smooth form $\digamma_1(\mathbf{u})$ of degree r on $\mathbb{R}^r$ with a compact support such that

$$\digamma_\epsilon = \pi^* \digamma_1(\frac{\mathbf{u}}{\epsilon}) \tag{II.2}$$

or abbreviated as

$$\digamma_\epsilon = \digamma_1(\frac{\mathbf{u}}{\epsilon})$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^r$ is the orthogonal projection, then $\digamma_\epsilon$ is called a blow-up form along $\mathbb{R}^r$ at $\mathbb{R}^{n-r}$.

Theorem II.2. *(Main theorem 2) Let c be a p dimensional regular cell in $\mathbb{R}^n$. Let ω_ϵ be a blow-up form of degree $r \le p$ in $\mathbb{R}^n$. Then the current*

$$T_c \wedge \omega_\epsilon \tag{II.3}$$

converges weakly to a current as $\epsilon \to 0$.

In the following, we give the technical detail of the proof. It consists of one lemma in set-theoretic limit and an estimate in functional analysis. The appendix includes another lemma which is mainly for the estimate in analysis.

II.2 proof

In the following, for an Euclidean space $\mathbb{R}^l$ with a coordinate $\mathbf{z}$, we'll abuse the notation to denote the volume form of a subspace with the concordant orientation and the volume density in Lebesgue integrals by the same expression $d\mu_{\mathbf{z}}$. The argument starts with a definition and a lemma about points and sets.

Definition II.3. *Let $W \subset \mathbb{R}^p$ be a subset in an Euclidean space with the origin $\mathbf{o}$. A point $\mathbf{a} \in \mathbb{R}^p$ is said to be a stable point of W if the line segment*

$$\{\mathbf{o} + t(\overrightarrow{\mathbf{oa}}),\ 0 < t \le 1\}$$

either lies in W completely or in W^c completely, where $\overrightarrow{\mathbf{oa}} \in T_{\mathbf{o}}\mathbb{R}^p = \mathbb{R}^p$ is the vector from $\mathbf{o}$ to $\mathbf{a}$, and W^c is the complement $\mathbb{R}^p \backslash W$. We denote the collection of stable points of W by $W_s^{\mathbf{o}}$.

Recall a regular cell c is a couple: a) oriented polyhedron $\Pi_p \subset \mathbb{R}^p$, b) a diffeomorphic embedding c of a neighborhood of Π to $\mathbb{R}^n$. Let $\mathbb{R}^r, \mathbb{R}^{p-r}, \mathbb{R}^{n-p}$ be subspaces of $\mathbb{R}^n$ with coordinates $\mathbf{u}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively such that

$$\mathbb{R}^n = \mathbb{R}^r \oplus \mathbb{R}^{p-r} \oplus \mathbb{R}^{n-p}. \tag{II.4}$$

Let

$$\eta : \mathbb{R}^n \quad \to \quad \mathbb{R}^p = \mathbb{R}^r \oplus \mathbb{R}^{p-r}$$

be the orthogonal projection to its subspace $\mathbb{R}^p$. Let $D_{\frac{1}{\epsilon}}$ for a positive ϵ be the linear transformation of $\mathbb{R}^n$ defined by the map

$$(\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) \quad \to \quad (\frac{\mathbf{u}}{\epsilon}, \mathbf{v}_1, \mathbf{v}_2). \tag{II.5}$$

In the context, we denote its restriction to subspaces also by $D_{\frac{1}{\epsilon}}$. All measures in the following are the Lebesgue measures on Euclidean spaces.

Lemma II.4. *Denote $W := \eta(C)$. There exists a subset $W_{fu} \subset W$ of measure 0 such that the set-theoretic limit (defined in §4, [1])*

$$\lim_{\epsilon \to 0} D_{\frac{1}{\epsilon}}\left(W \backslash W_{fu}\right) \tag{II.6}$$

exists ‡.

Proof. We denote

$$L := \mathbb{R}^{p-r}$$

The point $\mathbf{o} \in L$ should be viewed as the origin of the affine subspace $\mathbb{R}^r \oplus \mathbf{o}$ where $\mathbf{o} \in \mathbb{R}^{p-r}$ is a point, and partial scalar multiplication $D_{\frac{1}{\epsilon}}$ acts on it as the scalar multiplication. Let

$$W^{\mathbf{o}} = W \cap \left(\mathbb{R}^r \oplus \{\mathbf{o}\}\right).$$

Let $\mathcal{R}_{\mathbf{o}}$ be the ray

$$\{\mathbf{o} + t(\overrightarrow{\mathbf{oa}}) : \mathbf{a} \in W^{\mathbf{o}}, t > 0\}$$

that starts at the origin in the affine plane. Let

$$W^{\mathbf{o}}_{fu} \subset W^{\mathbf{o}}$$

denote the subset

$$\{\mathbf{a} \in W^{\mathbf{o}} : \mathcal{R}_{\mathbf{o}} \textit{ does not contain a stable point of } W^{\mathbf{o}}\}.$$

We divide W to three disjoint parts.

1) $W_{fu} = \bigcup_{\mathbf{o} \in L} W^{\mathbf{o}}_{fu}$, called the set of fully unstable points,

2) $W_s = \bigcup_{\mathbf{o} \in L} W^{\mathbf{o}}_{s}$, called the set of stable points,

‡For a family of sets S_ϵ, the existence of the set-theoretic limit means

$$\bigcap_{\epsilon_1 \leq 1} \bigcup_{\epsilon_2 \leq \epsilon_1} S_{\epsilon_2} = \bigcup_{\epsilon_1 \leq 1} \bigcap_{\epsilon_2 \leq \epsilon_1} S_{\epsilon_2}$$

3) W_{pu} is $W\backslash(W_{fu} \cup W_s)$, called the set of partially unstable points.

Next we blow-up each part by the scalar multiplication $D_{\frac{1}{\epsilon}}$ with $\epsilon \to 0$.

For the fully unstable points W_{fu}, we would like to show they are necessarily on the "boundary" which gives the measure 0. The following is the detail. The boundary of the polyhedron Π_p is defined by multiple hyperplanes. Hence the boundary of C is also defined by multiple hyperplanes H_j. On the other hand in the its target space, we let

$$\begin{array}{cccc} \nu : \mathbb{R}^r\backslash\{\mathbf{0}\} \oplus \mathbb{R}^{p-r} & \to & \mathbb{P}^{r-1} \times \mathbb{R}^{p-r} \\ (\mathbf{u}, \mathbf{v}_1) & \to & ([\mathbf{u}], \mathbf{v}_1) \end{array} \tag{II.7}$$

be the map that is the product of the projectivization map and the identity map (where $\mathbb{P}^{r-1}$ can be regarded as the real projectivization of $T_{\mathbf{0}}\mathbb{R}^r$, the set of directions). Fix a point $\mathbf{o} \in L$. Let $\mathbf{a} \in W^{\mathbf{o}}_{fu}$ other than $\mathbf{o}$. Since $\mathbf{a}$ is a fully unstable point, there are two sequences of points $\mathbf{p}_n, \mathbf{q}_n$ on the ray $\mathcal{R}_{\mathbf{o}}$ such that

$$\lim_{n\to\infty} \mathbf{p}_n = \mathbf{o} = \lim_{n\to\infty} \mathbf{q}_n$$

and

$$\mathbf{p}_n \notin W^{\mathbf{o}}, \mathbf{q}_n \in W^{\mathbf{o}}.$$

Thus the directions $\overrightarrow{\mathbf{op}_n}$ and $\overrightarrow{\mathbf{oq}_n}$, which are all parallel to the tangent vector $\overrightarrow{\mathbf{oa}}$ must lie on at least one nontrivial plane $\eta_*(H_j)$. Since a subplane properly contained in an Euclidean space has a measure 0, for each fixed $\mathbf{o}$, $\mathbb{P}(W^{\mathbf{o}}_{fu}\backslash\{\mathbf{o}\})$ has measure 0 in the manifold

$$\mathbb{P}(\mathbb{R}^r\backslash\{\mathbf{0}\}) \times \{\mathbf{o}\} \simeq \mathbb{P}^{r-1}$$

where $\mathbf{o}$ is fixed. Since

$$\mathbb{R}^r\backslash\{\mathbf{0}\} \to \mathbb{P}^{r-1}$$

is a bundle's projection, the inverse $W^{\mathbf{o}}_{fu}$ also has measure 0. To go further, we take the union over L to obtain $\nu(W_{fu}\backslash L) = \bigcup_{\mathbf{o}\in L} \mathbb{P}(W^{\mathbf{o}}_{fu}\backslash\{\mathbf{o}\})$ has measure 0 in the manifold

$$\mathbb{P}^{r-1} \times \mathbb{R}^{p-r}.$$

Due to the fibre bundle structure of the projectivization, we conclude W_{fu} in $\mathbb{R}^p$ has measure 0. Notice that $D_{\frac{1}{\epsilon}}$ is a linear transformation, $D_{\frac{1}{\epsilon}}(W_{fu})$ which is equal to W_{fu} also has measure 0. Therefore the limit is of 0. [§]

For stable points W_s, we consider the set $B_\epsilon = D_{\frac{1}{\epsilon}}(W_s)$. We would like to show B_ϵ as $\epsilon \to 0$ is a decreasing set. So it converges to a measurable set. The following is the detail. Let $\mathcal{R}_{\mathbf{o}}$ be the ray starting at $\mathbf{o} \in L$ and through a stable point $\mathbf{a} \in W^{\mathbf{o}}_s$ of $W^{\mathbf{o}}$ for an $\mathbf{o} \in L$. Since $\mathbf{a}$ is stable, the dilation by the scalar multiplication $D_{\frac{1}{\epsilon}}$ yields

$$D_{\frac{1}{\epsilon}}(\mathcal{R}_{\mathbf{o}} \cap W_s) \subset D_{\frac{1}{\epsilon'}}(\mathcal{R}_{\mathbf{o}} \cap W_s), \quad \text{for } \epsilon' < \epsilon < 1.$$

[§]But the set W_{fu} is not on the boundary of W.

Now taking the union over all the rays through stables points, we obtain

$$D_{\frac{1}{\epsilon}}(W_s) \subset D_{\frac{1}{\epsilon'}}(W_s), \quad \text{for } \epsilon' < \epsilon.$$

Therefore B_ϵ is a decreasing family of measurable sets. Let

$$B_0 := \cup_{\epsilon\in(0,1]}\left(D_{\frac{1}{\epsilon}}(W_s)\right). \tag{II.8}$$

Then set-theoretically the decreasing family yields

$$\lim_{\epsilon\to 0} B_\epsilon = B_0$$

and B_0 is measurable.

For partially unstable point W_{pu}, we consider the set $A_\epsilon = D_{\frac{1}{\epsilon}}(W_{pu})$. We would like to show A_ϵ as the set multiplied by $\dfrac{1}{\epsilon}$ will be pushed to ∞ as $\epsilon \to 0$. So it is empty. Here is the detail. If $\bigcap\limits_{\epsilon_1\leq 1}\bigcup\limits_{\epsilon_2\leq\epsilon_1} A_{\epsilon_2}$ is non-empty, there is a point

$$\mathbf{x} \in \bigcap_{\epsilon_1\leq 1}\bigcup_{\epsilon_2\leq\epsilon_1} A_{\epsilon_2}$$

i.e. $\mathbf{x} \in \bigcup\limits_{\epsilon_2\leq\epsilon_1} A_{\epsilon_2}$ for any $\epsilon_1 < 1$. So, there is a sequence of numbers ϵ_n such that $\lim\limits_{n\to\infty}\epsilon_n = 0$ and $D_{\epsilon_n}(\mathbf{x})$ lies in W_{pu}. Suppose that N is a number in the sequence such that $D_{\epsilon_N}(\mathbf{x}) \in W_{pu}$. By the definition of W_{pu}, there is a smaller $\epsilon_{N'} \neq 0$ such that $D_{\epsilon_{N'}}(\mathbf{x})$ is a stable point, i.e. $D_{\epsilon_{N'}}(\mathbf{x}) \in W_S$. Then all points $D_{\epsilon_n}(\mathbf{x})$ are stable whenever $\epsilon_n < \epsilon_{N'}$. But this contradicts the assertion above: there is a sequence of partially unstable points $\epsilon_n\mathbf{x}$ with $\epsilon_n \to 0$. Thus

$$\lim_{\epsilon\to 0} sup A_\epsilon = \bigcap_{\epsilon_1\leq 1}\bigcup_{\epsilon_2\leq\epsilon_1} A_{\epsilon_2} = \varnothing. \tag{II.9}$$

Therefore

$$\lim_{\epsilon\to 0} inf A_\epsilon \subset \lim_{\epsilon\to 0} sup A_\epsilon$$

is also empty. Hence $\lim\limits_{\epsilon\to 0} A_\epsilon$ exists and is equal to an empty set.

Combining the results for W_{fu}, W_s and W_{pu}, we complete the proof. □

Proof of Theorem II.2. We continue with all notations in Lemma II.4. Let ϕ be a test form of degree $p-r$ in $\mathbb{R}^n$. It amounts to show the convergence of the integral

$$\int_c \omega_\epsilon \wedge \phi \tag{II.10}$$

as $\epsilon \to 0$. Let $\mathbb{R}^r$ be the subspace with coordinates $\mathbf{u}$ such that the blow-up form is written as

$$\omega_\epsilon = \frac{1}{\epsilon^r} g(\frac{\mathbf{u}}{\epsilon}) d\mu_{\mathbf{u}} \tag{II.11}$$

where $g(\mathbf{u})$ is a C^∞ function of $\mathbb{R}^r$. Notice that the form $\omega_\epsilon \wedge \phi$ is the sum of simple forms in the coordinates of $\mathbb{R}^n$ that can be explicitly expressed. So, we'll focus on the integral of a single simple form.

We work with the simple form written as

$$\frac{1}{\epsilon^r} g(\frac{\mathbf{u}}{\epsilon}) \psi(\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} \tag{II.12}$$

where the volume forms $d\mu_{\mathbf{u}}, d\mu_{\mathbf{v}_1}$ determine two coordinate's planes

$$\mathbb{R}^r, \mathbb{R}^{p-r}$$

with coordinates $\mathbf{u}, \mathbf{v}_1$ respectively, and ψ is a C^∞ function on

$$\mathbb{R}^n = \mathbb{R}^r \oplus \mathbb{R}^{p-r} \oplus \mathbb{R}^{n-p}$$

that is the coefficient of the simple form $\psi d\mu_{\mathbf{v}_1}$ in the test form ϕ. Then the integral of (II.10) over $C := c(\Pi)$ is

$$\int_{D_{\frac{1}{\epsilon}}(C)} g(\mathbf{u}) \psi(\epsilon \mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} \tag{II.13}$$

where $\mathbf{u}$ is the new variable obtained from the old $\mathbf{u}$ divided by ϵ. Let K_1 be the support of $g(\mathbf{u})$, and K_2, K_3 be the bounded sets of $\mathbb{R}^{p-r}, \mathbb{R}^{n-p}$ such that C is contained in $\mathbb{R}^r \oplus K_2 \oplus K_3$. Then $\psi(\epsilon \mathbf{u}, \mathbf{v}_1, \mathbf{v}_2)$ uniformly converges to $\psi(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)$ in the bounded $K_1 \oplus K_2 \oplus K_3$. So, for any positive δ, we can find sufficiently small ϵ such that

$$|\psi(\epsilon \mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) - \psi(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)| \leq \delta. \tag{II.14}$$

Let c_ϵ be the composition

$$\overline{\Pi}_p \quad \xrightarrow{c} \quad \mathbb{R}^n \quad \xrightarrow{D_{\frac{1}{\epsilon}}} \quad \mathbb{R}^n. \tag{II.15}$$

Notice

$$D_{\frac{1}{\epsilon}}(C) \cap (K_1 \oplus K_2 \oplus K_3)$$

is a bounded set. Thus all coefficients of the form $c_\epsilon^*\big(g(u) d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1}\big)$ are bounded uniformly for all sufficiently small ϵ. Hence

$$\begin{aligned} &|\int_{D_{\frac{1}{\epsilon}}(C)} g(\mathbf{u}) \psi(\epsilon \mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} - \int_{D_{\frac{1}{\epsilon}}(C)} g(\mathbf{u}) \psi(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2) d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1}| \\ &\leq \delta M \end{aligned} \tag{II.16}$$

where M is a constant. For the integral

$$\int_{D_{\frac{1}{\epsilon}}(C)} g(\mathbf{u})\psi(\mathbf{0},\mathbf{v}_1,\mathbf{v}_2)d\mu_{\mathbf{u}}\wedge d\mu_{\mathbf{v}_1} \tag{II.17}$$

we make a change of variable from $\mathbf{u}$ to $\frac{\mathbf{u}}{\epsilon}$ to find (II.17) is equal to

$$\frac{1}{\epsilon^r}\int_C g(\frac{\mathbf{u}}{\epsilon})\psi(\mathbf{0},\mathbf{v}_1,\mathbf{v}_2)d\mu_{\mathbf{u}}\wedge d\mu_{\mathbf{v}_1} \tag{II.18}$$

Now we apply Lemma A.1, there is a compactly supported integrable function $\tilde{\xi}_\epsilon(\mathbf{u},\mathbf{v}_1)$ on $\mathbb{R}^p$ such that

$$\frac{1}{\epsilon^r}\int_C g(\frac{\mathbf{u}}{\epsilon})\psi(\mathbf{0},\mathbf{v}_1,\mathbf{v}_2)d\mu_{\mathbf{u}}\wedge d\mu_{\mathbf{v}_1} = \frac{1}{\epsilon^r}\int_W g(\frac{\mathbf{u}}{\epsilon})\tilde{\xi}_\psi(\mathbf{u},\mathbf{v}_1)d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1} \tag{II.19}$$

where W is the measurable set defined as in Lemma II.4, and the right hand side is a Lebesgue integral with the density measure $d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1}$, and $\tilde{\xi}_\psi(\mathbf{u},\mathbf{v}_1)$ in the integrand is a compactly supported L^1 function on $\mathbb{R}^p$. Furthermore, since $\psi(\mathbf{0},\mathbf{v}_1,\mathbf{v}_2)$ is a pullback function from $\mathbb{R}^{p-r}\oplus\mathbb{R}^{n-p}$, then $\tilde{\xi}_\psi(\mathbf{u},\mathbf{v}_1)$ is also a pullback of function $\xi_\psi(\mathbf{v}_1)$ from $\mathbb{R}^{p-r}$. So, in the following, we express the pullback function $\tilde{\xi}_\psi(\mathbf{u},\mathbf{v}_1)$ as $\xi_\psi(\mathbf{v}_1)$. Now changing the variables from $\frac{\mathbf{u}}{\epsilon}$ back to $\mathbf{u}$, we have

$$\begin{aligned} right\ hand\ side\ of(II.19) &= \textstyle\int_{\mathbb{R}^p}\chi_{D_{\frac{1}{\epsilon}}(W)}(\mathbf{u},\mathbf{v}_1)g(\mathbf{u})\xi_\psi(\mathbf{v}_1)d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1} \\ &= \textstyle\int_{\mathbb{R}^p}\chi_{D_{\frac{1}{\epsilon}}(W\setminus W_{fu})}(\mathbf{u},\mathbf{v}_1)g(\mathbf{u})\xi_\psi(\mathbf{v}_1)d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1} \end{aligned} \tag{II.20}$$

where $\chi_\bullet$ denotes the characteristic function of the set $\bullet$. Next for the Lebesgue integrals, we'll omit the notations for variables for the dominant convergence theorem. We'll see that the integrand in (II.20) satisfies

$$|\chi_{D_{\frac{1}{\epsilon}}(W\setminus W_{fu})}g\xi_\psi| \le |g\xi_\psi|$$

and $|g\xi_\psi|$ is an L^1 function on $\mathbb{R}^p$. The set-theoretic convergence in Lemma II.4 implies the $\chi_{D_{\frac{1}{\epsilon}}(W\setminus W_{fu})}g\xi_\psi$ point-wisely converges to the function

$$\chi_{B_0}g\xi_\psi.$$

By the dominant convergence theorem

$$\begin{aligned} &\lim_{\epsilon\to 0}\int_{\mathbb{R}^p}\chi_{D_{\frac{1}{\epsilon}}(W\setminus W_{fu})}g\xi_\psi d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1} = \int_{\mathbb{R}^p}\chi_{B_0}g\xi_\psi d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1} \\ &= \int_{B_0} g\xi_\psi d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1} \end{aligned} \tag{II.21}$$

Finally, combining (II.16) and (II.21), we obtain that

$$\lim_{\epsilon\to 0}\int_C \frac{1}{\epsilon^r} g(\frac{\mathbf{u}}{\epsilon})\psi(\mathbf{u},\mathbf{v}_1,\mathbf{v}_2)d\mu_{\mathbf{u}}\wedge d\mu_{\mathbf{v}_1} = \int_{B_0} g(\mathbf{u})\xi_\psi(\mathbf{v}_1)d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1}$$

(Note the left hand side is an integral of a differential form but the right is a Lebegue integral). We conclude

$$T_c \wedge \omega_\epsilon$$

converges to a functional as $\epsilon \to 0$. For the continuity of the functional, we see that if ϕ varies in a bounded set of forms to any orders, then in particular ϕ varies in the bounded set to the order of 0. Hence the formula (II.17) (as a number) is bounded. Then

$$\int_{B_0} g(\mathbf{u})\xi_\psi(\mathbf{u},\mathbf{v}_1)d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1}$$

as a number is bounded. So, the evaluation

$$\lim_{\epsilon\to 0}(T_c\wedge\omega_\epsilon)[\phi]$$

is also bounded. Hence the functional

$$\phi \to \lim_{\epsilon\to 0}(T_c\wedge\omega_\epsilon)[\phi]$$

defines a current. The proof is completed.

□

Appendix A Orthogonal projection of a cell

The integration of forms (II.10) is impossible in the traditional geometric analysis since the manifold's structure does not exist at the $\epsilon = 0$ (as that at ∞). Our idea is to convert it to a Lebesgue integral (see the right hand side of (II.19)) for the measure still exists there. The following lemma provides the basis to this conversion.

Lemma A.1. *Let $p \le n$ be two whole numbers. Let $\mathbb{R}^p, \mathbb{R}^{n-p}$ be subspaces of $\mathbb{R}^n$ such that $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^{n-p}$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^p$ be the orthogonal projection. Let c be a p-dimensional regular cell in $\mathbb{R}^n$, ψ a smooth function on $\mathbb{R}^n$. Then there is a compactly supported L^1 function ξ_ψ on $\mathbb{R}^p$ such that*

$$\pi(T_c\wedge\psi) = \xi_\psi \tag{A.1}$$

where $\pi(currents)$ denotes the pushforward on compactly supported currents, and ξ_ψ represents a current of degree 0.

Proof. Let μ be the Lebesgue measure on $\mathbb{R}^p$, ϕ a test function. Let $C = c(\Pi_p)$. We should note that since T_c is a current with a compact support, the pushforward $\pi(T_c \wedge \psi)$ is a well-defined 0-current. Hence it is both a distribution and a 0-current. So it can be evaluated in two different ways, and the evaluation of the distribution $\pi(T_c \wedge \psi)$ at ϕ is equal to the current's evaluation at forms,

$$\pi(T_c \wedge \psi)[\phi d\mu] \tag{A.2}$$

which has the integral estimate

$$\begin{aligned} \left| \pi(T_c \wedge \psi)[\phi d\mu] \right| &\leq \left| \int_C \psi \wedge \pi^*(\phi) \wedge \pi^*(d\mu) \right| \\ &\leq M ||\phi||_\infty \end{aligned} \tag{A.3}$$

where M is a constant independent of the test function and $||\bullet||_\infty = esssup|\bullet|$. Thus, $\pi(T_c \wedge \psi)$ as a distribution has order 0. Therefore it is a signed measure. Let $A \subset \mathbb{R}^p$ be a set of measure 0. Let $\overline{\pi} = \pi|_C$. So, $\overline{\pi}$ is a differential map between two manifolds of the same dimension p. Let

$$\overline{\pi}^{-1}(A) = E_1 \cup E_2$$

where E_1 is a set of critical points of $\overline{\pi}$, and $E_2 = \overline{\pi}^{-1}(A) \backslash E_1$. By the same estimate (A.3), we have

$$\left| \pi(T_c \wedge \psi)[A] \right| \leq M' | \textstyle\int_{E_1 + E_2} d\mu| \tag{A.4}$$

where M' is a constant, the integral is of the differential form $d\mu$. Since E_1 consists of critical points, the Jacobian of $\overline{\pi}$ is 0. Thus $\int_{E_1} d\mu = 0$. We let $E_2 = \cup_{i=1}^\infty E_2^i$ such that

$$\overline{\pi}|_{E_2^i} : E_2^i \to \overline{\pi}(E_2^i) \tag{A.5}$$

is diffeomorphic. Then each $\overline{\pi}(E_2^i)$ is contained in A. Thus $\mu(\overline{\pi}(E_2^i)) = 0$. Then

$$| \int_{E_2^i} d\mu| \leq | \int_{\overline{\pi}(E_2^i)} J d\mu| \leq k_i \mu(\overline{\pi}(E_2^i)) = 0$$

where J is the Jacobian of the map $\overline{\pi}|_{E_i}$ and k_i is the upper bound of $|J|$. Hence

$$\left| \pi(T_c \wedge \psi)[A] \right| \leq \sum_{i=1}^\infty | \int_{E_2^i} d\mu| = 0.$$

Thus the signed measure $\pi(T_c \wedge \psi)$ is absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}^p$. The Radon-Nikodym theorem ([1]) implies that the density function between the signed measure and the positive measure,

$$\xi_\psi = \frac{\pi(T_c \wedge \psi)}{\mu} \tag{A.6}$$

is an L^1 function. The numerator $\pi(T_c \wedge \psi)$ in the formula (A.6) indicates ξ_ψ has the bounded support $\pi(C)$. We complete the proof.

□

Example A.2. *If $\pi|_C : C \to \mathbb{R}^n$ is proper, then $\xi_1 = deg(\pi)\chi_{\pi(C)}$.*

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