SOME OPERATIONAL FORMULAS INVOLVING BESSEL NUMBERS

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ABSTRACT. In this brief note, we give three operational formulas that involve Bessel numbers. To the best of our knowledge, these formulas are new. We also derive another probably unknown formula from them. Finally, we present an example of application of one of these formulas.

Keywords: Bessel polynomial, Bessel number, primitive of order n of a function.

1 Introduction

1.1 Main results

It is well known that [5]:

$$(xp)^n = \sum_{k=0}^{n} S(n,k) x^k p^k \tag{1}$$

where $S(n,k)$ are the Stirling numbers of the second kind and $p = \frac{d}{dx}$ is the derivative operator.

In an attempt to obtain integral analogs of (1), we discovered the formulas:

$$I(xI)^{n-1}[f(x)] = \sum_{k=0}^{n-1} (-1)^k a(n-1,k) x^{n-k-1} F_{n+k}[f(x)] + \sum_{k=0}^{n-1} c_k \frac{x^{2k}}{(2k)!!} \tag{2}$$

$$p^{-1}(xp^{-1})^{n-1}[f(x)] = \Delta q_n[f(x)] \tag{3}$$

(4)

$$q_n[f(x)] = \sum_{k=0}^{n-1} (-1)^k a(n-1,k) x^{n-1-k} F_{n+k}[f(x)] - \sum_{k=1}^{n-1} q_{n-k}[f(x)](x_0) \frac{x^{2k}}{(2k)!!}$$

$$(x^{-1}p)^n = \sum_{k=0}^{n-1} (-1)^k a(n-1,k) x^{-n-k} p^{n-k} \tag{5}$$

where $I = \int . \, dx$, $p^{-1} = \int_{x_0}^{x} . \, dx$, $\Delta[f(x)] = f(x) - f(x_0)$, the c_k's are constants of integration, and $F_n[f(x)]$ called the primitive of order n of

$f(x)$ is the expression of the n-th indefinite integral of the function $f(x)$ excluding the constants of integration, that is, the case where the constants of integration are zero, it is given by the following formula (see [3]) :

$$F_n[f(x)](x) = \frac{x^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{F_1[x^k f(x)](x)}{(-x)^k} \tag{6}$$

Another formula which is the result of combining formula (2) with formula (5) is:

$$\mathrm{I} = \sum_{k=0}^{n-1}\sum_{j=0}^{n}(-1)^{k+j}a(n-1,k)a(n,j)x^{-n-k}p^{n-k}x^{n-j}F_{n+j} \tag{7}$$

where I stands for the identity operator.

We note that the numerical coefficients in formulas (2), (4), (5), and (7) are the Bessel coefficients [1], we also note that we can transform these coefficients into the Bessel numbers using relations (11) and (13).

We shall present the proofs in detail in Section 2. In Section 3, we see an interesting example of application of formula (3).

1.2 Bessel numbers

The Bessel numbers are reparametrized coefficients of Bessel polynomials. Choi and Smith [1] were the first to investigate these numbers from a combinatorial point of view; later, Han and Seo [4] showed that the Bessel numbers satisfy two properties of Stirling numbers: The two kinds of Bessel numbers are related by inverse formulas, and both Bessel numbers of the first kind and those of the second kind form log-concave sequences. Yang and Qiao [8] investigated Bessel numbers and Bessel matrices using exponential Riordan arrays and showed that Bessel numbers are a special case of the degenerate Stirling numbers. In 2022, Stenlund [7] gave new proofs for two identities that connect a sum containing both kinds of Stirling numbers with either the first or the second kind of Bessel numbers. In this note, we further contribute to establishing some operational formulas involving the Bessel numbers.

Recall that the Bessel polynomials $y_n(x)$ are the unique polynomial solutions to the second-order differential equation

$$x^2y_n'' + 2(x+1)y_n' = n(n+1)y_n \tag{8}$$

with the normalization $y_n(0) = 1$.

From the differential equation (8) one can easily derive the formula:

$$y_n(x) = \sum_{k=0}^{n}\frac{(n+k)!}{2^k k!(n-k)!}x^k \tag{9}$$

Let $a(n,k)$ denote the coefficient of x^k in the polynomial $y_n(x)$ [1, 6]. Set:

$$b(n,k) := \begin{cases} \frac{(-1)^{n-k}(2n-k-1)!}{2^{n-k}(n-k)!(k-1)!}, & \text{if } 1 \leq k \leq n \\ 0, & \text{if } 0 \leq n < k \end{cases} \tag{10}$$

that is :

$$b(n,k) = (-1)^{n-k}\ a(n-1, n-k) \tag{11}$$

We call the number $b(n,k)$ a Bessel number of the first kind. By convention, we put $b(0,k) = \delta_{0,k}$, where δ is the Kronecker delta.

The Bessel numbers of the second kind $B(n,k)$ are given by :

$$B(n,k) := \begin{cases} \frac{n!}{2^{n-k}(2k-n)!(n-k)!}, & \text{if } \lceil \frac{n}{2} \rceil \leq k \leq n \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

It is easily checked that:

$$B(n,k) = a(k, n-k) \tag{13}$$

2 Proofs

Lemma 2.1. *Let $n \in \mathbb{N}^*$. For all $1 \leq k \leq n$ we have:*

$$a(n,k) = \sum_{i=0}^{\min(k,n-1)} (n-i)^{(k-i)}\ a(n-1,i) \tag{14}$$

where $x^{(n)}$ represent the falling factorial $x(x-1)\ldots(x-n+1)$.

Proof. We have to show first the case where $k \leq n-1$, meaning the formula:

$$\forall\ 1 \leq k \leq n-1,\ a(n,k) = \sum_{i=0}^{k} (n-i)^{(k-i)}\ a(n-1,i) \tag{15}$$

By induction.

1. Base case: verify true for $k = 1$.

$$a(n,1) = \sum_{i=0}^{1} (n-i)^{(1-i)}\ a(n-1,i) = \frac{n(n+1)}{2}$$

2. Induction hypothesis: assume the statement is true until k.

$$a(n,k) = \sum_{i=0}^{k} (n-i)^{(k-i)}\ a(n-1,i)$$

3. Induction step : we will show that this statement is true for $(k+1)$.

We have to show the following statement to be true:

$$a(n,k+1)=\sum_{i=0}^{k+1}(n-i)^{(k+1-i)}\ a(n-1,i)$$

From [2], p. 23, we have:

$$\forall\ 1\leq k\leq n-1,\ \ a(n,k)=a(n-1,k)+(n-k+1)\ a(n,k-1) \tag{16}$$

Hence

$$\begin{aligned}
a(n,k+1) &= a(n-1,k+1)+(n-k)\ a(n,k)\\
&= a(n-1,k+1)+(n-k)\sum_{i=0}^{k}(n-i)^{(k-i)}\ a(n-1,i)\\
&= a(n-1,k+1)+\sum_{i=0}^{k}(n-i)^{(k+1-i)}\ a(n-1,i)\\
&= \sum_{i=0}^{k+1}(n-i)^{(k+1-i)}\ a(n-1,i)
\end{aligned}$$

which completes the induction.

The case $k=n$ may be proven by observing that $a(n,n)=a(n,n-1)$ [2]. □

Lemma 2.2. *Let $(\alpha,\beta)\in\mathbb{N}^2$, we have:*

$$Ix^{\alpha}F_{\beta}=\sum_{k=0}^{\alpha}(-1)^k\alpha^{(k)}x^{\alpha-k}F_{\beta+k+1}+C \tag{17}$$

where C is a constant of integration.

Proof. 1. Base case: verify true for $\alpha=0$.

$$Ix^{0}F_{\beta}=F_{\beta+1}+c=\sum_{k=0}^{0}(-1)^k0^{(k)}x^{0-k}F_{\beta+k+1}+c$$

2. Induction hypothesis : assume the statement is true until α.

$$Ix^{\alpha}F_{\beta}=\sum_{k=0}^{\alpha}(-1)^k\alpha^{(k)}x^{\alpha-k}F_{\beta+k+1}+C$$

3. Induction step : we will show that this statement is true for $(\alpha+1)$.
We have to show the following statement to be true:

$$Ix^{\alpha+1}F_{\beta}=\sum_{k=0}^{\alpha+1}(-1)^k(\alpha+1)^{(k)}x^{\alpha-k+1}F_{\beta+k+1}+D$$

where D is a constant of integration.

Applying integration by parts, we have:

$$\begin{aligned}
Ix^{\alpha+1}F_\beta &= x^{\alpha+1}F_{\beta+1} - (\alpha+1)Ix^{\alpha}F_{\beta+1} + K \\
&= x^{\alpha+1}F_{\beta+1} - (\alpha+1)\left(\sum_{k=0}^{\alpha}(-1)^k\alpha^{(k)}x^{\alpha-k}F_{\beta+k+2} + C\right) + K \\
&= x^{\alpha+1}F_{\beta+1} + \sum_{k=0}^{\alpha}(-1)^{k+1}(\alpha+1)^{(k+1)}x^{\alpha-k}F_{\beta+k+2} + D \\
&= x^{\alpha+1}F_{\beta+1} + \sum_{k=1}^{\alpha+1}(-1)^{k}(\alpha+1)^{(k)}x^{\alpha-k+1}F_{\beta+k+1} + D \\
&= \sum_{k=0}^{\alpha+1}(-1)^{k}(\alpha+1)^{(k)}x^{\alpha-k+1}F_{\beta+k+1} + D
\end{aligned}$$

Hence, the lemma is proven by induction. □

Theorem 2.3. *Let $f(x)$ be a function of x, for any $n \in \mathbb{N}^*$ we have that:*

$$I(xI)^{n-1}[f(x)] = \sum_{k=0}^{n-1}(-1)^k a(n-1,k)x^{n-k-1}F_{n+k}[f(x)] + \sum_{k=0}^{n-1} c_k \frac{x^{2k}}{(2k)!!} \tag{18}$$

Proof. 1. Base case: verify true for $n = 1$.

$$I = F_1 + c_0 = \sum_{k=0}^{0}(-1)^k a(0,k)x^{0-k}F_{1+k} + \sum_{i=0}^{0} c_i \frac{x^{2i}}{(2i)!!}$$

2. Induction hypothesis : assume the statement is true until n.

$$I(xI)^{n-1} = \sum_{k=0}^{n-1}(-1)^k a(n-1,k)x^{n-k-1}F_{n+k} + \sum_{k=0}^{n-1} c_k \frac{x^{2k}}{(2k)!!}$$

3. Induction step : we will show that this statement is true for $(n+1)$. We have to show the following statement to be true:

$$I(xI)^{n} = \sum_{k=0}^{n}(-1)^k a(n,k)x^{n-k}F_{n+k+1} + \sum_{k=0}^{n} b_k \frac{x^{2k}}{(2k)!!}$$

By applying the induction hypothesis,

$$\begin{aligned}
I(xI)^n &= (Ix)I(xI)^{n-1} \\
&= \sum_{k=0}^{n-1}(-1)^k a(n-1,k)Ix^{n-k}F_{n+k} + \sum_{i=0}^{n-1} c_i \frac{Ix^{2i+1}}{(2i)!!}
\end{aligned}$$

Using Lemma 2.2, we have :

$$Ix^{n-k}F_{n+k} = \sum_{i=0}^{n-k}(-1)^i(n-k)^{(i)}x^{n-k-i}F_{n+k+i+1} + C$$

Substituting back and simplifying, we get:

$$I(xI)^n = \sum_{k=0}^{n-1}\sum_{i=0}^{n-k}(-1)^{k+i}(n-k)^{(i)}a(n-1,k)x^{n-k-i}F_{n+1+k+i} + D + \sum_{k=1}^{n} c_{k-1}\frac{x^{2k}}{(2k)!!}$$

Setting $j = k + i$ in the double sum, we get:

$$I(xI)^n = \sum_{k=0}^{n-1}\sum_{j=k}^{n}(-1)^j(n-k)^{(j-k)}a(n-1,k)x^{n-j}F_{n+1+j} + D + \sum_{k=1}^{n} c_{k-1}\frac{x^{2k}}{(2k)!!}$$

Now we need to invert the order of summation. We have:

$$\sum_{i=0}^{n-1}\sum_{j=i}^{n} a_{i,j} = \sum_{j=0}^{n}\sum_{i=0}^{\min(j,n-1)} a_{i,j} \tag{19}$$

Hence, applying this formula to interchange the order of summation, we get,

$$I(xI)^n = \sum_{j=0}^{n}(-1)^j\left(\sum_{k=0}^{\min(j,n-1)}(n-k)^{(j-k)}\ a(n-1,k)\right)x^{n-j}F_{n+1+j} + D + \sum_{k=1}^{n} c_{k-1}\frac{x^{2k}}{(2k)!!}$$

Using Lemma 2.1, we get,

$$I(xI)^n = \sum_{j=0}^{n}(-1)^j a(n,j)x^{n-j}F_{n+1+j} + D + \sum_{k=1}^{n} c_{k-1}\frac{x^{2k+1}}{(2k)!!}$$

Also, let us consider the set of constants b_i such that $b_0 = D$ and, for $i \geq 1$, $b_i = c_{i-1}$. Hence,

$$I(xI)^n = \sum_{j=0}^{n}(-1)^j a(n,j)x^{n-j}F_{n+1+j} + \sum_{k=0}^{n} b_k\frac{x^{2k}}{(2k)!!}$$

Therefore, the theorem is proven by induction. □

It can also be proven that:

$$(Ix)^n[f(x)] = \sum_{k=0}^{n}(-1)^k a(n,k)x^{n-k}F_{n+k}[f(x)] + \sum_{k=0}^{n-1} c_k \frac{x^{2k}}{(2k)!!} \tag{20}$$

Lemma 2.4. *Let $(\alpha, \beta) \in \mathbb{N}^2$. We have:*

$$p^{-1}x^{\alpha}F_{\beta} = \Delta \sum_{k=0}^{\alpha}(-1)^k \alpha^{(k)} x^{\alpha-k}F_{\beta+k+1} \tag{21}$$

Proof. 1. Base case: verify true for $\alpha = 0$.

$$p^{-1}x^{0}F_{\beta} = \Delta \sum_{k=0}^{0}(-1)^k 0^{(k)} x^{0-k}F_{\beta+k+1}$$

2. Induction hypothesis : assume the statement is true until α.

$$p^{-1}x^{\alpha}F_{\beta} = \Delta \sum_{k=0}^{\alpha}(-1)^k \alpha^{(k)} x^{\alpha-k}F_{\beta+k+1}$$

3. Induction step : we will show that this statement is true for $(\alpha + 1)$. We have to show the following statement to be true:

$$p^{-1}x^{\alpha+1}F_{\beta} = \Delta \sum_{k=0}^{\alpha+1}(-1)^k (\alpha+1)^{(k)} x^{\alpha+1-k}F_{\beta+k+1}$$

Applying integration by parts, we have:

$$\begin{aligned}
p^{-1}x^{\alpha+1}F_{\beta} &= \Delta x^{\alpha+1}F_{\beta+1} - (\alpha+1)p^{-1}x^{\alpha}F_{\beta+1} \\
&= \Delta x^{\alpha+1}F_{\beta+1} - (\alpha+1)\Delta \sum_{k=0}^{\alpha}(-1)^k \alpha^{(k)} x^{\alpha-k}F_{\beta+k+2} \\
&= \Delta x^{\alpha+1}F_{\beta+1} + \Delta \sum_{k=0}^{\alpha}(-1)^{k+1} (\alpha+1)^{(k+1)} x^{\alpha-k}F_{\beta+k+2} \\
&= \Delta x^{\alpha+1}F_{\beta+1} + \Delta \sum_{k=1}^{\alpha+1}(-1)^{k} (\alpha+1)^{(k)} x^{\alpha-k+1}F_{\beta+k+1} \\
&= \Delta \sum_{k=0}^{\alpha+1}(-1)^{k} (\alpha+1)^{(k)} x^{\alpha+1-k}F_{\beta+k+1}
\end{aligned}$$

Hence, the lemma is proven by induction. ∎

Theorem 2.5. *Let $f(x)$ be a function of x, for any $n \in \mathbb{N}^*$ we have that:*

$$p^{-1}(xp^{-1})^{n-1}[f(x)] = \Delta q_n[f(x)] \tag{22}$$

where:

$$q_n[f(x)] = \sum_{k=0}^{n-1}(-1)^k a(n-1,k)x^{n-1-k}F_{n+k} - \sum_{k=1}^{n-1} q_{n-k}[f(x)](x_0)\frac{x^{2k}}{(2k)!!} \tag{23}$$

Proof. 1. Base case: verify true for $n = 1$.

$$p^{-1}[f(x)] = \Delta q_1[f(x)]$$

2. Induction hypothesis : assume the statement is true until n.
3. Induction step : we will show that this statement is true for $(n+1)$.
We have to show the following statement to be true:

$$p^{-1}(xp^{-1})^{n}[f(x)] = \Delta q_{n+1}[f(x)]$$

where

$$q_{n+1}[f(x)] = \sum_{j=0}^{n}(-1)^j a(n,j)x^{n-j}F_{n+1-j}[f(x)] - \sum_{k=1}^{n} q_{n+1-k}[f(x)](x_0)\frac{x^{2k}}{(2k)!!}$$

By applying the induction hypothesis,

$$\begin{aligned} p^{-1}(xp^{-1})^n &= p^{-1}x\Delta q_n \\ &= p^{-1}xq_n - p^{-1}xq_n(x_0) \\ &= \sum_{k=0}^{n-1}(-1)^k a(n-1,k)p^{-1}x^{n-k}F_{n+k} - \sum_{k=0}^{n-1} q_{n-k}(x_0)\frac{p^{-1}x^{2k+1}}{(2k)!!} - p^{-1}xq_n(x_0) \end{aligned}$$

Using Lemma 2.4, we have:

$$p^{-1}x^{n-k}F_{n+k} = \Delta \sum_{i=0}^{n-k}(-1)^i (n-k)^{(i)} x^{n-k-i}F_{n+k+i+1}$$

Substituting back and simplifying, we obtain :

$$p^{-1}(xp^{-1})^n = \Delta \sum_{k=0}^{n-1}\sum_{j=k}^{n}(-1)^j (n-k)^{(j-k)} a(n-1,k)x^{n-j}F_{n+1-j} - \Delta\sum_{k=1}^{n} q_{n+1-k}(x_0)\frac{x^{2k}}{(2k)!!}$$

Applying formula (19) and Lemma 2.1 we get :

$$p^{-1}(xp^{-1})^n = \Delta \sum_{j=0}^{n} (-1)^j a(n,j) x^{n-j} F_{n+1-j} - \Delta \sum_{k=1}^{n} q_{n+1-k}(x_0) \frac{x^{2k}}{(2k)!!}$$

The case for $(n+1)$ is proven. Hence, the theorem is proven by induction. □

Theorem 2.6. *Let $n \in \mathbb{N}^*$ we have that:*

$$(x^{-1}p)^n = \sum_{k=0}^{n-1} (-1)^k \ a(n-1,k) x^{-n-k} p^{n-k} \tag{24}$$

Proof. A proof can be done by simple induction using the following rule taken from [2], p. 23:

$$\forall\ 1 \le k \le n-1,\ a(n,k) = a(n-1,k) + (n+k-1)\ a(n-1,k-1) \tag{25}$$

□

Corollary 2.6.1. *For all $n \in \mathbb{N}^*$ we have that:*

$$\mathrm{I} = \sum_{k=0}^{n-1} \sum_{j=0}^{n} (-1)^{k+j} a(n-1,k) a(n,j) x^{-n-k} p^{n-k} x^{n-j} F_{n+j} \tag{26}$$

where I *is the identity operator.*

Proof. Applying formula (24) to the formula (20), we obtain the above formula. □

3 An example of application

In this section, we prove the following result:

Result 1. *For all $n \in \mathbb{N}^*$, we have :*

(27)

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^n (2k+1)(k+1)(k+2)\cdots(k+n)} = \frac{\pi}{4} \sum_{k=0}^{n-1} (-1)^k a(n-1,k) x^{n-1-k} \alpha_{n+k}(1)$$

$$-\ln(2) \sum_{k=0}^{n-1} (-1)^k a(n-1,k) x^{n-1-k} \beta_{n+k}(1) + \sum_{k=0}^{n-1} (-1)^k a(n-1,k) x^{n-1-k} \gamma_{n+k}(1)$$

where α_n, β_n and γ_n are the polynomials associated with the n-th primitive of $\arctan x$.

Proof. We use the formula :

$$p^{-1}(xp^{-1})^{n-1}[\arctan x] = \sum_{k=0}^{n-1}(-1)^k a(n-1,k)x^{n-1-k}F_{n+k}[\arctan x] - \sum_{k=0}^{n-2} a(k,k)F_{2k+1}[\arctan x](0)\frac{x^{2(n-k-1)}}{(2(n-k-1))!!} \tag{28}$$

which results from formula (22) by setting $x_0 = 0$ and $f(x) = \arctan x$. Since the formula contains $F_n[\arctan x]$, we first have to calculate it, we have:

$$F_0[\arctan x](x) = \arctan x$$
$$F_1[\arctan x](x) = x\arctan x - \frac{1}{2}\ln(1+x^2)$$
$$F_2[\arctan x](x) = \left(\frac{x^2}{2} - \frac{1}{2}\right)\arctan x - \frac{x}{2}\ln(1+x^2) + \frac{x}{2}$$
$$F_3[\arctan x](x) = \left(\frac{x^3}{6} - \frac{x}{2}\right)\arctan x - \left(\frac{x^2}{4} - \frac{7}{12}\right)\ln(1+x^2) + \frac{5x^2}{12}$$
$$F_4[\arctan x](x) = \left(\frac{x^4}{24} - \frac{x^2}{4} + \frac{25}{24}\right)\arctan x - \left(\frac{x^3}{12} - \frac{7x}{12}\right)\ln(1+x^2) + \left(\frac{x^3}{24} - \frac{x}{24}\right)$$

This data suggests that:

$$F_n[\arctan x](x) = \alpha_n(x)\arctan x - \beta_n(x)\ln(1+x^2) + \gamma_n(x)$$

where α_n, β_n and γ_n are found using formulas (6), (31) and (32), so the right-hand side of (28) will be equal to :

$$\arctan x\sum_{k=0}^{n-1}(-1)^k a(n-1,k)x^{n-1-k}\alpha_{n+k}(x) - \ln(1+x^2)\sum_{k=0}^{n-1}(-1)^k a(n-1,k)x^{n-1-k}\beta_{n+k}(x) + \sum_{k=0}^{n-1}(-1)^k a(n-1,k)x^{n-1-k}\gamma_{n+k}(x)$$

and the left-hand side equals :

$$p^{-1}(xp^{-1})^{n-1}[\arctan x] = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}p^{-1}(xp^{-1})^{n-1}(x^{2k+1}) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+2n}}{2^n(2k+1)(k+1)(k+2)\cdots(k+n)}$$

Setting $x = 1$ in both sides, we get the desired formula. □

4 Appendix

We prove the following two propositions:

Proposition 1. *For all $n \in \mathbb{N}^*$, we have :*

$$F_1\left(\frac{x^{2n+1}}{1+x^2}\right) = \frac{1}{2}\left((-1)^n \ln(1+x^2) + \sum_{k=1}^{n}(-1)^{n-k}\frac{x^{2k}}{k}\right) \tag{29}$$

$$F_1\left(\frac{x^{2n}}{1+x^2}\right) = (-1)^n \arctan x + \sum_{k=1}^{n}(-1)^{n-k}\frac{x^{2k-1}}{2k-1} \tag{30}$$

Proof. It can easily be checked that the two formulas are correct for $n = 1$. Suppose that they are correct for n, and let us prove them correct for $n+1$. We have:

$$\begin{aligned} F_1\left(\frac{x^{2n+3}}{1+x^2}\right) &= F_1\ x^{2n+1} - F_1\left(\frac{x^{2n+1}}{1+x^2}\right) \\ &= \frac{x^{2n+2}}{2n+2} - \frac{1}{2}\left((-1)^n \ln(1+x^2) + \sum_{k=1}^{n}(-1)^{n-k}\frac{x^{2k}}{k}\right) \\ &= \frac{1}{2}\left((-1)^{n+1} \ln(1+x^2) + \sum_{k=1}^{n+1}(-1)^{n+1-k}\frac{x^{2k}}{k}\right) \end{aligned}$$

$$\begin{aligned} F_1\left(\frac{x^{2n+2}}{1+x^2}\right) &= F_1\ x^{2n} - F_1\left(\frac{x^{2n}}{1+x^2}\right) \\ &= \frac{x^{2n+1}}{2n+1} - (-1)^n \arctan x - \sum_{k=1}^{n}(-1)^{n-k}\frac{x^{2k-1}}{2k-1} \\ &= (-1)^{n+1} \arctan x - \sum_{k=1}^{n+1}(-1)^{n+1-k}\frac{x^{2k-1}}{2k-1} \end{aligned}$$

The proposition is proven by induction. □

Proposition 2. *For all $n \in \mathbb{N}^*$, we have :*

(31)

$$F_1(x^{2n} \arctan x) = \frac{1}{2n+1}\left(x^{2n+1} \arctan x - \frac{(-1)^n}{2}\ln(1+x^2) - \sum_{k=1}^{n}(-1)^{n-k}\frac{x^{2k}}{2k}\right)$$

(32)

$$F_1(x^{2n-1}\arctan x) = \frac{1}{2n}\left(\left(x^{2n} - (-1)^n\right)\arctan x - \sum_{k=1}^{n}(-1)^{n-k}\frac{x^{2k-1}}{2k-1}\right)$$

Proof. The above formulas are obtained using integration by parts and formulas (29) and (30). □

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