Five Hard Problems with a Simple Solution

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Abstract. We present a collection of five nontrivial exercises in number theory (Questions 1–4) and graph theory (Question 5). These problems can be efficiently solved using insights and shortcuts derived from the author's previously published papers. This preprint invites readers to test their expertise in these fields and assess their ability to independently solve the proposed exercises.
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MSC2020: 11A07, 68R10 (Primary) 11F33 (Secondary).

1 The problems

Question 1. Which is the maximum value of the positive integer n (let us call it $n_{\max}$) such that ${}^{1000000000}3333704193 \equiv {}^{31415926535897932384626433832795}3333704193 \pmod{10^n}$, where ${}^{m}3333704193$ means $3333704193^{3333704193^{3333704193^{\cdot^{\cdot^{\cdot}}}}}$ m-times (e.g., ${}^{3}3 = 3^{3^3} = 3^{27} = 7625597484987$)?

Question 2. Which is the smallest positive integer n (let us indicate it as $n_{\min}$) such that, in the decimal numeral system, ${}^{n}202520252025 \equiv {}^{n+2025}202520252025 \pmod{10^{202552022025}}$ (where ${}^{n}202520252025$ indicates the n-th tetration of 202520252025, i.e., $202520252025^{202520252025^{202520252025^{\cdot^{\cdot^{\cdot}}}}}$ n-times)?

Question 3. Which 4-digit number can be formed by juxtaposing (from left to right) the four distinct congruence classes modulo 10 of the differences between the 4000000025-th rightmost digit of ${}^{1000000006}267785184193$ and the 4000000025-th rightmost digit of ${}^{1000000007}267785184193$, the 4000000029-th rightmost digit of ${}^{1000000007}267785184193$ and the 4000000029-th rightmost digit of ${}^{1000000008}267785184193$, the 4000000033-th rightmost digit of ${}^{1000000008}267785184193$ and the 4000000033-th rightmost digit of ${}^{1000000009}267785184193$, and lastly the congruence class modulo 10 of the difference between the 4000000037-th rightmost digit of ${}^{1000000009}267785184193$ and the 4000000037-th rightmost digit of ${}^{1000000010}267785184193$?

Question 4. Which is the congruence class modulo $10^{16309690970750}$ of the difference ${}^{2718281828459}2922943 - {}^{3141592653589}2922943$, where ${}^{2718281828459}2922943$ means $2922943^{2922943^{2922943^{\cdot^{\cdot^{\cdot}}}}}$ 2718281828459-times (e.g., ${}^{3}3 = 3^{3^3} = 3^{27} = 7625597484987$)?

Question 5. Let the two grids $G_1 := \{\{0,1,2\} \times \{0,1,2\} \times \{0,1,2\}\} \subset \mathbb{R}^3$ and $G_2 := \{\{4,5\} \times \{4,5\} \times \{4,5\}\} \subset \mathbb{R}^3$ be given. Which is the minimum number of edges that a closed polygonal chain (i.e., a circuit) must have to join all the 27 points of G_1 first and then all the 8 points of G_2, returning to the starting point with its last line segment?

2 The solutions

Answer 1. The value of n such that ${}^{1000000000}3333704193 \equiv {}^{3141592653589793238462643383279 5}3333704193 \pmod{10^n}$ and ${}^{1000000000}3333704193 \not\equiv {}^{31415926535897932384626433832795}3333704193 \pmod{10^{n+1}}$ is 9000000009. So the solution to this problem is 9000000009.

Explanation of Answer 1. We know that the *congruence speed* (see Definition 1.1 of [3]) of the tetration base 3333704193 is certainly stabile at height $\nu_5(3333704193^2+1)+2 = 10+2 = 12$ since $3333704193^2+1 = 11113583646425781250 = 5^{10} \cdot 2 \cdot 42793 \cdot 13296929$ (see Definition 2.1, p. 447, of Reference [3]). Thus, let us calculate the number of stable digits of ${}^{12}3333704193$ (i.e., the least significant frozen digits of 3333704193 at height 12 that will not change by moving to height 13) so that we will use the knowledge of the constant congruence speed of 3333704193 to finally compute the exact number of the rightmost frozen digits of ${}^{1000000000}3333704193$.

Now, the constant congruence speed of 3333704193 is equal to 9 by Equation (16) of "Number of stable digits of any integer tetration" [3] (since $V(3333704193) = \nu_2(3333704193-1)$), while ${}^{12}3333704193 \equiv {}^{13}3333704193 \pmod{10^{117}}$ and ${}^{12}3333704193 \not\equiv {}^{13}3333704193 \pmod{10^{118}}$ (we can do this by hand, in a couple of minutes, by iterating Hensel's lifting lemma on the free online version of WolframAlpha) implies that ${}^{1000000000}3333704193 \equiv {}^{31415926535897932384626433832795}3333704193 \pmod{10^{9\cdot(1000000000-12)+117}}$ and ${}^{1000000000}3333704193 \not\equiv {}^{31415926535897932384626433832795}3333704193 \pmod{10^{9\cdot(1000000000-12)+118}}$.

Therefore, the correct answer to this original problem is $n_{\max} = 9 \cdot (1000000000-12)+117 = 9000000009$ (indeed, $n_{\max} = 9 \cdot (1000000000+1)$ and thus $9 \mid n_{\max}$ as expected since $1000000000 \geq \nu_5(3333704193^2+1)+2$).

Answer 2. The smallest $n \in \mathbb{Z}^+$ such that ${}^{n}202520252025$ is congruent to ${}^{n+1}202520252025$ modulo $10^{202552022025}$ is 67506750674. In detail, $n_{\min} = 67506750674$ is the correct answer to this question since ${}^{67506750674-1}202520252025 \not\equiv {}^{67506750674}202520252025 \pmod{10^{202552022025}}$ and ${}^{67506750674}202520252025 \equiv {}^{67506750674+1}202520252025 \pmod{10^{202552022025}}$ implies that ${}^{67506750674-1}202520252025 \not\equiv {}^{67506750674+2025}202520252025 \pmod{10^{202552022025}}$ and ${}^{67506750674}202520252025 \equiv {}^{67506750674+2025}202520252025 \pmod{10^{202552022025}}$.

Thus, $n_{\min} = 67506750674$.

Explanation of Answer 2. We use the constancy of the *congruence speed* (see Definition 1.1 of [3]) of the tetration base 202520252025 to easily solve the present problem, proving that $n_{\min} = 67506750674$.

The fifth line of Equation (16) of Reference [3] provides the exact number of the new rightmost frozen digits of ${}^{m}202520252025$ at any height $m \in \mathbb{N}$ such that $m \geq \nu_2(202520252025^2-1)-1+2$ (since taking an hyperexponent greater than or equal to $\tilde{\nu}(a)+2$, see Definition 1.1 of Reference [3], is a sufficient condition for the constancy of the congruence speed of any tetration base not a multiple of 10, there are only a few cases that we need to directly check in order to provide the whole map of the congruence speed of 20252025202).

More specifically, $(\nu_2(202520252025^2-1)-1)+2 = 4+2$ so that ${}^{6}202520252025 \pmod{10^{3\cdot(6+1)}}$ will not be a thread, and here are the last 30 digits of

$^{1}202520252025,^{2}202520252025,\ldots,^{6}202520252025$:

000000000000000000202520252025 → 0000000000000000002025202520**25** (height 1),
409532443620264530181**884765625** (height 2),
347573653794825077**056884765625** (height 3),
691678420640528**202056884765625** (height 4),
451657668687**403202056884765625** (height 5),
538815871**812403202056884765625** (height 6).

The above confirms that the congruence speed of 202520252025 is 2 at height 1, 7 at height 2, and 3 for each (integer) hyperexponent greater than 2.

Thus, $n_{\min} = \lceil \frac{202552022025-9}{3} \rceil + 2$.

Hence, $n_{\min} = \frac{202552022025-9}{3} + 2 = 67517340672 + 2$.

Therefore, the solution is 67517340674.

Answer 3. The correct answer is 2684 (since the congruence class modulo 10 of the difference between the 4000000025-th rightmost digit of $^{1000000006}267785184193$ and the 4000000025-th rightmost digit of $^{1000000007}267785184193$ is 2, the congruence class modulo 10 of the difference between the 4000000029-th rightmost digit of $^{1000000007}267785184193$ and the 4000000029-th rightmost digit of $^{1000000008}267785184193$ is 6, and so forth).

Explanation of Answer 3. This result is verifiable by observing that the constant congruence speed of the tetration base 267785184193 is equal to 4 (since 267785184193 is congruent to 13 modulo 20 and $\frac{84193-04193}{10^4} \neq 5$ so that line 13 of Equation (16) of [3] implies that $V(267785184193) = \nu_5(267785184193^2 + 1) = 4$).

Then, we can use the sufficient condition $\nu_5(267785184193^2 + 1) + 2$ in order to find a small hyperexponent $\bar{b}(202520252025) = \nu_5(267785184193^2 + 1) + 2 = 6$ of 267785184193 which guarantees that the congruence class modulo 10 of the difference between the rightmost non-stable digit of $^{\bar{b}+4\cdot k}202520252025$ and the rightmost non-stable digit of $^{\bar{b}+1+4\cdot k}202520252025$ will not change for every $k = 0, 1, 2, 3, 4, \ldots$. Now, since these congruence classes form a 4-iteration cycle (it is a general property holding for any tetration base not a multiple of 10 that the author discussed in Chapters 3, 4, 6, and 7 of the 2011 book "La strana coda della serie $n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}$"), we only need to repeat the process above for $^{\bar{b}+1}202520252025$, $^{\bar{b}+2}202520252025$, and $^{\bar{b}+3}202520252025$.

Then, we only need to compute the differences between the 4000000025-th, 4000000029-th, 4000000033-th, 4000000037-th righmost digits of $^{6}202520252025$ and $^{7}202520252025$, $^{7}202520252025$ and $^{8}202520252025$, $^{8}202520252025$ and $^{9}202520252025$, $^{9}202520252025$ and $^{10}202520252025$ (respectively), that are not stable at the given heights.

Here is the list of the last 50 digits of $^{1}202520252025$, $^{2}202520252025$, ..., $^{10}202520252025$:

00000000000000000000000000000000000000267785184 0**193** (height 1),
7274677478731023078605132705566487300131318**9344193** (height 2),
6970554517069889275015781013215417815 1**469989344193** (height 3),
469305770558072353835611106431453 7**0215469989344193** (height 4),
2985003507212416357496415566 82**00090215469989344193** (height 5),

$86572083424338844790891220\mathbf{65380009021546998934 4193}$ (height 6),
$871606568792472508233 8\mathbf{990865380009021546998934 4193}$ (height 7),
$85662154799964302 4\mathbf{81629908653800090215469989344193}$ (height 8),
$974838496077746\mathbf{497681629908653800090215469989344193}$ (height 9),
$3944653269\mathbf{66424976816299086538000902154699893441 93}$ (height 10).

Hence, we get the cycle $[2, 6, 8, 4]$ which uniquely identifies, for any given $k \in \mathbb{N}_0$, the four distinct congruence classes modulo 10 of the difference between the rightmost non-stable digit of ${}^{6+4\cdot k}202520252025$ and the corresponding digit of ${}^{7+4\cdot k}202520252025$ (and this congruence class modulo 10 is always 2), $\ldots$, and so on up to the congruence class modulo 10 of the difference between the rightmost non-stable digit of ${}^{9+4\cdot k}202520252025$ and the corresponding digit of ${}^{10+4\cdot k}202520252025$ (and this congruence class modulo 10 is always 4).

Therefore, the solution to this original problem is the number 2684.

Answer 4. The answer is $5 \cdot 10^{16309690970749}$ since ${}^{2718281828459}2922943 \equiv {}^{3141592653589}2922943 \pmod{10^{16309690970749}}$ (the congruence speed of the tetration base 2922943 is 0 at height 1, 7 at heights 2 to 8, and finally it stabilizes at the constant congruence speed value $V(2922943) = 6$ which characterizes every hyperexponent of 2922943 at or above 9) [3] while the absolute value of the difference between the rightmost non-stable digit of ${}^{\bar{b}}2922943$ and the rightmost non-stable digit of ${}^{\bar{b}+1}2922943$ is equal to 5 for any integer $\bar{b} \geq 9$.

Explanation of Answer 4. Trivial result (see [4]).

Answer 5. The minimum-link closed polygonal chain that visits all the points of G_1 and then all the points of G_2 in $\mathbb{R}^3$ consists of 18 connected line segments.

Explanation of Answer 5. A constructive proof is given since, for each $k \in \mathbb{N} - \{0, 1\}$, the provided upper bound matches the trivial lower bound that follows by combining the general solution $3 \cdot 2^{k-2}$ for any $\{0, 1\}^k$ grid [5] and the general solution $\frac{3^k-1}{2}$ for any $\{0, 1, 2\}^k$ grid [1].

In detail, here we are considering the case $k = 3$ so that we need at least $\frac{3^3-1}{2} + 3 \cdot 2^{3-1} - 1$ line segments to join all the vertices of $G_1 \cup G_2$ through a single polygonal chain (given the fact that $\{0, 1\}^3$ has no more than 2 collinear points and that we can use the last line that solves G_1 to fit 2 more points of G_2), so 6 points of G_2 remain. Then, the proof of Lemma 1 of [5] is sufficient to guarantee no covering trail for $G_1 \cup G_2$ with less than 18 line segments.

Now, we observe that a closed polygonal chain satisfying our constraints actually exists since $P_{18} := (0, 1, 0)$-$(0, 3, 0)$-$(3, 0, 3)$-$(0, 0, 0)$-$(0, 0, 3)$-$(3, 3, 0)$-$(0, 0, 0)$-$(0, 3, 3)$-$(3, 0, 0)$-$(0, 3, 0)$-$(0, 0, 3)$-$(3, 0, 0)$-$(0, 0, 0)$-$(6, 6, 6)$-$(2, 4, 4)$-$(5, 4, \frac{11}{2})$-$(5, 4, 3)$-$(5, 6, 5)$-$(0, 1, 0)$ is such that $\{\{0, 1, 2\}^3 \cup \{4, 5\}^3\} \in P_{18}$.

Therefore, the correct answer is 18 (see Figures 1 and 2).

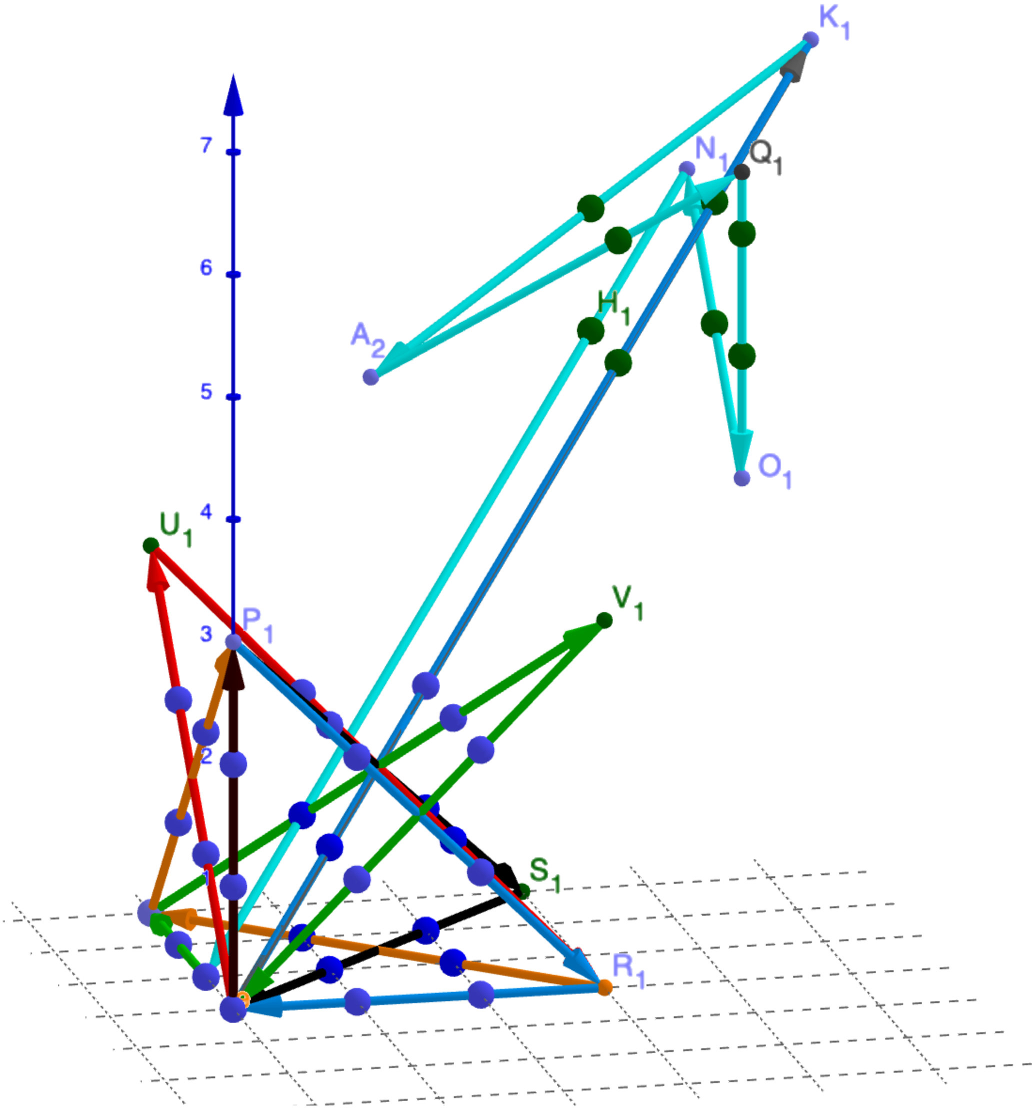

Figure 1. P_{18}, perspective 1.

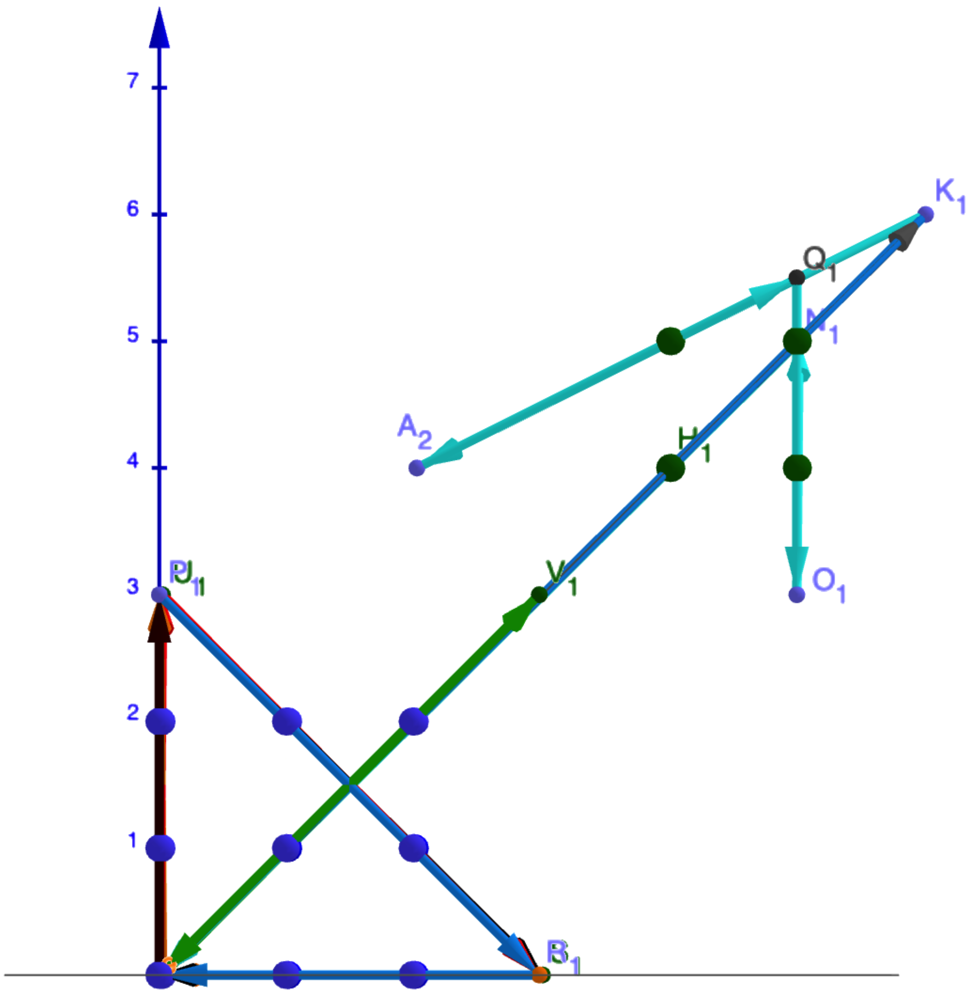

Figure 2. P_{18}, perspective 2.

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