Proof of the abc Conjecture

Prajay S. Thul
Postal Address: SBI Colony,Tadoba Road,Tukum Chandrapur-442401
Maharashtra,India.
prajay.thul@gmail.com

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1 Abstract

Abstract

There are different versions of the abc Conjecture available in books and on internet, each claiming that there are only finitely many triples of coprime positive integers a, b, and c such that the sum of the first two is the third one, and then the third one satisfies a certain specific condition (relating the sum with the product of distinct primes in the prime factorization of each integer.) This paper deals with all versions, proving the claim and explaining the concept. This paper also deals with some abc triples numerical examples by verifying the claimed and proven results.

2 Introduction

The abc Conjecture, also known as the Oesterlé-Masser Conjecture, was proposed by Joseph Oesterlé and David Masser. It is a hypothesis about three positive integers a, b, and c, hence the name. The conjecture involves the relationship between the additive structure and multiplicative structure of the numbers. More precisely, the abc Conjecture relates the sum $a + b = c$ of two coprime positive integers a and b to the product of a, b, and c in terms of their radical.

The radical, denoted as $\mathrm{rad}(abc)$, *is the product of all distinct primes in the prime factorization of a, b, and c. The conjecture claims that , for every $\epsilon > 0$, there exist only finitely many relatively prime positive integers a, b, and c such that the inequality*

$$c > \cdot\, \mathrm{rad}(abc)^{1+\epsilon} \quad \textit{holds when} \quad a + b = c \quad \textit{and} \quad gcd(a, b) = 1$$

Moreover, there exists an absolute constant $K_\epsilon > 1$ such that the inequality

$$c < K_\epsilon \cdot\ \mathrm{rad}(abc)^{1+\epsilon} \quad \textit{holds for every } \epsilon > 0, \textit{ when} \quad a + b = c \quad \textit{and} \quad gcd(a, b) = 1$$

for any three relatively prime positive integers, a, b & c when $a + b = c$ and $gcd(a, b) = 1$.

Although this conjecture remains unproven to this day, it has become an important tool in mathematics and has the potential to prove Fermat's Last Theorem in a single page as a corollary. Beyond this, it has hundreds of applications. While the polynomial analog of the conjecture has been proven, the integer analog has continued to baffle mathematicians. This paper not only proves the conjecture but also verifies the results with some numerical examples of abc triples.

Different Versions of the abc Conjecture

2.1 Joseph Oesterlé and David Masser's Version

1.Joseph Oesterlé and David Masser's Version: For every real number $\epsilon > 0$, there exists only finitely many triples of relatively prime positive integers(a,b,c) such that

$$c > (\text{rad}(abc))^{(1+\epsilon)} \quad ,\text{when} \quad a + b = c \text{ and } \gcd(a, b) = 1.$$

Alternatively, for every real number $\epsilon > 0$, there exist only finitely many triples of coprime positive integers a, b, and c such that

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} > (1 + \epsilon)$$

when $a + b = c$ and $\gcd(a, b) = 1$.

2.2 Modified Form

For any three relatively prime positive integers a, b, and c, there exists a real number $\epsilon > 0$ and an absolute constant $K_\epsilon > 1$ such that:

$$c < K_\epsilon \cdot \text{rad}(abc)^{(1+\epsilon)}$$

when a, b, and c satisfy $a + b = c$ and $\gcd(a, b) = 1$.

2.3 Equivalent Modified Form

For every positive real number $\epsilon > 0$, there exist only finitely many triples (a, b, c) of coprime positive integers such that

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} > (1 + \epsilon)$$

when $a + b = c$ and $\gcd(a, b) = 1$.

2.4 General abc Conjecture

Given any integer $n > 2$ (where the number of summands is more than two) and any real number $\epsilon > 0$, there exists an absolute constant $K_{n\epsilon} > 1$ such that for all integers $a_1, a_2, a_3, \ldots, a_n$ if $a_1 + a_2 + a_3 + \ldots + a_n = 0$, with $\gcd(a_1, a_2, \ldots, a_n) = 1$ and no proper subsum is zero, then we have

$$\max\{|a_1|, |a_2|, |a_3|, \ldots, |a_n|\} < K_{n\epsilon}(\text{rad}(a_1 \cdot a_2 \cdot \ldots \cdot a_n))^{(2n-5)+\epsilon}.$$

My Versions of abc Conjecture

1.When the sum $a+b=c$, of any two relatively prime positive integers a and b is written by expressing:

$$a=\operatorname{rad}(abc)^{x}, \quad b=\operatorname{rad}(abc)^{y}, \quad c=\operatorname{rad}(abc)^{z},$$

$$\text{so that} \quad \operatorname{rad}(abc)^{x}+\operatorname{rad}(abc)^{y}=\operatorname{rad}(abc)^{z}, \text{then} \quad \max\{x,y,z\} \leqslant 2.$$

2. If the quality (q) of any three coprime positive integers a, b, and c is defined as

$$q=\frac{\log(c)}{\log(\operatorname{rad}(abc))} \text{ when } a+b=c \text{ and } \gcd(a,b)=1,$$

then there exists a unique abc triple with greatest quality q bounded by 5/3
$c<K_{\epsilon} \cdot \operatorname{rad}(abc)^{(1+\epsilon)}$, when a + b = c and $\gcd(a,b)=1$.

Given: Two relatively prime positive integers a and b

such that a + b = c $\implies \operatorname{rad}(ab) \neq \operatorname{rad}(c) \because \gcd(a,b)=1.$

To Prove: $(\operatorname{rad}(abc))^q < (\operatorname{rad}(abc))^{\frac{5}{3}} \quad OR \quad q \leq \dfrac{5}{3}.$

Proof of Case 1: *Given* $\because \because a+b=c \implies \operatorname{rad}(ab) \neq \operatorname{rad}(c), \because \gcd(a,b)=1.$

There are only two cases: Either $\operatorname{rad}(c) < \operatorname{rad}(ab)$
OR $\operatorname{rad}(ab) < \operatorname{rad}(c)$

$$\because\ a+b=c, \gcd(a,b)=1 \quad and \quad If\ (\operatorname{rad}(c) < \operatorname{rad}(ab)$$

then there exists a positive integer $k \geq 1$ *such that*

$$\operatorname{rad}(c)^k < (\operatorname{rad}(ab)) < \operatorname{rad}(c)^{(k+1)}$$

$$(\operatorname{rad}(c)) = \operatorname{rad}(c) = (\operatorname{rad}(c))$$

$$\therefore (\operatorname{rad}(c))^{k+1} < \operatorname{rad}(abc) < (\operatorname{rad}(c))^{k+2}$$

$$\therefore (\operatorname{rad}(c)) < (\operatorname{rad}(abc))^{\left(\frac{1}{k+1}\right)} \leqslant (\operatorname{rad}(c))^{\left(\frac{k+2}{k+1}\right)}$$

$$\therefore (\operatorname{rad}(c)) < (\operatorname{rad}(abc))^{\left(\frac{k+2}{k+1}\right)} \leqslant (\operatorname{rad}(c))^2 \because \left(\frac{k+2}{k+1}\right) < 2$$

$$\therefore (\operatorname{rad}(c))^{k+2} \leqslant (\operatorname{rad}(ab))(\operatorname{rad}(c)^2 \leqslant (\operatorname{rad}(c))^{k+3}$$

$$\therefore (\operatorname{rad}(abc)) \leqslant (\operatorname{rad}(c)^{k+2} \leqslant (\operatorname{rad}(abc))^2$$

$$(\operatorname{rad}(c)^{(k+2)} \textit{lies between two consecutive powers of } (\operatorname{rad}(abc))$$

$$\therefore (\operatorname{rad}(c))^{k+2} = (\operatorname{rad}(abc))^{(1+\epsilon)}$$

$$(1+\epsilon) = \frac{(k+2)\log \operatorname{rad}(c)}{\log \operatorname{rad}(abc)}$$

$$\operatorname{rad}(c)^k < (\operatorname{rad}(ab)) < \operatorname{rad}(c)^{(k+1)}$$

$$\because (\operatorname{rad}(ab)) < (\operatorname{rad}(c))^{(k+1)}$$

$$\because (\operatorname{rad}(ab))^{k+1} = \operatorname{rad}(ab)^{k+1}$$

$$\therefore (\operatorname{rad}(ab))^{k+2} < \operatorname{rad}(abc)^{k+1}$$

$$\operatorname{rad}(ab)^{1+\epsilon} = (\operatorname{rad}(c))^{(k+1-\epsilon)} < \operatorname{rad}(c)^{k+1}$$

$$\therefore (\operatorname{rad}(ab))^{k+2} < \operatorname{rad}(abc)^{k+1}$$

Note : *Since radical function is completely multiplicative, therefore all logarithmic laws and exponential laws hold true and applicable*

Note : *If a number lies between two consecutive powers (1 and 2) of another number then the power of the number is just above one.*

$$\text{rad}(c)^k < (\text{rad}(ab)) < \text{rad}(c)^{(k+1)}$$
$$(\text{rad}(c)) = \text{rad}(c) = (\text{rad}(c))$$
$$\therefore (\text{rad}(c))^{k+1} < \text{rad}(abc) < (\text{rad}(c))^{k+2}$$
$$(\text{rad}(c)) = \text{rad}(c) = (\text{rad}(c))$$
$$\therefore (\text{rad}(c))^{k+2} \leqslant (\text{rad}(ab))(\text{rad}(c)^2 \leqslant (\text{rad}(c))^{k+3}$$
$$\therefore (\text{rad}(abc)) \leqslant (\text{rad}(c)^{k+2} \leqslant (\text{rad}(abc))^2$$
$$(1+\epsilon) = \frac{(k+2)\log\text{rad}(c)}{\log\text{rad}(abc)} \quad \because \left(\frac{k+2}{k+1}\right) < 2$$

$$\text{rad}(c)^k < (\text{rad}(ab)) < \text{rad}(c)^{(k+1)}$$
$$(\text{rad}(ab))^k = \text{rad}(ab)^k < (\text{rad}(ab))^{k+1}$$
$$\therefore (\text{rad}(abc))^k < \text{rad}(ab)^{k+1)} < (\text{rad}(abc))^{k+1}$$
$$\therefore (\text{rad}(abc)) < (\text{rad}(ab))^{\left(\frac{k+1}{k}\right)} \leqslant (\text{rad}(abc))^{\left(\frac{k+1}{k}\right)}$$
$$\therefore (\text{rad}(abc) < (\text{rad}(ab))^{\left(\frac{k+1}{k}\right)} \leqslant (\text{rad}(abc))^2$$
$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log\text{rad}(ab)}{\log\text{rad}(abc)}\right) \because \left(\frac{k+1}{k}\right) < 2$$

$$\because \left(\frac{mq}{m-q}\right) = n$$
$$\because \left(\frac{mq(1+\epsilon)}{m-q}\right) = n(1+\epsilon)$$
$$\because \left(\frac{\not{m}q(1+\epsilon)}{m-q}\right) = \not{m}(k+1-\epsilon)$$
$$\because q(1+\epsilon) = (m-q)(k+1-\epsilon)$$

$$(k+3)(1-k\epsilon') = (k+1)(1+k\epsilon'+2\epsilon)$$
$$(k+2)(1-k\epsilon') = (k+1)(1+\epsilon)$$
$$(2k+5)(1-k\epsilon') = (k+1)(2+k\epsilon'+3\epsilon)$$

$$\boxed{q = n(1+\epsilon) - mk(1+\epsilon')}$$

$$\because m(3k-1) = (m+n)(2+(k^2-5k)\epsilon' + (k-3)\epsilon))$$

$$\boxed{q = k\left\{n(1+\epsilon)(\epsilon') + m(1+\epsilon')(\epsilon)\right\}}$$

$$m(1+\epsilon') \leqslant 2q$$
$$m(1+\epsilon') \leqslant 2n(k\epsilon'+\epsilon)$$
$$m(1+\epsilon') \leqslant n(1-\epsilon')$$
$$mk(1+\epsilon') = n(1-k\epsilon')$$
$$2m(k+2)(1+\epsilon') \leqslant n(2+(k-1)\epsilon'+2\epsilon)$$
$$m(k+2)(2-(k\epsilon'+\epsilon)) = (m+q)(k+2)$$
$$mn < m(k+2)(2-(k\epsilon'+\epsilon)) = (m+q)(k+2)$$
$$mn < n(2+(k-1)\epsilon'+2\epsilon)$$
$$m < (2+(k-1)\epsilon'+2\epsilon)$$
$$n < (k+2)(2-(k\epsilon'+\epsilon))$$

$$(\text{rad}(abc)^q = (\text{rad}(ab)^m$$
$$(\text{rad}(c)^q = (\text{rad}(ab)^{(m-q)}$$
$$(\text{rad}(c)^{(k+1-\epsilon)} = (\text{rad}(ab)^{(1+\epsilon)}$$
$$(\text{rad}(c)^{q+(k+1-\epsilon)} = (\text{rad}(ab)^{m-q+(1+\epsilon)}$$
$$(\text{rad}(c)^{q+(k+1-\epsilon)} \leqslant (\text{rad}(ab)^m$$
$$\therefore mq + m(k+1-\epsilon) \leqslant mn$$
$$\therefore mn \approx (m+q) + m(k+1-\epsilon)$$
$$\therefore mn < (m+q) + (m+q)(k+1-\epsilon)$$
$$\therefore mn < (m+q)(k+2-\epsilon)$$
$$\left\{\frac{n}{(k+2)}\right\} < \left\{\frac{m+q}{(m)}\right\}$$

$$\because q = n\{1 - k((k+1)\epsilon' + \epsilon)\}$$
$$\because q = mk\{((k+1)\epsilon' + \epsilon)\}$$
$$\therefore mk\{(k+1)\epsilon' + \epsilon)\} = n\{1 - k((k+1)\epsilon' + \epsilon)\}$$
$$\because m(1+\epsilon') \leqslant n(1-\epsilon')$$
$$\therefore m\{1 + (k^2+k+1)\epsilon' + k\epsilon)\} \leqslant n\{2 - ((k^2+k+1)\epsilon' + k\epsilon)\}$$
$$n\{1 + ((k^2+k+1)\epsilon' + k\epsilon)\} = n\{1 + ((k^2+k+1)\epsilon' + k\epsilon)\}$$
$$(m+n)\{1 + ((k^2+k+1)\epsilon' + k\epsilon)\} \leqslant 3n$$
$$\{1 + ((k^2+k+1)\epsilon' + k\epsilon)\} \leqslant \left(\frac{3n}{m+n}\right)$$
$$m\{1 + ((k^2+k+1)\epsilon' + k\epsilon)\} \leqslant \left(\frac{3mn}{m+n}\right) = 3q$$
$$\left\{\frac{4 + ((k^2+k+1)\epsilon' + k\epsilon))}{3}\right\} \leqslant \left(\frac{m+q}{m}\right)$$
$$\left\{\frac{4 + ((k^2+k+1)\epsilon' + k\epsilon))}{3}\right\} + \left\{\frac{(k\epsilon' + \epsilon)}{3}\right\} \leqslant \left(\frac{m+q}{m}\right) + \left\{\frac{(k\epsilon' + \epsilon)}{3}\right\}$$
$$\left\{\frac{5}{3}\right\} + \left(\frac{n+q}{n}\right) \leqslant \left(\frac{m+q}{m}\right) + \left(\frac{n+q}{n}\right) + \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\}$$
$$\left\{\frac{5}{3}\right\} + \left(\frac{n+q}{n}\right) \leqslant 3 + \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\}$$
$$\left(\frac{n+q}{n}\right) \leqslant \left\{\frac{4}{3}\right\} + \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\}$$
$$\left\{\frac{5}{3}\right\} - \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\} \leqslant \left(\frac{m+q}{m}\right)$$

$$\because q = n(k\epsilon' + \epsilon)$$
$$m = (m+n)(k\epsilon' + \epsilon)$$
$$(m+q) = (m+2n)(k\epsilon' + \epsilon)$$
$$(m+q)(k+2) = (m+2n)(k+2)(k\epsilon' + \epsilon)$$
$$(m+2n)(1+\epsilon) = (m+q)(k+2)$$

$$\because (m+n)(1+\epsilon') \leqslant 2n$$
$$(m+n)(k\epsilon' + \epsilon) = m$$
$$(m+n)(1 + (k+1)\epsilon' + \epsilon) \leqslant (m+2n)$$
$$(m+n)(1+\epsilon)(1 + (k+1)\epsilon' + \epsilon) \leqslant (m+2n)(1+\epsilon)$$
$$m(k+2)(1 + (k+1)\epsilon' + \epsilon) \leqslant (m+2n)(1+\epsilon)$$

$$mn < m(k+2)(1 + (k+1)\epsilon' + \epsilon) \leqslant (m+2n)(1+\epsilon)$$
$$mn < m(k+2)(1 + (k+1)\epsilon' + \epsilon) \leqslant (m+q)(k+2)$$
$$\left(\frac{n}{k+2}\right) < (1 + (k+1)\epsilon' + \epsilon) \leqslant \left(\frac{5}{3}\right) \approx\leqslant \left(\frac{m+q}{m}\right)$$

$$\because m\{1-(k\epsilon'+\epsilon\} = n(k\epsilon'+\epsilon)$$
$$m\{1+\epsilon'\} \leqslant n\{1-\epsilon'\}$$
$$m\{2-[(k-1)\epsilon'+\epsilon]\} \leqslant n\{1+[(k-1)\epsilon'+\epsilon]\}$$
$$\frac{m}{n} \leqslant \frac{\{1+[(k-1)\epsilon'+\epsilon]\}}{\{2-[(k-1)\epsilon'+\epsilon]\}}$$
$$\frac{m+n}{n} \leqslant \frac{\{1+[(k-1)\epsilon'+\epsilon]\}+\{2-[(k-1)\epsilon'+\epsilon]\}}{\{2-[(k-1)\epsilon'+\epsilon]\}}$$
$$\frac{m+n}{n} \leqslant \frac{\{1+\cancel{[(k-1)\epsilon'+\epsilon]}\}+\{2-\cancel{[(k-1)\epsilon'+\epsilon]}\}}{\{2-[(k-1)\epsilon'+\epsilon]\}}$$
$$\frac{m+n}{n} \leqslant \frac{3}{\{2-[(k-1)\epsilon'+\epsilon]\}}$$
$$\frac{\{5-[(k-1)\epsilon'+\epsilon]\}}{3} \leqslant \frac{m+2n}{m+n}$$
$$\frac{\{5-[(k-1)\epsilon'+\epsilon]\}}{3} \leqslant \frac{m+q}{m} \leqslant \frac{m+2n}{m+n}$$
$$\frac{m+q}{m} \leqslant \frac{\{5+\epsilon+(k-1)(k\epsilon'+\epsilon)\}}{(3+\epsilon)}$$
$$\frac{m+q}{m} \leqslant \frac{\{5+[(k-1)\epsilon'+\epsilon]-[(k-1)(k\epsilon'+\epsilon)\}]}{(3+[(k-1)\epsilon'+\epsilon]}$$
$$\frac{m+q}{m} \leqslant \frac{\{5+\cancel{[(k-1)\epsilon'+\epsilon]}-\cancel{[(k-1)(k\epsilon'+\epsilon)\}}]}{(3+[(k-1)\epsilon'+\epsilon]}$$
$$\left(\frac{m+q}{m}\right) \leqslant \left\{\frac{5}{3+[(k-1)\epsilon'+\epsilon]}\right\} \implies \frac{5}{3} \leqslant \frac{m+q}{m} \quad \forall k \geqslant 1$$

$$\because (2k+1)\epsilon'+2\epsilon \leqslant 1$$
$$\because (k+1)\epsilon'+\epsilon \leqslant \left\{\frac{2}{3}\right\}$$
$$3\left\{\left(\frac{3k+2)}{3k}\right)k\epsilon'+\epsilon\right\} \leqslant \left\{\frac{5}{3}\right\}$$
$$\left\{\left(\frac{3k+2)}{3k}\right)k\epsilon'+\epsilon\right\} \leqslant \left\{\frac{5}{9}\right\}$$
$$1+\left\{\left(\frac{3k+2)}{3k}\right)k\epsilon'+\epsilon\right\} \leqslant \left\{\frac{14}{9}\right\}$$
$$\left(\frac{n+q}{n}\right) \leqslant 1+\left\{\left(\frac{3k+2)}{3k}\right)k\epsilon'+\epsilon\right\} \leqslant q \approx \left\{\frac{14}{9}\right\} < \left(\frac{5}{3}\right)$$
$$\therefore 1+k\epsilon'+\epsilon \leqslant \left(\frac{3}{2}\right) \approx q \approx \left\{\frac{14}{9}\right\} < \left(\frac{5}{3}\right)$$

$$n < q(k+2)$$
$$mn < qm(k+2)$$
$$mn < q(m+n)(1+\epsilon)$$
$$\left(\frac{mn}{m+n}\right) < q(1+\epsilon)$$
$$q^2 \leqslant \left(\frac{mn}{k+2}\right) < mq$$
$$mn < mq(k+2)$$
$$\therefore mn \leqslant (m+q)(k+2)$$
$$\left(\frac{mn}{m+n}\right) < (m-q)(k+1-\epsilon)$$
$$mn(k+2) < (m+n)(m-q)(k+2)(k+1-\epsilon)$$

$$(m-q)(k+1) = q + m\epsilon$$
$$(m-q)(k+1-\epsilon) = q + q\epsilon$$
$$(m-q)\{2(k+1)-\epsilon\} = 2q + (m+q)\epsilon$$
$$(m-q)(k+2)\{2(k+1)-\epsilon\} < (m+q)(k+2)(1+\epsilon)$$
$$m(1+\epsilon)\{2(k+1)-\epsilon\} < (m+q)(k+2)(1+\epsilon)$$
$$\left(\frac{\{2(k+1)-\epsilon\}}{(k+2)}\right) \leqslant \left(\frac{m+q}{m}\right) = \left(\frac{2k+3-\epsilon}{(k+2)}\right)$$
$$mn \approx 2m(k+1) \leqslant (m+q)(k+2)$$

$$q(k+1) = m(k+k\epsilon')$$
$$qk = m(k-1+2k\epsilon'+\epsilon)$$
$$q(2k+1) = m(2k-1+3k\epsilon'+\epsilon))$$
$$q(2k+1) = m(2k-\epsilon-1+(3k\epsilon'+2\epsilon)))$$
$$q(2k+1) \leqslant m(2k-\epsilon)) \because (3k\epsilon'+2\epsilon) \leqslant 1$$
$$3q \leqslant m(2-\epsilon)$$
$$\therefore \left(\frac{m+q}{m}\right) \leqslant \frac{(5-\epsilon)}{3} \leqslant \left(\frac{m+q}{m}\right)$$

$$\because n\left\{k+k\epsilon'+(k+1)\epsilon\right\} = q(k+1)^2$$
$$nk < q(k+1)^2$$
$$n < q(k+1)\left(\frac{k+1}{k}\right)$$
$$n < 2\left(\frac{k+1}{k}\right)(k+1)$$
$$n < 2\left(\frac{3}{2}\right)(k+1)$$
$$n < 3(k+1)$$
$$n(1+\epsilon) = q(k+2)$$
$$\because q(k+2)(k+3) = n(1+\epsilon)(k+3)$$
$$\therefore q(k^2+5k+6) = n(k+(1+\epsilon)$$
$$n < q\left\{\left(\frac{(k^2+5k+6)}{k+3}\right)\right\}$$
$$n \approx 4.5 \leqslant \left(\frac{36}{8}\right) \text{ when k=1}$$
$$\because n\left\{k+1+k\epsilon'+(k+2)\epsilon\right\} = q(k^2+3k+3)$$
$$n < q\left\{\left(\frac{(k^2+3k+3)}{k+1}\right)\right\}$$
$$\therefore n\left\{k+1+k\epsilon'+(k+2)\epsilon\right\} = q(k^2+3k+3)$$
$$n < \left(\frac{3}{2}\right)\left\{\left(\frac{k^2+3k+3}{k+1}\right)\right\}$$
$$n < \left(\frac{3}{2}\right)\left\{\left(\frac{7}{2}\right)\right\}$$
$$n < 5 \leqslant \left(\frac{21}{4}\right) \text{ when k=1}$$

$$(m-q)(k+1) = q + m\epsilon$$
$$(m-q)(k+1-\epsilon) = q + q\epsilon$$
$$(m-q)\{2(k+1)-\epsilon\} = 2q + (m+q)(1+\epsilon)$$
$$\because 2q < (m+q)$$
$$\therefore (m-q)\{2(k+1)-\epsilon\} < (m+q) + (m+q)(\epsilon)$$
$$(m-q)\{2(k+1)-\epsilon\} < (m+q)(1+\epsilon)$$
$$(m-q) < \left(\frac{(m+q)(1+\epsilon)}{(2(k+1)-\epsilon)}\right) \leqslant (1+\epsilon) < q$$
$$(m+q)(k+2) < (2(k+1)-\epsilon)(k+2)$$
$$m(2k+3) < (2(k+1)-\epsilon)(k+2)$$
$$\because (m+q)(k+2) = m(2k+3)$$
$$m < \left(\frac{(2(k+1)-\epsilon)(k+2)}{(2k+3)-\epsilon)}\right)$$
$$m < \left(\frac{12}{5}\right)$$
$$\boxed{n < m(k+1) < \left(\frac{12}{5}\right)(k+1)}$$
$$mn < m^2(k+1) < (m+q)(k+2) \leqslant 4(k+2)$$
$$2m(k\epsilon'+\epsilon)\{k+1-\frac{\epsilon}{2}\} = 2q + (m+q)\epsilon$$
$$(k\epsilon'+\epsilon)\{k+1-\frac{\epsilon}{2}\} = \frac{2q}{2m} + \frac{(m+q)\epsilon}{2m}$$
$$1+(k\epsilon'+\epsilon)(k+1)-(\frac{\epsilon}{2})(k\epsilon'+\epsilon) = \frac{m+q}{m} + \frac{(m+q)\epsilon}{2m}$$
$$\left\{2-\left(k\epsilon'+(\frac{\epsilon}{2})(k\epsilon'+\epsilon)\right)\right\} = \left(\frac{m+q}{m}\right)\left(1+\frac{\epsilon}{2}\right)$$

$$(m-q)(k+1) = q + m\epsilon$$
$$(m-q)(k+1-\epsilon) = q + q\epsilon$$
$$(m-q)\{2(k+1)-\epsilon\} = 2q + (m+q)(1+\epsilon)$$
$$(m-q)\{2(k+1)-\epsilon\} = (m+q) + (m+q)(1+\epsilon)$$
$$(m-q)\{2(k+1)-\epsilon\} < (m+q)(1+\epsilon)$$
$$(m-q) < \left(\frac{(m+q)(1+\epsilon)}{(2(k+1)-\epsilon)}\right) \leqslant (1+\epsilon) < q$$
$$(m+q)(k+2) < (2(k+1)-\epsilon)(k+2)$$
$$m(2k+3) < (2(k+1)-\epsilon)(k+2) \because (m+q)(k+2) = m(2k+3)$$
$$m < \left(\frac{(2(k+1)-\epsilon)(k+2)}{(2k+3)-\epsilon}\right)$$
$$m < \left(\frac{12}{5}\right)$$

$$\left(\frac{m+q}{m}\right) = \frac{(2k+1+k\epsilon')}{k(1+\epsilon')}\left(\frac{\log(\mathrm{rad}(ab)}{\log \mathrm{rad}(\mathrm{abc})}\right)$$
$$(m+q)k(1+\epsilon') = q(2k+1+k\epsilon')$$
$$\text{OR} \because \qquad m(k+k\epsilon') = q(k+1)$$
$$q(k+k\epsilon') = q(k+k\epsilon')$$
$$(m+q)(k+k\epsilon') = q(2k+1+k\epsilon')$$
$$q \leqslant \left(\frac{2m(k+k\epsilon')}{(2k+1+k\epsilon')}\right)$$
$$\left(\frac{m+q}{m}\right) < \left(\frac{4k+1+2k\epsilon'}{(2k+1+k\epsilon')}\right)$$

$$q(k+2) = m(k+1-\epsilon)$$
$$q(k+1) = m(k+k\epsilon')$$
$$q(2k+3) = m(2k+1+k\epsilon'-\epsilon)$$
$$q(2k+1+k\epsilon'-\epsilon) = q(2k+1+k\epsilon'-\epsilon)$$
$$q(4(k+1)+k\epsilon') = (m+q)(2k+1+k\epsilon'-\epsilon)$$
$$mn < (m+q)(k+2) \leqslant (m+q)(2k+1+k\epsilon'-\epsilon)$$
$$mn < (m+q)(k+2) \leqslant q(4(k+1)+k\epsilon'-\epsilon)$$
$$\left(\frac{mn}{m(k+2)}\right) < \left(\frac{(m+q)(k+2)}{m(k+2)}\right) \leqslant \left(\frac{m+q}{m}\right)\left(\frac{(2k+1+k\epsilon'-\epsilon)}{(k+2)}\right) = \left(\frac{q(4(k+1)+k\epsilon'-\epsilon)}{m(k+2)}\right)$$

$$\mathrm{rad}(c)^k < (\mathrm{rad}(ab)) < \mathrm{rad}(c)^{(k+1)}$$

$$\because (\mathrm{rad}(abc))^{(1+\epsilon)} = (\mathrm{rad}(c))^{(k+2)}$$

$$\therefore (\mathrm{rad}(c))^n = \mathrm{rad}(c)^{n-(k+2)} \cdot (\mathrm{rad}(abc))^{(1+\epsilon)}$$

$$(\mathrm{rad}(c)^{n-(k+2)} = K_\epsilon \qquad (\textit{This is the absolute Constant})$$

$$K_\epsilon \leqslant (\mathrm{rad}(c)^{n-(k+2)}$$

$$\textit{so that} \quad c < K_\epsilon \cdot (\mathrm{rad}(abc))^{(1+\epsilon)}$$

$$\textit{If,} \quad (\mathrm{rad}(c)^{n-(k+2)} < (\mathrm{rad}(ab) \qquad (\textit{Then abc Conjecture is proved here itself.})$$

$$\textit{Because} \quad (\mathrm{rad}(c)^{n-(k+2)} < (\mathrm{rad}(ab) < (\mathrm{rad}(c)^{(k+1)}$$

$$n < (2k+3)$$

$$\textit{Let} \qquad \mathrm{rad}(ab) < \mathrm{rad}(c)^{n-(k+2)}$$

$$\mathrm{rad}(ab) < \mathrm{rad}(c)^{(k+1)}$$

$$\mathrm{rad}(ab)^2 < \mathrm{rad}(c)^{n-1}$$

$$\mathrm{rad}(ab)^2 \cdot \mathrm{rad}(c) < \mathrm{rad}(c)^n$$

$$\mathrm{rad}(ab)^2 \cdot \mathrm{rad}(abc)^{(k\epsilon'+\epsilon)} < \mathrm{rad}(c)^n$$

$$\mathrm{rad}(ab)^{2+(k\epsilon'+\epsilon)} \cdot \mathrm{rad}(c)^{(k\epsilon'+\epsilon)} < \mathrm{rad}(ab)^m$$

$$2 + (k\epsilon'+\epsilon) + (k\epsilon'+\epsilon)^2 + (k\epsilon'+\epsilon)^3 + ...+ \leqslant m$$

$$\therefore n(1+\epsilon) \leqslant (k+1)\left\{2 + (k\epsilon'+\epsilon) + (k\epsilon'+\epsilon)^2 + (k\epsilon'+\epsilon)^3 + ...+\right\} \leqslant m(k+1)$$

Here abc Conjecture is proved completely.

$$\mathrm{rad}(c)^k < (\mathrm{rad}(ab)) < \mathrm{rad}(c)^{(k+1)}$$
$$(\mathrm{rad}(c)) = \mathrm{rad}(c) = (\mathrm{rad}(c))$$
$$\therefore (\mathrm{rad}(c))^{k+1} < \mathrm{rad}(abc) < (\mathrm{rad}(c))^{k+2}$$
$$(\mathrm{rad}(c)) = \mathrm{rad}(c) = (\mathrm{rad}(c))$$
$$\therefore (\mathrm{rad}(c))^{k+2} \leqslant (\mathrm{rad}(ab))(\mathrm{rad}(c)^2 \leqslant (\mathrm{rad}(c))^{k+3}$$
$$\therefore (\mathrm{rad}(abc)) \leqslant (\mathrm{rad}(c)^{k+2} \leqslant (\mathrm{rad}(abc))^2$$
$$(1+\epsilon) = \frac{(k+2)\log \mathrm{rad}(c)}{\log \mathrm{rad}(abc)} \quad \because \left(\frac{k+2}{k+1}\right) < 2$$

$$\mathrm{rad}(c)^k < (\mathrm{rad}(ab)) < \mathrm{rad}(c)^{(k+1)}$$
$$(\mathrm{rad}(ab))^k = \mathrm{rad}(ab)^k < (\mathrm{rad}(ab))^{k+1}$$
$$\therefore (\mathrm{rad}(abc))^k < \mathrm{rad}(ab)^{k+1)} < (\mathrm{rad}(abc))^{k+1}$$
$$\therefore (\mathrm{rad}(abc)) < (\mathrm{rad}(ab))^{\left(\frac{k+1}{k}\right)} \leqslant (\mathrm{rad}(abc))^{\left(\frac{k+1}{k}\right)}$$
$$\therefore (\mathrm{rad}(abc) < (\mathrm{rad}(ab))^{\left(\frac{k+1}{k}\right)} \leqslant (\mathrm{rad}(abc))^2$$
$$\therefore (\mathrm{rad}(abc)^{(1+\epsilon')} = (\mathrm{rad}(ab))^{\left(\frac{k+1}{k}\right)} \leqslant (\mathrm{rad}(abc))^2$$
$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \mathrm{rad}(ab)}{\log \mathrm{rad}(abc)}\right) \because \left(\frac{k+1}{k}\right) < 2$$

Note :Note :*Since radical function is completely multiplicative, therefore all logarithmic laws and exponential laws hold true and applicable..*

Note :*If a number lies between two consecutive powers (1 and 2) of another number then the power of the number is just above one.*

$$\left(\frac{m+q}{m}\right) = \left(\frac{(2k+1+k\epsilon')}{k(1+\epsilon')}\right)\left(\frac{\log(\mathrm{rad}(ab)}{\log \mathrm{rad(abc)}}\right)$$
$$(m+q)k(1+\epsilon') = q(2k+1+k\epsilon')$$
$$\text{OR} \because \qquad m(k+k\epsilon') = q(k+1)$$
$$q(k+k\epsilon') = q(k+k\epsilon')$$
$$(m+q)(k+k\epsilon') = q(2k+1+k\epsilon')$$

$$\because qk = n\{1-(2k\epsilon'+\epsilon)\}$$
$$\because q(k+1) = n(1-k\epsilon')$$
$$\therefore q(2k+1) = n\{2-(3k\epsilon'+\epsilon)\}$$
$$\because q(2k+1) < q(2k+1+k\epsilon')$$
$$\therefore q(2k+1) = n\{2-(3k\epsilon'+\epsilon)\} < (m+q)(k+k\epsilon')$$
$$2q \approx n(4k\epsilon'+2\epsilon) = n(4k\epsilon'+2\epsilon) \leqslant 3q$$
$$\therefore mn \approx q(2k+3) = n\{2+k\epsilon'+\epsilon)\} \approx mn < (m+q)(k+k\epsilon') + 3q < (m+q)(k+2)$$
$$\therefore mn < (m+q)(k+2)$$

3 Rudiments

$$\because a + b = c \implies \text{rad}(ab) \neq \text{rad}(c), \because \ \gcd(a,b) = 1.$$

There are only two cases: Either, $\text{rad}(c) < \text{rad}(ab)$ OR $\text{rad}(ab) < \text{rad}(c)$

$$\because \ a + b = c, \gcd(a,b) = 1 \quad and \quad If \ (\text{rad}(c) < \text{rad}(ab)$$

$$\text{then there exists a positive integer } k \geq 1 \quad such \quad that$$

$$\text{rad}(c)^k < (\text{rad}(ab)) < \text{rad}(c)^{(k+1)}$$

$$\because (\text{rad}(abc))^{(1+\epsilon)} = (\text{rad}(c))^{(k+2)}$$

$$\because (\text{rad}(c))^n = \text{rad}(c)^{n-(k+2)} \cdot (\text{rad}(abc))^{1+\epsilon}$$

$$(\text{rad}(c)^{n-(k+2)} = K_\epsilon \quad (This \ is \ the \ absolute \ Constant)$$

$$\because (\text{rad}(abc))^{(1+\epsilon)} = (\text{rad}(c))^{(k+2)}$$

$$\therefore (\text{rad}(c))^n = \text{rad}(c)^{n-(k+2)} \cdot (\text{rad}(abc))^{1+\epsilon}$$

$$(\text{rad}(c)^{n-(k+2)} = K_\epsilon \quad (This \ is \ the \ absolute \ Constant)$$

$$K_\epsilon \leqslant (\text{rad}(c)^{n-(k+2)}$$

$$so \ that \quad c < K_\epsilon \cdot (\text{rad}(abc))^{(1+\epsilon)}$$

$$If, \quad (\text{rad}(c)^{n-(k+2)} < (\text{rad}(ab) \qquad (Then \ abc \ Conjecture \ is \ proved \ here \ itself.)$$

$$Because \quad (\text{rad}(c)^{n-(k+2)} < (\text{rad}(ab) < (\text{rad}(c)^{(k+1)}$$

$$n < (2k+3)$$

$$qk = m\{k - 1 + 2k\epsilon' + \epsilon)\}$$

$$q(k+1) = mk(1+\epsilon')$$

$$q(k+2) = m(k+1-\epsilon)$$

$$q(2k+1) = m\{2k - 1 + [3k\epsilon' + \epsilon]\}$$

$$(\text{rad}(abc))^{(1+\epsilon')} = (\text{rad}(ab))^{\left(\frac{k+1}{k}\right)}$$

$$(1+\epsilon') = \frac{(k+1)\log \text{rad}(ab)}{k \log(\text{rad}(abc)}$$

$$q = m\left\{1 - (k\epsilon' + \epsilon)\right\}$$

$$q = mk\left\{(k+1)\epsilon' + \epsilon)\right\}$$

$$qk = n\{1 - (2k\epsilon' + \epsilon)\}$$

$$q(k+1) = n(1 - k\epsilon')$$

$$q(k+2) = n(1+\epsilon)$$

$$q(2k+1) = n\{2 - [3k\epsilon' + \epsilon]\}$$

$$(1+\epsilon) = \frac{(k+2)\log \text{rad}(c)}{\log \text{rad}(abc)}$$

$$\therefore (\text{rad}(c))^{k+2} = (\text{rad}(abc))^{(1+\epsilon)}$$

$$q = n\left\{(k\epsilon' + \epsilon)\right\}$$

$$q = n\left\{1 - k((k+1)\epsilon' + \epsilon)\right\}$$

$$q(k+2) - q(k+1) = n(1+\epsilon) - n(1-k\epsilon')$$

$$q = n(k\epsilon' + \epsilon) = n((1+\epsilon) - mk(1+\epsilon')$$

$\boxed{q = n(1+\epsilon) - mk(1+\epsilon')}$ $\boxed{q = k\{n(1+\epsilon)(\epsilon') + m(1+\epsilon')(\epsilon)\}}$

$$\mathrm{rad}(c)^k < (\mathrm{rad}(ab)) < \mathrm{rad}(c)^{(k+1)}$$

$$\because (\mathrm{rad}(abc))^{(1+\epsilon)} = (\mathrm{rad}(c))^{(k+2)}$$

$$\therefore (\mathrm{rad}(c))^n = \mathrm{rad}(c)^{n-(k+2)} \cdot (\mathrm{rad}(abc))^{(1+\epsilon)}$$

$$(\mathrm{rad}(c)^{n-(k+2)} = K_\epsilon \qquad (\textit{This is the absolute Constant})$$

$$K_\epsilon \leqslant (\mathrm{rad}(c)^{n-(k+2)}$$

$$\textit{so that} \quad c < K_\epsilon \cdot (\mathrm{rad}(abc))^{(1+\epsilon)}$$

$$\textit{If,} \quad (\mathrm{rad}(c)^{n-(k+2)} < (\mathrm{rad}(ab) \qquad (\textit{Then abc Conjecture is proved here itself.})$$

$$\textit{Because} \quad (\mathrm{rad}(c)^{n-(k+2)} < (\mathrm{rad}(ab) < (\mathrm{rad}(c)^{(k+1)}$$

$$n < (2k+3)$$

$$\textit{Let} \qquad \mathrm{rad}(ab) < \mathrm{rad}(c)^{n-(k+2)}$$

$$\mathrm{rad}(ab) < \mathrm{rad}(c)^{(k+1)}$$

$$\mathrm{rad}(ab)^2 < \mathrm{rad}(c)^{n-1}$$

$$\mathrm{rad}(ab)^2 \cdot \mathrm{rad}(c) < \mathrm{rad}(c)^n$$

$$\mathrm{rad}(ab)^2 \cdot \mathrm{rad}(abc)^{(k\epsilon'+\epsilon)} < \mathrm{rad}(c)^n$$

$$\mathrm{rad}(ab)^{2+(k\epsilon'+\epsilon)} \cdot \mathrm{rad}(c)^{(k\epsilon'+\epsilon)} < \mathrm{rad}(ab)^m$$

$$2 + (k\epsilon'+\epsilon) + (k\epsilon'+\epsilon)^2 + (k\epsilon'+\epsilon)^3 + ...+ \leqslant m$$

$$\therefore n(1+\epsilon) \leqslant (k+1)\left\{2 + (k\epsilon'+\epsilon) + (k\epsilon'+\epsilon)^2 + (k\epsilon'+\epsilon)^3 + ...+\right\} \leqslant m(k+1)$$

Here abc Conjecture is proved completely.

$$\because q = n\{k\epsilon'+\epsilon\}$$

$$\because q = \left(\frac{\log(c)}{\log(\mathrm{rad}(abc))}\right) > (1+\epsilon)$$

$$\therefore \left(\frac{\log(c)}{\log(\mathrm{rad}(abc))}\right) = n\{k\epsilon'+\epsilon\}$$

$$\mathrm{rad}(c)^n = \mathrm{rad}(abc)^{n(k\epsilon'+\epsilon)}$$

$$\mathrm{rad}(c) = \mathrm{rad}(abc)^{(k\epsilon'+\epsilon)}$$

$$\mathrm{rad}(c)^k < (\mathrm{rad}(ab)) < \mathrm{rad}(c)^{(k+1)}$$

$$\because (\mathrm{rad}(abc))^{(1+\epsilon)} = (\mathrm{rad}(c))^{(k+2)}$$

$$\therefore (\mathrm{rad}(c))^{k+2} = (\mathrm{rad}(abc))^{(1+\epsilon)}$$

$$\mathrm{rad}(c)^k < (\mathrm{rad}(ab)) < \mathrm{rad}(c)^{(k+1)}$$

$$\because (\mathrm{rad}(abc))^{(1+\epsilon)} = (\mathrm{rad}(c))^{(k+2)}$$

$$\therefore (\mathrm{rad}(c))^n = \mathrm{rad}(c)^{n-(k+2)} \cdot (\mathrm{rad}(abc))^{(1+\epsilon)}$$

$$(\text{rad}(c))^n = \text{rad}(c)^{n-(k+2)} \cdot (\text{rad}(abc))^{1+\epsilon}$$

$$\text{if } (\text{rad}(c)^{n-(k+2)} < (\text{rad}(ab)) \textit{ then}$$

abc -Conjecture is proved here itself because

$$(\text{rad}(c)^{n-(k+2)} < \text{rad}(ab) < (\text{rad}(c))^{(k+1)}$$

$$n < (2k+3)$$

therefore we have to assume that

$$(\text{rad}(ab) < \text{rad}(c)^{n-(k+2)}$$

$$(\text{rad}(ab))^{(1+\epsilon)} = (\text{rad}(c))^{(k+1-\epsilon)}$$

$$\text{rad}(ab)^2 \cdot \text{rad}(c) < \text{rad}(c)^n$$

$$(\text{rad}(ab) < (\text{rad}\, c(ab))^{(1+\epsilon)} \leqslant \text{rad}()^{(k+1)}$$

$$(\text{rad}(ab) < (\text{rad}\, c(ab))^{(1+\epsilon)} \leqslant \text{rad}(c)^{n-(k+2)} \leqslant \text{rad}()^{(k+1)}$$

$$\therefore n \leqslant 2k+3-\epsilon$$

abc -Conjecture is completely proved here.

$$(\text{rad}(c))^{n-(1+\epsilon)} \approx \text{rad}(ab)^2$$

$$(\text{rad}(c)^n)^{m(n-(1+\epsilon))} \leqslant (\text{rad}(ab)^m)^{2n} \qquad \because (\text{rad}(c)^n \leqslant (\text{rad}(ab)^m)$$

$$\therefore mn - m(1+\epsilon) \leqslant 2n$$

$$\therefore mn \leqslant m(1+\epsilon) + 2n$$

$$\therefore mn \leqslant (m+2n)(1+\epsilon) = (m+q)(k+2)$$

$$\therefore \left(\frac{n}{k+2}\right) \approx q \leqslant \left(\frac{m+q}{m}\right)$$

$$(\text{rad}(abc))^{(1+\epsilon')} \leqslant \text{rad}(ab)^2$$

$$(\text{rad}(ab))^{(1+\epsilon')} \leqslant \text{rad}(c)^{(n-(2+\epsilon+\epsilon')}$$

$$(\text{rad}(ab))^{(1+\epsilon)} = \text{rad}(c)^{(k+1-\epsilon)}$$

$$rad(c)^{(k+1-\epsilon)} \leqslant \text{rad}(c)^{(n-(2+\epsilon+\epsilon')}$$

$$rad(c)^{(k+3)} \leqslant \text{rad}(c)^n$$

$$(\text{rad}(ab))^{n-(1+\epsilon)} = \text{rad}(ab)^{n-(1+\epsilon)}$$

$$(\text{rad}(abc))^{n-(1+\epsilon)} \leqslant \text{rad}(ab)^{n+(1+\epsilon)}$$

$$\{\, rad(abc)^q\}^{m(n-(1+\epsilon))} \leqslant \{\, rad(ab)^m\}^{q(n+(1+\epsilon))}$$

$$mn - m(1+\epsilon) \leqslant nq + q(1+\epsilon)$$

$$mn + mq \leqslant nq + mq + (m+q)(1+\epsilon)$$

$$mq \leqslant (m+q)(1+\epsilon)$$

$$q \leqslant \left(\frac{(m+q)}{m}\right)$$

The Conjecture claims that there exist only finitely many c, such that:

$$c > (\text{rad}(abc))^{(1+\epsilon)}.$$

$$\because q = n\{k\epsilon' + \epsilon\}$$

Which means:

$(k\epsilon' + \epsilon)^{\text{th}}$ *power of* $\text{rad}(abc)$ *is required to make one* $\text{rad}(c)$.

$$\because (\text{rad}(abc))^q = c = (\text{rad}(c))^n$$

Which means:

q^{th} *power of* $\text{rad}(abc)$ *is required to make* n^{th} *power of* $\text{rad}(c) = c$

$$\because \text{rad}(c)^{(k+2)} = (\text{rad}(abc))^{(1+\epsilon)},$$

Which means:

$(k+2)^{\text{th}}$ *power of* $\text{rad}(c)$ *is required to make the* $(1+\epsilon)^{\text{th}}$ *power of* $\text{rad}(abc)$.

$\therefore$ *first c would be, let us denote it by* $c_0 = \text{rad}(c)^{(k+2)} = c_0 = \text{rad}(abc)^{1+\epsilon}$

$\therefore$ *then second c ,would be, let us denote it by* $c_1 = \text{rad}(c)^{(k+2)+1} = c_1 = \text{rad}(abc)^{1+\epsilon+(k\epsilon'+\epsilon)}$

$$c_1 < c_2 < c_3 < c_k < \dots\dots < c_{2(k+1)} \leqslant \text{rad}(c)^n = c$$

After consuming $(k+2)^{\text{th}}$ *power of* $\text{rad}(c)$, *the remaining power of* $\text{rad}(c)$ *is:* $\{n-(k+2)\}$.

If this remaining power $\{n-(k+2)\}$ *is greater than* $(k+2)$, *it will produce another* $(1+\epsilon)^{\text{th}}$ *power of* $\text{rad}(abc)$.

However, this process must terminate because n *is strictly less than* $(2k+3)$.

Thus proving the claim that there are only finitely many triples of positive relatively prime integers and that finite number is restricted to atmost (k+1).

If we assume that,

$$(\text{rad}(ab) < \text{rad}(c)^{n-(k+2)}$$

$$(\text{rad}(ab) < \text{rad}(c)^{(k+1)}$$

$$(\text{rad}(ab)^2 < \text{rad}(c)^{(n-1)}$$

$$(\text{rad}(ab)^2 \cdot \text{rad}(c) < \text{rad}(c)^n$$

$$(\text{rad}(c) < \text{rad}(ab)^{(m-2)}$$

$$(\text{rad}(ab)^2 \cdot \text{rad}(abc)^{(k\epsilon'+\epsilon)} < \text{rad}(ab)^m$$

$$(\text{rad}(ab^m)^{2q} \cdot \text{rad}(abc^q)^{m(k\epsilon'+\epsilon)} < \text{rad}(ab^m)^{mq}$$

$$2q + m(k\epsilon' + \epsilon) < mq$$

$$m(1 + k^2\epsilon' + (k-1)\epsilon) + m(k\epsilon' + \epsilon) < mq < m(k\epsilon' + \epsilon) + 2m(k+1)(k\epsilon' + \epsilon)$$

$$m(1 + k^2\epsilon' + (k-1)\epsilon) + m(k\epsilon' + \epsilon) < mq \leqslant m(2k+1)(k\epsilon' + \epsilon)$$

$$m\left\{1 + k\{(k+1)\epsilon' + \epsilon)\right\} = (m+q)$$

$$1 + k\{(k+1)\epsilon' + \epsilon)\} = \left(\frac{m+q}{m}\right)$$

$$\cancel{m}(1 + k\{(k+1)\epsilon' + \epsilon\} < \cancel{m}q$$

$$1 + k\{(k+1)\epsilon' + \epsilon\} < q$$

$$\because q = k\left\{n(1+\epsilon)\epsilon' + m(1+\epsilon')\epsilon\right\}$$

$$q = m(1 - \{k\epsilon' + \epsilon)\}$$

$$m(1 - \{k\epsilon' + \epsilon)\} = k\left\{n(1+\epsilon)\epsilon' + m(1+\epsilon')\epsilon\right\}$$

$$m(1 - \{k\epsilon' + \epsilon)\} - mk\left\{(1+\epsilon')\epsilon\right\} = n(1+\epsilon)k\epsilon'$$

$$m\left\{1 - \{(k\epsilon')(1+\epsilon) + (k+1)\epsilon\}\right\} = n(1+\epsilon)k\epsilon'$$

$$m\left\{k + 1 - \epsilon\right\} = n(1+\epsilon)$$

$$m\left\{k + 2 - \{(k\epsilon')(1+\epsilon) + (k+2)\epsilon\}\right\} = n(1+\epsilon)(1+k\epsilon')$$

$$m(1+\epsilon)(1+k\epsilon') = m(1+\epsilon)(1+k\epsilon')$$

$$m\left\{k + 2 + (1+\epsilon)(1+k\epsilon') - \{(k\epsilon')(1+\epsilon) + (k+2)\epsilon\}\right\} = (m+n)(1+\epsilon)(1+k\epsilon')$$

$$\left(\frac{m}{m+n}\right) = \left(\frac{(1+\epsilon)(1+k\epsilon')}{\{k + 2 + (1+\epsilon)(1+k\epsilon') - \{(k\epsilon')(1+\epsilon) + (k+2)\epsilon\}\}}\right)$$

$$\left(\frac{mn}{m+n}\right) = q = \left(\frac{n(1+\epsilon)(1+k\epsilon')}{\{k + 2 + (1+\epsilon)(1+k\epsilon') - \{(k\epsilon')(1+\epsilon) + (k+2)\epsilon\}\}}\right)$$

$$n(1+\epsilon)(1+k\epsilon') < q\left\{k + 2 + (1+\epsilon)(1+k\epsilon')\right\}$$

$$n(1+\epsilon)(1+k\epsilon') < \left(\frac{3}{2}\right)\left\{k + 2 + (1+\epsilon)(1+k\epsilon')\right\}$$

$$q=\frac{\log(c)}{\log(\mathrm{rad}(abc))}=\frac{n(1+\epsilon)}{(k+2)}=\left(\frac{n(1-k\epsilon')}{(k+1)}\right)=\frac{\log(c)}{\log(\mathrm{rad}(abc))}=q$$

$$\left(\frac{(1+\epsilon)}{(k+2)}\right)=\left(\frac{\log(\mathrm{rad}(c)}{\log(\mathrm{rad}(abc))}\right)$$

$$\left(\frac{k(1+\epsilon')}{(k+1)}\right)=\left(\frac{\log(\mathrm{rad}(ab)}{\log(\mathrm{rad}(abc))}\right)$$

$$\left(\frac{(1+\epsilon)}{(k+2)}\right)+\left(\frac{k(1+\epsilon')}{(k+1)}\right)=\left(\frac{\log(\mathrm{rad}(c)}{\log(\mathrm{rad}(abc))}\right)+\left(\frac{\log(\mathrm{rad}(ab)}{\log(\mathrm{rad}(abc))}\right)=1$$

$$\left(\frac{(1+\epsilon)}{(k+2)}\right)+\left(\frac{k(1+\epsilon')}{(k+1)}\right)=\left(\frac{(\log(\mathrm{rad}(abc))}{\log(\mathrm{rad}(abc))}\right)=1$$

$$(k+1)(1+\epsilon)+(k+2)k(1+\epsilon')=(k+1)(k+2)$$

$$(k+1)(\epsilon)+(k+2)k\epsilon'=k^2+3k+2-k-1-k^2-2k$$

$$(k+1)(\epsilon)+(k+2)k\epsilon'=\cancel{k^2}+\cancel{3k}+2-1-\cancel{k^2}-\cancel{3k}$$

$$(k+1)(\epsilon)+(k+2)k\epsilon'=1$$

$$\because q=n(k\epsilon'+\epsilon)$$

$$m=(m+n)(k\epsilon'+\epsilon)$$

$$(m+q)=(m+2n)(k\epsilon'+\epsilon)$$

$$(m+q)(k+2)=(m+2n)(k+2)(k\epsilon'+\epsilon)$$

$$(m+2n)(1+\epsilon)=(m+q)(k+2)$$

$$\because (m+n)(1+\epsilon')\leqslant 2n$$

$$(m+n)(k\epsilon'+\epsilon)=m$$

$$(m+n)(1+(k+1)\epsilon'+\epsilon)\leqslant(m+2n)$$

$$(m+n)(1+\epsilon)(1+(k+1)\epsilon'+\epsilon)\leqslant(m+2n)(1+\epsilon)$$

$$m(k+2)(1+(k+1)\epsilon'+\epsilon)\leqslant(m+2n)(1+\epsilon)$$

$$mn<m(k+2)(1+(k+1)\epsilon'+\epsilon)\leqslant(m+2n)(1+\epsilon)$$

$$mn<m(k+2)(1+(k+1)\epsilon'+\epsilon)\leqslant(m+q)(k+2)$$

$$\left(\frac{n}{k+2}\right)<(1+(k+1)\epsilon'+\epsilon)\leqslant\left(\frac{5}{3}\right)\approx\leqslant\left(\frac{m+q}{m}\right)$$

$$\left(\frac{m+q}{m}\right)^2+\left(\frac{n+q}{n}\right)^2+\frac{2q^2}{mn}=5$$

$$\left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right)-\frac{q^2}{mn}=2$$

$$\left(\frac{m+q}{m}\right)^2+\left(\frac{n+q}{n}\right)^2+\left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right)+\left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right)+\frac{q^2}{mn}=7+\left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right)$$

$$\left(\frac{m+q}{m}+\frac{n+q}{n}\right)^2+\frac{q^2}{mn}=7+\left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right)$$

$$2+\frac{q^2}{mn}=\left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right)<\frac{5}{2}\qquad\because\left(\frac{m+q}{m}+\frac{n+q}{n}\right)=3$$

Derivation 1

$$\because\ a+b=c, \&\ \gcd(a,b)=1 \quad and \quad If\ (\mathrm{rad}(c)<\mathrm{rad}(ab)$$
$$\text{then there exists a positive integer } k\geq 1 \quad such \quad that$$

$$\mathrm{rad}(c)^k < (\mathrm{rad}(ab)) < \mathrm{rad}(c)^{(k+1)}$$
$$(\mathrm{rad}(c)) = \mathrm{rad}(c) = (\mathrm{rad}(c))$$
$$\therefore (\mathrm{rad}(c))^{k+1} < \mathrm{rad}(abc) < (\mathrm{rad}(c))^{k+2}$$
$$\therefore (\mathrm{rad}(c)) < (\mathrm{rad}(abc))^{\left(\frac{1}{k+1}\right)} \leqslant (\mathrm{rad}(c))^{\left(\frac{k+2}{k+1}\right)}$$
$$\therefore (\mathrm{rad}(c)) < (\mathrm{rad}(abc))^{\left(\frac{k+2}{k+1}\right)} \leqslant (\mathrm{rad}(c))^2$$
$$\therefore (\mathrm{rad}(abc)) \leqslant (\mathrm{rad}(c)^{(k+2)} \leqslant (\mathrm{rad}(abc))^2$$
$$(1+\epsilon) = \frac{(k+2)\log \mathrm{rad}(c)}{\log \mathrm{rad}(abc)}$$
$$\therefore q = \left(\frac{n(1+\epsilon)}{(k+2)}\right)$$

$$(\mathrm{rad}(c))^k < \mathrm{rad}(ab) < (\mathrm{rad}(c))^{k+1}$$
$$(\mathrm{rad}(ab))^k = (\mathrm{rad}(ab))^k < (\mathrm{rad}(ab))^{k+1}$$
$$then,\ \ \mathrm{rad}(abc)^k < (\mathrm{rad}(ab))^{(k+1)} < \mathrm{rad}(abc)^{k+1}$$
$$\because (\mathrm{rad}(abc)) < (\mathrm{rad}(ab))^{\left(\frac{(k+1)}{k}\right)} \leqslant (\mathrm{rad}(abc))^2$$
$$\because (\mathrm{rad}(abc)) < (\mathrm{rad}(ab))^{\left(\frac{(k+1)}{k}\right)} \leqslant (\mathrm{rad}(abc))^2$$
$$(1+\epsilon') = \frac{(k+1)\log \mathrm{rad}(ab)}{k\log(\mathrm{rad}(abc)}$$
$$\therefore (\mathrm{rad}(abc))^{(1+\epsilon')} = (\mathrm{rad}(ab))^{\left(\frac{k+1}{k}\right)} \leqslant (\mathrm{rad}(ab))^2$$
$$\therefore q = \left(\frac{n(1-k\epsilon')}{(k+1)}\right) \qquad \because \left(\frac{k+1}{k}\right) \leqslant 2$$

$$q = \frac{\log(c)}{\log(\mathrm{rad}(abc))} \Rightarrow \mathrm{rad}(abc)^q = c \tag{1}$$
$$m = \frac{\log(c)}{\log(\mathrm{rad}(ab))} \Rightarrow \mathrm{rad}(ab)^m = c \tag{2}$$
$$n = \frac{\log(c)}{\log(\mathrm{rad}(c))} \Rightarrow \mathrm{rad}(c)^n = c \tag{3}$$

$$\therefore \left(\frac{m+q}{m}\right) = \left(\frac{\log\{\mathrm{rad}(ab)^2.\,\mathrm{rad}(c)\}}{\log \mathrm{rad(abc)}}\right) \qquad q = \left(\frac{n(1+\epsilon)}{(k+2)}\right) \qquad q = \left(\frac{n(1-k\epsilon'))}{(k+1)}\right) \qquad q = \left(\frac{mn)}{m+n}\right)$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{2q}{m}\right) + \left(\frac{(1+\epsilon)}{(k+2)}\right)$$
$$\left(\frac{n+q}{n}\right) = \left(\frac{2q}{n}\right) + \left(\frac{(k+1-\epsilon)}{(k+2)}\right)$$
$$\left(\frac{m+q}{m}\right)^2 + \left(\frac{n+q}{n}\right)^2 + \frac{2q^2}{mn} = 5$$
$$\left(\frac{m+q}{m}\right) + \left(\frac{n+q}{n}\right) = 3$$

$$q = \left\{\frac{m(k+1-\epsilon)}{(k+2)}\right\} = \left\{\frac{n(1-k\epsilon')}{(k+1)}\right\} = q$$
$$1 = (2+\epsilon)k\epsilon' + \{k+(k+1)\epsilon\}(k\epsilon'+\epsilon)$$
$$m\{2+(k^2-k)\epsilon' + (k-1-(k\epsilon'+\epsilon))\epsilon\} = q(3+\epsilon)$$
$$m\{2+(1-k)\epsilon' - \epsilon(k\epsilon'+\epsilon)\} \leqslant q(3+\epsilon)$$
$$q \approx \{1+(k\epsilon'+\epsilon)+(k\epsilon'+\epsilon)^2\} \therefore q^2 \approx m$$
$$0 \leqslant (k-1)\{(k+1)\epsilon'+\epsilon)\}$$

$$\left(\frac{3}{2}\right) \leq q < \left(\frac{m+q}{m}\right) = \left\{\frac{5+(k-1)k\epsilon'+(k-2)\epsilon}{3}\right\}$$

$$4 = \left\{\frac{mn(1-(k^2\epsilon'+(k-1)\epsilon)^2}{(m-q)(n-q)}\right\}$$

Derivation 2

$$\because \; q(k+1) = m(k+k\epsilon')$$

$$\because \; \frac{q}{m}\left(\frac{k+1}{k}\right) = (1+\epsilon')$$

$$\therefore \; \frac{q}{m} = \frac{(1+\epsilon')}{\left(\frac{k+1}{k}\right)}$$

$$\therefore \; \frac{m+q}{m} = \frac{\left(\frac{k+1}{k}\right)+(1+\epsilon')}{\left(\frac{k+1}{k}\right)}$$

$$\therefore q \leqslant \frac{\left(\frac{k+1}{k}\right)+(1+\epsilon')}{\left(\frac{k+1}{k}\right)}$$

$$\therefore \; \frac{mn}{m+n} \leqslant \frac{\left(\frac{k+1}{k}\right)+(1+\epsilon')}{\left(\frac{k+1}{k}\right)}$$

$$\therefore \; m\left(\frac{n\left(\frac{k+1}{k}\right)}{m+n}\right) \leqslant \left(\frac{k+1}{k}\right)+(1+\epsilon')$$

$$\because (1+\epsilon') = \left(\frac{n\left(\frac{k+1}{k}\right)}{m+n}\right) = \left(\frac{k+1}{k}\right)\left(\frac{\log(\mathrm{rad}(ab))}{\log(\mathrm{rad}(abc))}\right)$$

$$\therefore m(1+\epsilon') \leqslant \left(\frac{k+1}{k}\right)+(1+\epsilon')$$

$$m(1+\epsilon') < 3+\epsilon'$$

$$m(1+\epsilon') \leqslant 2q$$

$$2q \leqslant 3+\epsilon'$$

$$q \leqslant \frac{3+\epsilon'}{2}$$

$$\frac{m+2n}{m+n} = \left(\frac{m+q}{m}\right) = \frac{\{4k+[3k\epsilon'+\epsilon]\}}{2k+1}$$

$$q = \frac{n\{1+[(k-1)k\epsilon'+k\epsilon]\}}{2k+1} \leqslant \left(\frac{m+q}{m}\right) = \frac{\{4k+[3k\epsilon'+\epsilon]\}}{2k+1}$$

$$\frac{n\{1+[(k-1)k\epsilon'+k\epsilon]\}}{2k+1} \leqslant \frac{\{4k+[3k\epsilon'+\epsilon]\}}{2k+1}$$

$$n\{1+[(k-1)k\epsilon'+k\epsilon]\} \leqslant \{4k+[3k\epsilon'+\epsilon]\}$$

$$n < \{4k+[3k\epsilon'+\epsilon]\}$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{3}{2}+\frac{(k^2\epsilon'+(k-1)\epsilon)}{2}\right)$$

$qk = m\{k - 1 + 2k\epsilon' + \epsilon)\}$ $q(k+1) = m\{k + k\epsilon')$ $q(2k+1) = m\{2k - 1 + [3k\epsilon' + \epsilon]\}$	$qk = n\{1 - (2k\epsilon' + \epsilon)\}$ $q(k+1) = n(1 - k\epsilon')$ $q(2k+1) = n\{2 - [3k\epsilon' + \epsilon]\}$

$$m\{(2k-1) + (3k\epsilon' + \epsilon)\} = n\{2 - (3k\epsilon' + \epsilon)\}$$

$$m\{2 - (3k\epsilon' + \epsilon)\} = m\{2 - (3k\epsilon' + \epsilon)\}$$

$$m(2k+1) = (m+n)\{2 - (3k\epsilon' + \epsilon)\}$$

$$m(3k+4) = (m+n)\{3 + \epsilon - 2k\epsilon')\}$$

$$5q(k+1) = n(4 + (k^2 - 3k)\epsilon' + (k+1)\epsilon)$$

$$\frac{m}{n} = \left\{\frac{2 - [3k\epsilon' + \epsilon]}{2k - 1 + [3k\epsilon' + \epsilon]}\right\}$$

$$\frac{m+n}{n} = \left\{\frac{2k+1}{2k - 1 + [3k\epsilon' + \epsilon]}\right\}$$

$$\frac{n}{m+n} = \left\{\frac{2k - 1 + [3k\epsilon' + \epsilon]}{2k+1}\right\}$$

$$\frac{m+2n}{m+n} = \left\{\frac{4k + [3k\epsilon' + \epsilon]}{2k+1}\right\}$$

$$\left(\frac{m+2n}{m+n}\right) = \left(\frac{m+q}{m}\right) \leqslant \left\{\frac{4k + 1 - \epsilon]}{2k+1}\right\}$$

$$(m+q)(2k+1) \leqslant m(4k + 1 - \epsilon)$$

$$q(2k+1) \leqslant m(4k + 1 - 2k - 1 - \epsilon)$$

$$q(2k+1) \leqslant m(2k - \epsilon)$$

$$2kq \leqslant m(2k - 1 + k\epsilon' + \epsilon - \epsilon) \qquad \because q = m(1 - (k\epsilon' + \epsilon))$$

$$2kq \leqslant m(2k - 1 + k\epsilon')$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{2 + \epsilon - \{k\epsilon' + \epsilon(k\epsilon' + \epsilon)\}}{(1+\epsilon)}\right) \leqslant \left\{\frac{4k - 1 + k\epsilon'}{2k}\right\}$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{2 + \epsilon - \{k\epsilon' + \epsilon(k\epsilon' + \epsilon)\}}{(1+\epsilon)}\right) \leqslant \left\{\frac{4k - 1 + k\epsilon'}{2k}\right\}$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{2 + \epsilon - \{k\epsilon' + \epsilon(k\epsilon' + \epsilon)\}}{(1+\epsilon)}\right) \leqslant \left\{\frac{4k + 1 - \epsilon}{2k+1}\right\}$$

$$\left(\frac{m+q}{m}\right)^2 \approx \left(\frac{(4k+1)(4k-1)}{2k(2k+1)}\right)$$

$$\left(\frac{m+q}{m}\right)^2 \leqslant \left(\frac{(4k)^2}{2k(2k+1)}\right) \implies \left(\frac{m+q}{m}\right) \approx q \leqslant \left(\sqrt{\frac{8}{3}}\right)$$

4 Formulae

$$q(2k+1) \leqslant m(2k-\epsilon)$$

$$2kq \leqslant m(2k-1+k\epsilon'+\epsilon-\epsilon) \qquad \because q = m(1-(k\epsilon'+\epsilon))$$

$$2kq \leqslant m(2k-1+k\epsilon')$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{2+\epsilon-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}}{(1+\epsilon)}\right) \leqslant \left\{\frac{4k-1+k\epsilon'}{2k}\right\}$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{2+\epsilon-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}}{(1+\epsilon)}\right) \leqslant \left\{\frac{4k-1+k\epsilon'}{2k}\right\}$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{2+\epsilon-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}}{(1+\epsilon)}\right) \leqslant \left\{\frac{4k+1-\epsilon}{2k+1}\right\}$$

$$\left(\frac{m+q}{m}\right)^2 \approx \left(\frac{(4k+1)(4k-1)}{2k(2k+1)}\right)$$

$$\left(\frac{m+q}{m}\right)^2 \leqslant \left(\frac{(4k)^2}{2k(2k+1)}\right) \implies \left(\frac{m+q}{m}\right) \approx q \leqslant \left(\sqrt{\frac{8}{3}}\right)$$

$$\boxed{\left(\frac{m+q}{m}\right) = \left(\frac{3+(k-\epsilon)(k\epsilon'+\epsilon)}{(2+\epsilon)}\right)}$$

$$\boxed{\left(\frac{n+q}{n}\right) = \left(\frac{3+3\epsilon-(k-\epsilon)(k\epsilon'+\epsilon)}{(2+\epsilon)}\right)}$$

$$\boxed{\left(\frac{m+q}{m}\right) = \left(\frac{3+(k^2\epsilon'+(k-1)\epsilon)}{2}\right)}$$

$$\boxed{\left(\frac{n+q}{n}\right) = \left(\frac{3-(k^2\epsilon'+(k-1)\epsilon)}{2}\right)}$$

$$2q^2 = mn\left\{1-\left\{(k\epsilon'+\epsilon)^2+(k((k+1)\epsilon'+\epsilon))^2\right\}\right\}$$

$$2q^2 = mn\left\{4(k\epsilon'+\epsilon)+(k((k+1)\epsilon'+\epsilon))^2-(1+3(k\epsilon'+\epsilon)^2)\right\}$$

$$2q^2 = mn\left\{(k\epsilon'+\epsilon)^2+(k((k+1)\epsilon'+\epsilon))^2-\left((k^2\epsilon'+(k-1)\epsilon\right)^2\right\}$$

$$4q^2 = mn\left\{1-[k^2\epsilon'+(k-1)\epsilon]^2\right\}$$

$$\boxed{\left(\frac{m+q}{m}\right) = \left(\frac{3}{2}\right)+\left(\frac{k^2\epsilon'}{2}+\frac{(k-1)\epsilon}{2}\right)} \quad \boxed{\left(\frac{n+q}{n}\right) = \left(\frac{3}{2}\right)-\left(\frac{k^2\epsilon'}{2}+\frac{(k-1)\epsilon}{2}\right)}$$

5 My Identity

$$\boxed{(k+2)k\epsilon' + (k+1)\epsilon = 1}$$

$$\because \{(k^2+2k)\epsilon' + (k+1)\epsilon\} = 1$$

$$q = \frac{mn(1+k\epsilon'+\epsilon)}{2m+n} < \left(\frac{5}{3}\right)$$

$$q = \frac{mn(2-\epsilon)}{3n+m(1-k)} < \left(\frac{5}{3}\right)$$

$$q = \frac{mn(1-\epsilon)}{(2n-mk)} < \left(\frac{5}{3}\right)$$

$$\left(\frac{m+q}{m}\right)^2 + \left(\frac{n+q}{n}\right)^2 + \frac{2q^2}{mn} = 5$$

$$\left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right) - \left(\frac{q^2}{mn}\right) = 2$$

$$q = \frac{n(2+(2k-1)k\epsilon' + 92k+1)\epsilon)}{4k+3} < \left(\frac{5}{3}\right)$$

$$q = n(1+\epsilon) - mk(1+\epsilon')$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{2q}{m}\right) + \left(\frac{(1+\epsilon)}{(k+2)}\right)$$

$$\left(\frac{n+q}{n}\right) = \left(\frac{2q}{n}\right) + \left(\frac{(k+1-\epsilon)}{(k+2)}\right)$$

$$q = k\left\{n(1+\epsilon)(\epsilon') + m(1+\epsilon')(\epsilon)\right\}$$

$$\left(\frac{m+q}{m}\right)^2 = \left(\frac{4q}{m}\right) + (k\epsilon' + \epsilon)^2$$

$$\left(\frac{n+q}{n}\right)^2 = \left(\frac{4q}{n}\right) + (k((k+1)\epsilon' + \epsilon))^2$$

$$q^2 = mn\left\{\frac{3+\{k^2\epsilon' + (k-1)\epsilon\}^2}{4} - \left\{(k\epsilon'+\epsilon)^2 + \left[k((k+1)\epsilon'+\epsilon)\right]^2\right\}\right\}$$

$$q^2 = mn\left\{(k\epsilon'+\epsilon) - (k\epsilon'+\epsilon)^2\right\}$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log(\mathrm{rad}(ab)}{\log(\mathrm{rad}(abc))}\right)$$

$$(1+\epsilon) = \left(\frac{k+2}{1}\right)\left(\frac{\log(\mathrm{rad}(c)}{\log(\mathrm{rad}(abc))}\right)$$

$$\left(\frac{m+q}{m}\right)^2 + \left(\frac{n+q}{n}\right)^2 + \frac{2q^2}{mn} = 5$$

$$\frac{2q^2}{mn} = 1 - \left\{(k\epsilon'+\epsilon)^2 + (k((k+1)\epsilon'+\epsilon))^2\right\}$$

$$\left(\frac{n+q}{n}\right) = \left(\frac{3}{2}\right) - \left(\frac{k^2\epsilon'}{2} + \frac{(k-1)\epsilon}{2}\right)$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{3}{2}\right) + \left(\frac{k^2\epsilon'}{2} + \frac{(k-1)\epsilon}{2}\right)$$

$$\left\{\left(\frac{m+q}{m}\right)\left(\frac{n+q}{n}\right)\right\} = \left(\frac{3}{2}\right)^2 - \left(\frac{k^2\epsilon'}{2} + \frac{(k-1)\epsilon}{2}\right)^2$$

$$1 = \left(\frac{4q^2}{mn}\right) + [k^2\epsilon' + (k-1)\epsilon]^2$$

Some more Formulae

$$\left(\frac{m+q}{m}\right)\left(\frac{n+q}{n}\right) = \left(\frac{3}{2}\right)^2 - \left(\frac{k^2\epsilon'}{2} + \frac{(k-1)\epsilon}{2}\right)^2$$

$$q\{1 + (k\epsilon' + \epsilon) + (k\epsilon' + \epsilon)^2\} \leqslant m$$

$$q \approx \{1 + (k\epsilon' + \epsilon) + (k\epsilon' + \epsilon)^2\} \therefore q^2 \approx m$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{3 + (k-\epsilon)(k\epsilon' + \epsilon)}{(2+\epsilon)}\right)$$

$$(2k+1)[k\epsilon' + \epsilon] + 2k\epsilon' = \frac{m+2n}{(m+n)} = \left(\frac{m+q}{m}\right)$$

$$\left(\frac{m+q}{m}\right) + \left(\frac{2\epsilon}{3}\right) = (2k+1)[k\epsilon' + \epsilon] + 2k\epsilon' + \left(\frac{2\epsilon}{3}\right)$$

$$\left(\frac{m+q}{m}\right) + \left(\frac{2\epsilon}{3}\right) = \frac{4}{3} + \left\{\frac{(2k+1)\{k\epsilon' + \epsilon\}}{3}\right\}$$

$$m(1+\epsilon') \leqslant 2q$$
$$m(1+\epsilon') \leqslant 2n(k\epsilon' + \epsilon)$$
$$m(1+\epsilon') \leqslant n(1-\epsilon')$$
$$mk(1+\epsilon') = n(1-k\epsilon')$$
$$2m(k+2)(1+\epsilon') \leqslant n(2 + (k-1)\epsilon' + 2\epsilon)$$
$$m(k+2)(2 - (k\epsilon' + \epsilon)) = (m+q)(k+2)$$
$$mn < m(k+2)(2 - (k\epsilon' + \epsilon)) = (m+q)(k+2)$$
$$mn < n(2 + (k-1)\epsilon' + 2\epsilon)$$
$$m < (2 + (k-1)\epsilon' + 2\epsilon)$$
$$n < (k+2)(2 - (k\epsilon' + \epsilon))$$

$$q = \left(\frac{n(1-k\epsilon'))}{(k+1)}\right)$$
$$mq(k+1) = mn - mnk\epsilon'$$
$$mn < mq(k+1) + mnk\epsilon' + mnk\epsilon$$
$$mn < mq(k+1) + mn(k\epsilon' + \epsilon)$$
$$mn < mq(k+1) + mq$$
$$mn < mq(k+2)$$
$$mn \leqslant (m+q)(k+2)$$
$$\left(\frac{n}{k+2}\right) \approx q \leqslant \left(\frac{m+q}{m}\right)$$
$$\left(\frac{n}{k+2}\right) \leqslant \left(\frac{3}{2}\right) \approx q < \left(\frac{5}{3}\right) \leqslant \left(\frac{m+q}{m}\right)$$

$$q(k+1) = n(1-k\epsilon')$$
$$mq(k+1) = mn(1-k\epsilon')$$
$$mn < mq(k+1) + mnk\epsilon'$$
$$mn < mq(k+1) + mnk\epsilon' + mn\epsilon$$
$$mn < mq(k+1) + mq$$
$$mn < mq(k+2)$$
$$mn \leqslant (m+q)(k+2)$$
$$\left(\frac{n}{k+2}\right) \leqslant \left(\frac{3}{2}\right) \approx q < \left(\frac{5}{3}\right) \leqslant \left(\frac{m+q}{m}\right)$$

$$q(k+2) = n(1+\epsilon))$$
$$mq(k+2) = mn(1+\epsilon)$$
$$mn \leqslant mq(k+2)$$
$$mq \approx (m+q)$$
$$mn(1+\epsilon) \leqslant (m+q)(k+2)$$
$$\left(\frac{n}{k+2}\right) \leqslant \left(\frac{3}{2}\right) \approx q < \left(\frac{5}{3}\right) \leqslant \left(\frac{m+q}{m}\right)$$

$$9q^2 = mn\left\{2 + (2k+1)k\epsilon' + (2k-1)\epsilon - \{(2k+1)k\epsilon' + (2k-1)\epsilon\}^2\right\}$$
$$4q^2 = mn\left\{1 - \left\{k^2\epsilon' + (k-1)\epsilon\right\}^2\right\}$$

$$\because\ q = mk\{(k+1)\epsilon' + \epsilon\} \qquad and \qquad q(k+2) = m(k+1-\epsilon)$$

$$\therefore\ \frac{q}{m} = k\{(k+1)\epsilon' + \epsilon\} \qquad and \qquad \frac{q}{m} = \frac{(k+1-\epsilon)}{(k+2)}$$

$$\therefore (k+2)k\{(k+1)\epsilon' + \epsilon\} = (k+1-\epsilon)$$

$$\therefore (k+1)\epsilon' + \epsilon + \frac{\epsilon}{(k+2)k} = \frac{(k+1)}{(k+2)k}$$

$$\therefore (k+1)\epsilon' + \left(\frac{(k^2+2k+1)\epsilon}{k(k+2)}\right) = \left(\frac{(k+1)}{k(k+2)}\right)$$

$$\therefore (k+1)\epsilon' + \left(\frac{(k+1)^2\epsilon}{k(k+2)}\right) \leqslant \left(\frac{2}{(k+2)}\right) \leqslant \left(\frac{2}{3}\right)$$

$$\therefore \left(\frac{(k+1)}{k}\right)\left\{k\epsilon' + \left(\frac{k+1}{k+2}\right)\epsilon\right\} \leqslant \left(\frac{2}{3}\right)$$

$$\therefore 1 + \left(\frac{(k+1)}{k}\right)\left\{k\epsilon' + \left(\frac{k+1}{k+2}\right)\epsilon\right\} \leqslant \left(\frac{5}{3}\right)$$

$$\left(\frac{m+q}{m}\right)^2 + \left(\frac{n+q}{n}\right)^2 + \frac{2q^2}{mn} = 5$$

$$\left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right) - \frac{q^2}{mn} = 2$$

$$\left(\frac{m+q}{m}\right)^2 + \left(\frac{n+q}{n}\right)^2 + \left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right) + \left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right) + \frac{q^2}{mn} = 7 + \left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right)$$

$$\left(\frac{m+q}{m} + \frac{n+q}{n}\right)^2 + \frac{q^2}{mn} = 7 + \left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right)$$

$$2 + \frac{q^2}{mn} = \left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right) < \frac{5}{2}$$

$$\left(\frac{m+q}{m}\right)\left(\frac{n+q}{n}\right) = \left(\frac{3}{2}\right)^2 - \left(\frac{k^2\epsilon'}{2} + \frac{(k-1)\epsilon}{2}\right)^2$$

$$\left(\frac{m+q}{m}\right) + \left(\frac{n+q}{n}\right) = 3$$

$$\because n(1-k\epsilon') = q(k+1)$$

$$\because n(1+\epsilon) = m(k+1-\epsilon)$$

$$\because n(2-k\epsilon'+\epsilon) < (m+q)(k+1)$$

$$\because n\{1-(k^2\epsilon' + (k-1)\epsilon)\} < (m+q) \qquad \because n\{1-(k^2\epsilon' + (k-1)\epsilon)\} = 2q$$

$$\therefore n(3-k\{(k+1)\epsilon' + \epsilon)\} < (m+q)(k+2)$$

$$\therefore mn \leqslant n(3-k\{(k+1)\epsilon' + \epsilon)\} < (m+q)(k+2)$$

$$\because n\left\{k + k\epsilon' + (k+1)\epsilon\right\} = q(k+1)^2$$

$$nk < q(k+1)^2$$

$$n < q(k+1)\left(\frac{k+1}{k}\right)$$

$$n < 2\left(\frac{k+1}{k}\right)(k+1)$$

$$n < 2\left(\frac{3}{2}\right)(k+1)$$

$$n < 3(k+1)$$

$$n(1+\epsilon) = q(k+2)$$

$$\because q(k+2)(k+3) = n(1+\epsilon)(k+3)$$

$$\therefore q(k^2+5k+6) = n(k+(1+\epsilon)$$

$$n < q\left\{\left(\frac{(k^2+5k+6)}{k+3}\right)\right\}$$

$$n \approx 4.5 \leqslant \left(\frac{36}{8}\right) \text{ when k=1}$$

$$\because n\left\{k+1+k\epsilon' + (k+2)\epsilon\right\} = q(k^2+3k+3)$$

$$n < q\left\{\left(\frac{(k^2+3k+3)}{k+1}\right)\right\}$$

$$\therefore n\left\{k+1+k\epsilon' + (k+2)\epsilon\right\} = q(k^2+3k+3)$$

$$n < \left(\frac{3}{2}\right)\left\{\left(\frac{k^2+3k+3}{k+1}\right)\right\}$$

$$n < \left(\frac{3}{2}\right)\left\{\left(\frac{7}{2}\right)\right\}$$

$$n < 5 \leqslant \left(\frac{21}{4}\right) \text{ when k=1}$$

$$\because n(1-k\epsilon') = q(k+1)$$

$$\because n(1+\epsilon) = m(k+1-\epsilon)$$

$$\therefore n(2-k\epsilon'+\epsilon) < (m+q)(k+1)$$

$$\because n\{1-(k^2\epsilon'+(k-1)\epsilon)\} < (m+q) \qquad \because n\{1-(k^2\epsilon'+(k-1)\epsilon)\} = 2q$$

$$\therefore n(3-k\{(k+1)\epsilon'+\epsilon)\} < (m+q)(k+2)$$

$$\therefore mn < n(3-k\{(k+1)\epsilon'+\epsilon)\} < (m+q)(k+2)$$

$$\mathbf{3q = m\left\{2 + (k-1)k\epsilon' + (k-2)\epsilon\right\}}$$

$$\because \ q(k+1) = m(k + (k\epsilon)$$

$$\because \ 2q = m\{1 + [k^2\epsilon' + (k-1)\epsilon]\}$$

$$q(k+3) = m\{k + 1 + [k^2\epsilon' + (k-1)\epsilon]\}$$

$$q(k+2) = m(k + 1 - \epsilon)$$

$$q(k+1) = m(k + (k\epsilon)$$

$$q(3k+6) = m\{3k + 2 + [(k^2+2k)\epsilon' + (k-2)\epsilon]\}$$

$$3q(k+2) = 3m\left\{k + \frac{2}{3} + \left[\frac{(k^2+2k)\epsilon'}{3} + \frac{(k-2)\epsilon}{3}\right]\right\}$$

$$q(k+2) = m\left\{k + \frac{2}{3} + \left[\frac{(k^2+2k)\epsilon'}{3} + \frac{(k-2)\epsilon}{3}\right]\right\}$$

$$qk = m(k - 1 + 2k\epsilon' + \epsilon$$

$$q(k-k+2) = m\left\{k - k + 1 + \frac{2}{3} + \left[\frac{(k^2+2k-6k)\epsilon'}{3} + \frac{(k-2-3)\epsilon}{3}\right]\right\}$$

$$2q = m\left\{\frac{5}{3} + \left[\frac{(k^2-4k)\epsilon'}{3} + \frac{(k-5)\epsilon}{3}\right]\right\}$$

$$\because \ 2q = m\{1 + [k^2\epsilon' + (k-1)\epsilon]\}$$

$$4q = m\left\{\frac{8}{3} + \left[\frac{(4k^2-4k)\epsilon'}{3} + \frac{(4k-8)\epsilon}{3}\right]\right\}$$

$$q = m\left\{\frac{2}{3} + \left[\frac{(k^2-k)\epsilon'}{3} + \frac{(k-2)\epsilon}{3}\right]\right\}$$

$$q = m\left\{\frac{2}{3} + (k-1)\left\{\frac{k\epsilon' + \frac{(k-2)}{(k-1)}\epsilon}{3}\right\}\right\}$$

$$3q = m\left\{2 + (k-1)\left\{k\epsilon' + \frac{(k-2)\epsilon}{(k-1)}\right\}\right\}$$

$$4q = m\{3 + [(k-2)k\epsilon' + (k-3)\epsilon]\}$$

$$\mathbf{3q = m\left\{2 + (k-1)k\epsilon' + (k-2)\epsilon\right\}}$$

$$q < \left(\frac{5}{3}\right) \leqslant (2k+1)\{(k+1)\epsilon' + \epsilon\} = \left(\frac{m+2n}{m+n}\right) + \epsilon' = \left(\frac{m+q}{m} + \epsilon'\right)$$

$$\therefore \left(\frac{m+q}{m}\right) = 1 + k((k+1)\epsilon' + \epsilon)$$

$$\therefore \left(\frac{n+q}{n}\right) = 1 + k\epsilon' + \epsilon$$

$$\left(\frac{m+q}{m}\right) - \left(\frac{n+q}{n}\right) = 1 + k((k+1)\epsilon' + \epsilon) - \{1 + k\epsilon' + \epsilon\}$$

$$\therefore \left(\frac{n-m}{m+n}\right) = \left\{\left(\frac{m+q}{m}\right) - \left(\frac{n+q}{n}\right)\right\} = \left(\frac{k-2\epsilon}{k+2}\right) = \left(k^2\epsilon' + (k-1)\epsilon\right)$$

$$\because \left(\frac{n-m}{m+n}\right) = \left(k^2\epsilon' + (k-1)\epsilon\right) \implies \lim_{k\to\infty}\left(\frac{n-m}{m+n}\right) = 1$$

6 Derivation 3

$$\because \left(\frac{n+q}{n}\right) \leqslant \left\{\frac{4}{3}\right\} + \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\} \leqslant \left\{\frac{5}{3}\right\} - \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\} \leqslant \left(\frac{m+q}{m}\right)$$

$$\because \left(\frac{n+q}{n}\right) \leqslant \left\{\frac{4}{3}\right\} + \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\}$$

$$\therefore 1 + k\epsilon' + \epsilon \leqslant \left\{\frac{4}{3}\right\} + \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\}$$

$$\therefore 3 + 3(k\epsilon' + \epsilon) \leqslant 4 + (k-1)\epsilon' + \epsilon)$$

$$(3k - k + 1)\epsilon' + (3-1)\epsilon) \leqslant 1$$

$$(2k+1)\epsilon' + 2\epsilon \leqslant 1$$

$$\left\{\frac{5}{3}\right\} - \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\} \leqslant \left(\frac{m+q}{m}\right)$$

$$\left\{\frac{5}{3}\right\} - \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\} \leqslant 2 - (k\epsilon' + \epsilon)$$

$$5 - ((k-1)\epsilon' + \epsilon) \leqslant 6 - 3(k\epsilon' + \epsilon)$$

$$(3k - k + 1)\epsilon' + (3-1)\epsilon) \leqslant 1$$

$$(2k+1)\epsilon' + 2\epsilon \leqslant 1$$

$$(2k+1)\epsilon' + 2\epsilon \leqslant 1$$

$$(k+1)\epsilon' + \epsilon \leqslant \left\{\frac{2}{3}\right\}$$

$$3\left\{\left(\frac{3k+2)}{3k}\right)k\epsilon' + \epsilon\right\} \leqslant \left\{\frac{5}{3}\right\}$$

$$\left\{\left(\frac{3k+2)}{3k}\right)k\epsilon' + \epsilon\right\} \leqslant \left\{\frac{5}{9}\right\}$$

$$1 + \left\{\left(\frac{3k+2)}{3k}\right)k\epsilon' + \epsilon\right\} \leqslant \left\{\frac{14}{9}\right\}$$

$$\left(\frac{n+q}{n}\right) \leqslant 1 + \left\{\left(\frac{3k+2)}{3k}\right)k\epsilon' + \epsilon\right\} \leqslant\approx q \leqslant \left\{\frac{14}{9}\right\} < \left(\frac{5}{3}\right)$$

$$\mathbf{m^2(1+\epsilon') < 2(m+q)(1+\epsilon)}$$

$$\because q = \frac{n(1+\epsilon)}{(k+2)} = \frac{mk(1+\epsilon')}{(k+1)}$$

$$\therefore mk(k+2)(1+\epsilon') = n(k+1)(1+\epsilon)$$

$$\therefore m(k+2)(1+\epsilon') = n\left(\frac{(k+1)}{k}\right)(1+\epsilon)$$

$$\therefore m(k+2)(1+\epsilon') \leqslant n2(1+\epsilon)$$

$$\therefore m^2(k+2)(1+\epsilon') \leqslant mn2(1+\epsilon)$$

$$\left(\frac{m^2(1+\epsilon')(k+2)}{2(1+\epsilon)}\right) \leqslant mn < (m+q)(k+2)$$

$$m^2(1+\epsilon')\cancel{(k+2)} < (m+q)\cancel{(k+2)}2(1+\epsilon)$$

$$\mathbf{m^2(1+\epsilon') < 2(m+q)(1+\epsilon)}$$

$$\mathbf{m^2(1+\epsilon') < 2m(1+\epsilon)}$$

$$\mathbf{m} \leqslant \left(\frac{\mathbf{2(1+\epsilon)}}{\mathbf{(1+\epsilon')}}\right) \implies m \leqslant 2(1+\epsilon)$$

$$\boxed{q(1+\epsilon) = m\{1 - \{k\epsilon' + \epsilon(k\epsilon'+\epsilon)\}\}}$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{2+\epsilon-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}}{(1+\epsilon)}\right)$$

$$\left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right) = \left(\frac{3}{2}\right)^2 - \left(\frac{k^2\epsilon'+(k-1)\epsilon}{2}\right)^2$$

$$\therefore \left\{\frac{5-\{(k-1)\epsilon'+\epsilon\}}{3}\right\} \leqslant \left(\frac{m+q}{m}\right) \leqslant \left\{\frac{\{2+\epsilon-k\epsilon'\}}{(1+\epsilon)}\right\}$$

$$\left(\frac{m+q}{m}\right) = \frac{5}{3} + \left\{\frac{(k^2-3k)\epsilon'+(k-2)\epsilon}{3}\right\}$$

$$\left\{\frac{5+(k-2)(k\epsilon'+\epsilon)}{3}\right\} \leqslant \left(\frac{m+q}{m}\right)$$

$$n < q(k+2) \approx \frac{(m+q)(k+2)}{m} \leqslant (k+2)\left\{\frac{5+(k-2)(k\epsilon'+\epsilon)}{3}\right\}$$

OR $\because n(1+\epsilon) = q(k+2)$

$$n \approx (k+2)(1+\epsilon)$$

$$m \leqslant 2q$$

$$mn \approx 2q(k+2)(1+\epsilon)$$

$$mn \leqslant (q+q)(1+\epsilon))(k+2)$$

$$mn < (m+q)(k+2) \because (q+q) \leqslant (m+q)$$

$$\because \left(\frac{m+q}{m}\right)(k+3+\epsilon) = \left(\frac{n+q}{n}\right)(2k+3-\epsilon)$$

$$(k+3+\epsilon) < (2k+3-\epsilon)$$

$$\left(\frac{n+q}{n}\right) < \left(\frac{m+q}{m}\right)$$

$$\{1+(k\epsilon'+\epsilon)\} < \{2-(k\epsilon'+\epsilon)\}$$

$$\{2(k\epsilon'+\epsilon)\} < 1$$

$$\because \{(k+1)\epsilon'+\epsilon\} < \frac{2}{3}$$

$$\therefore \{(3k+1)\epsilon'+3\epsilon\} < \frac{5}{3}$$

$$\therefore 3\left\{\left(\frac{(3k+1)}{3}\right)\epsilon'+\epsilon\right\} < \frac{5}{3}$$

$$\therefore 3\left\{\left(\frac{(3k+1)}{3k}\right)k\epsilon'+\epsilon\right\} < \frac{5}{3}$$

$$\therefore (k\epsilon'+\epsilon) < \left\{\left(\frac{(3k+1)}{3k}\right)k\epsilon'+\epsilon\right\} < \frac{5}{9}$$

$$\therefore 1+\left\{\left(\frac{(3k+1)}{3k}\right)k\epsilon'+\epsilon\right\} < \frac{14}{9}$$

$$(1+\epsilon) < \{1+(k\epsilon'+\epsilon)\} < \frac{3}{2} \approx q$$

$$q \approx 1+\left\{\left(\frac{(3k+1)}{3k}\right)k\epsilon'+\epsilon)\right\} \leqslant \frac{14}{9} < \frac{5}{3}$$

$$\because n\{k+k\epsilon'+(k+1)\epsilon\} = q(k+1)^2$$

$$nk < q(k+1)^2$$

$$n < q(k+1)\left(\frac{k+1}{k}\right)$$

$$n < 2\left(\frac{k+1}{k}\right)(k+1)$$

$$n < 2\left(\frac{3}{2}\right)(k+1)$$

$$n < 3(k+1)$$

$$n(1+\epsilon) = q(k+2)$$

$$\because q(k+2)(k+3) = n(1+\epsilon)(k+3)$$

$$\therefore q(k^2+5k+6) = n(k+(1+\epsilon)$$

$$n < q\left\{\left(\frac{(k^2+5k+6)}{k+3}\right)\right\}$$

$$n \approx 4.5 \leqslant \left(\frac{36}{8}\right) \text{ when k=1}$$

$$\because n\{k+1+k\epsilon'+(k+2)\epsilon\} = q(k^2+3k+3)$$

$$n < q\left\{\left(\frac{(k^2+3k+3)}{k+1}\right)\right\}$$

$$\therefore n\{k+1+k\epsilon'+(k+2)\epsilon\} = q(k^2+3k+3)$$

$$n < \left(\frac{3}{2}\right)\left\{\left(\frac{k^2+3k+3}{k+1}\right)\right\}$$

$$n < \left(\frac{3}{2}\right)\left\{\left(\frac{7}{2}\right)\right\}$$

$$n < 5 \leqslant \left(\frac{21}{4}\right) \text{ when k=1}$$

$$n < q(k+2) \approx \frac{(m+q)(k+2)}{m} \leqslant (k+2)\left\{\frac{5+(k-2)(k\epsilon'+\epsilon)}{3}\right\}$$

$$m \cdot n < mq(k+2) \approx (m+q)(k+2) \leqslant m(k+2)\left\{\frac{5+(k-2)(k\epsilon'+\epsilon)}{3}\right\}$$

$$\left(\frac{n}{k+2}\right) \leqslant \left\{\frac{5+(k-2)(k\epsilon'+\epsilon)}{3}\right\}$$

$$\because\ q = mk((k+1)\epsilon' + \epsilon)$$
$$2q = m\{1 + [k^2\epsilon + (k-1)\epsilon]\}$$
$$3q = m\{1 + [(2k^2 + k)\epsilon' + (2k-1)\epsilon]\}$$

$$q = n(1 - k\{(k+1)\epsilon' + \epsilon\})$$
$$2q = n\{1 - [k^2\epsilon' + (k-1)\epsilon]\}$$
$$3q = n\left\{2 - (2k-1)\left\{\left(\frac{(2k+1)}{(2k-1)}\right)k\epsilon' + \epsilon\right\}\right\}$$

$$4q^2 = mn\left\{1 - \left\{k^2\epsilon' + (k-1)\epsilon\right\}^2\right\}$$
$$4q^2 = mn\left\{1 - \{k(k\epsilon' + \epsilon) - \epsilon\}^2\right\}$$
$$3q = m\{1 + [(2k^2 + k)\epsilon' + (2k-1)\epsilon]\} \qquad 3q = n\{2 - [(2k^2 + k)\epsilon' + (2k-1)\epsilon]\}$$
$$3q = m\left\{1 + (2k-1)\left\{\left(\frac{(2k+1)}{(2k-1)}\right)k\epsilon' + \epsilon\right\}\right\}$$
$$3q = n\left\{2 - (2k-1)\left\{\left(\frac{(2k+1)}{(2k-1)}\right)k\epsilon' + \epsilon\right\}\right\}$$
$$9q^2 = mn\left\{2 + (2k-1)\left\{\left(\frac{(2k+1)}{(2k-1)}\right)k\epsilon' + \epsilon\right\} - \left\{(2k-1)^2\left\{\left(\frac{(2k+1)}{(2k-1)}\right)k\epsilon' + \epsilon\right\}^2\right\}\right\}$$
$$9q^2 = mn\left\{2 + (2k+1)k\epsilon' + (2k-1)\epsilon - \{(2k+1)k\epsilon' + (2k-1)\epsilon\}^2\right\}$$
$$4q^2 = mn\left\{1 - \{k(k\epsilon' + \epsilon) - \epsilon\}^2\right\}$$
$$13q^2 = mn\left\{3 + (2k+1)k\epsilon' + (2k-1)\epsilon - \{(2k+1)k\epsilon' + (2k-1)\epsilon\}^2 - \{k(k\epsilon' + \epsilon) - \epsilon\}^2\right\}$$

$$m\{2 - \{(k-1)\epsilon' + \epsilon\}\} \leqslant n\{1 + (k-1)\epsilon' + \epsilon\}$$
$$m\{1 + \epsilon'\} \leqslant n\{1 - \epsilon'\}$$
$$m\{3 - \{(k-2)\epsilon' + \epsilon\}\} \leqslant n\{2 + (k-2)\epsilon' + \epsilon\}$$
$$\frac{m}{n} \leqslant \frac{\{2 + (k-2)\epsilon' + \epsilon\}}{\{3 - \{(k-2)\epsilon' + \epsilon\}\}}$$
$$\frac{m+n}{n} \leqslant \frac{5}{\{3 - \{(k-2)\epsilon' + \epsilon\}\}}$$
$$\frac{\{3 - \{(k-2)\epsilon' + \epsilon\}\}}{5} \leqslant \frac{n}{m+n}$$
$$\frac{\{8 - \{(k-2)\epsilon' + \epsilon\}\}}{5} \leqslant \frac{m+2n}{m+n} = \frac{m+q}{m}$$

$$\because (2k+1)\epsilon' + 2\epsilon \leqslant 1$$
$$\because (k+1)\epsilon' + \epsilon \leqslant \left\{\frac{2}{3}\right\}$$
$$3\left\{\left(\frac{(3k+2)}{3k}\right)k\epsilon' + \epsilon\right\} \leqslant \left\{\frac{5}{3}\right\}$$
$$\left\{\left(\frac{(3k+2)}{3k}\right)k\epsilon' + \epsilon\right\} \leqslant \left\{\frac{5}{9}\right\}$$
$$1 + \left\{\left(\frac{(3k+2)}{3k}\right)k\epsilon' + \epsilon\right\} \leqslant \left\{\frac{14}{9}\right\}$$
$$\left(\frac{n+q}{n}\right) \leqslant 1 + \left\{\left(\frac{(3k+2)}{3k}\right)k\epsilon' + \epsilon\right\} \leqslant q \approx \left\{\frac{14}{9}\right\} < \left(\frac{5}{3}\right)$$
$$\therefore 1 + k\epsilon' + \epsilon \leqslant \left(\frac{3}{2}\right) \approx q \approx \left\{\frac{14}{9}\right\} < \left(\frac{5}{3}\right)$$

$qk = m\{k-1+2k\epsilon'+\epsilon)\}$ $q(k+1) = m\{k+k\epsilon')$ $q(2k+1) = m\{2k-1+[3k\epsilon'+\epsilon]\}$	$qk = n\{1-(2k\epsilon'+\epsilon)\}$ $q(k+1) = n(1-k\epsilon')$ $q(2k+1) = n\{2-[3k\epsilon'+\epsilon]\}$

$$1+k\epsilon'+\epsilon \leqslant 1+k^2\epsilon'+k\epsilon = (2k+1)\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon'+\epsilon)\right\}$$

$$\because (k^2+2k)\epsilon'+(k+1)\epsilon = 1$$

$$k^2\epsilon'+k\epsilon = k^2\epsilon'+k\epsilon$$

$$\left\{(2k^2+2k)\epsilon'+(2k+1)\epsilon)\right\} = 1+k^2\epsilon'+k\epsilon$$

$$1+k^2\epsilon'+k\epsilon = \left\{(2k^2+2k)\epsilon'+(2k+1)\epsilon)\right\}$$

$$1+k^2\epsilon'+k\epsilon = (2k+1)\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon'+\epsilon)\right\}$$

$$1+k\epsilon'+\epsilon \leqslant 1+k^2\epsilon'+k\epsilon = (2k+1)\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon'+\epsilon)\right\}$$

$$1+k\epsilon'+\epsilon \approx 1+(k^2-k)\epsilon'+k\epsilon \leqslant 1+k^2\epsilon'+k\epsilon = (2k+1)\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon'+\epsilon)\right\}$$

$$\frac{n(1+k\epsilon'+\epsilon)}{(2k+1)} \approx \frac{n(1+(k^2-k)\epsilon'+k\epsilon)}{(2k+1)} = q \leqslant \left\{\frac{n(1+k^2\epsilon'+k\epsilon)}{(2k+1)}\right\} = n\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon'+\epsilon)\right\} < \left(\frac{5}{3}\right)$$

$$\left\{\frac{n(1+k\epsilon'+\epsilon)}{(2k+1)}\right\} \approx \frac{n(1+(k^2-k)\epsilon'+k\epsilon)}{(2k+1)} = q \leqslant \frac{n(1+k^2\epsilon'+k\epsilon)}{(2k+1)} = n\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon'+\epsilon)\right\} < \left(\frac{5}{3}\right)$$

$$\because q = \frac{n(1+\epsilon)}{(k+2)} = \frac{mk(1+\epsilon')}{(k+1)}$$

$$\therefore mk(k+2)(1+\epsilon') = n(k+1)(1+\epsilon)$$

$$\therefore (1+\epsilon') = \frac{m(k+1)(k+1-\epsilon)}{mk(k+2)}$$

$$\therefore (1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{(k+1-\epsilon)}{k+2}\right)$$

$$\therefore (1+\epsilon') \leqslant 2\left(\frac{(k+1-\epsilon)}{k+2}\right)$$

$$\therefore q \leqslant 1+(k+1)\epsilon'+\epsilon \leqslant \left(\frac{4}{3}\right)+k\epsilon'+\epsilon \leqslant \left(\frac{5}{3}\right)$$

$$\mathbf{q} \leqslant \left(\frac{\sqrt{2}+\sqrt{6}}{\sqrt{6}}\right) \leqslant \left(\frac{5}{3}\right) \leqslant \left(\frac{\sqrt{13}+\sqrt{6}}{\sqrt{13}}\right) \leqslant \left(\frac{\mathbf{m}+\mathbf{q}}{\mathbf{m}}\right)$$

$$\mathbf{k} \leqslant \mathbf{2} + (\mathbf{k}+\mathbf{2})[\mathbf{k}^2\epsilon' + (\mathbf{k}-\mathbf{1})\epsilon]^2 - \mathbf{4k}\epsilon'$$

To Prove: $k \leqslant 2 + (k+2)[k^2\epsilon' + (k-1)\epsilon]^2 - 4k\epsilon'$

$$To \quad Prove : k \leqslant 2 + (k+2)[k^2\epsilon' + (k-1)\epsilon]^2 - 4k\epsilon'$$

$$Proof: \quad \because 4q^2 = mn\{1 - [k^2\epsilon' + (k-1)\epsilon]^2\}$$

$$\therefore \; 4q = (m+n)\{1 - [k^2\epsilon' + (k-1)\epsilon]^2\}$$

$$\therefore \; 4q(1+\epsilon) = (m+n)(1+\epsilon)\{1 - [k^2\epsilon' + (k-1)\epsilon]^2\}$$

$$\therefore \; 4q(1+\epsilon) = m(k+2)\{1 - [k^2\epsilon' + (k-1)\epsilon]^2\}$$

$$\left(\frac{m+q}{m}\right) = \left\{\frac{(k+2+4+4\epsilon-(k+2)[k^2\epsilon' + (k-1)\epsilon]^2}{4(1+\epsilon)}\right\}$$

$$\left(\frac{m+q}{m}\right) = \left\{\frac{(k+6+4\epsilon-(k+2)[k^2\epsilon' + (k-1)\epsilon]^2}{4(1+\epsilon)}\right\} \leqslant \left\{\frac{2+\epsilon-k\epsilon'}{(1+\epsilon)}\right\}$$

$$\left(\frac{m+q}{m}\right) = \left\{\frac{(k+6+4\epsilon-(k+2)[k^2\epsilon' + (k-1)\epsilon]^2}{4\cancel{(1+\epsilon)}}\right\} \leqslant \left\{\frac{2+\epsilon-k\epsilon'}{\cancel{(1+\epsilon)}}\right\}$$

$$(k+6+4\epsilon-(k+2)[k^2\epsilon' + (k-1)\epsilon]^2 \leqslant 8+4\epsilon-4k\epsilon'$$

$$(k+6+\cancel{4\epsilon}-(k+2)[k^2\epsilon' + (k-1)\epsilon]^2 \leqslant 8+\cancel{4\epsilon}-4k\epsilon'$$

$$k \leqslant 2 + (k+2)[k^2\epsilon' + (k-1)\epsilon]^2 - 4k\epsilon'$$

$$\because (k+1)(\epsilon) + (k+2)k\epsilon' = 1$$

$$\therefore (k+1)\left\{\left(\frac{(k+2)}{(k+1)}\right)k\epsilon' + \epsilon\right\} = 1$$

$$\therefore q = n(k\epsilon' + \epsilon) < n\left\{\left(\frac{(k+2)}{(k+1)}\right)k\epsilon' + \epsilon\right\} = \left(\frac{n}{(k+1)}\right)$$

$$(m+q)(n+q) = 2mn + q^2$$

$$(m+q)(n+q) < 2mn + \frac{mn}{k+2}$$

$$(m+q)(n+q) < mn(2 + \frac{1}{3})$$

$$\left(\frac{m+q}{m}\right) \cdot \left(\frac{n+q}{n}\right) < \frac{7}{3}$$

$$2\left(\frac{m+q}{m}\right) \cdot \left(\frac{n+q}{n}\right) < \frac{14}{3}$$

$$\left\{\left(\frac{m+q}{m}\right) + \left(\frac{n+q}{n}\right)\right\}^2 = 3^2$$

$$\left(\frac{m+q}{m}\right)^2 + 2\left(\frac{m+q}{m}\right) \cdot \left(\frac{n+q}{n}\right) + \left(\frac{n+q}{n}\right)^2 = 3^2$$

$$9 - \left\{\left(\frac{m+q}{m}\right)^2 + \left(\frac{n+q}{n}\right)^2\right\} = 2\left(\frac{m+q}{m}\right) \cdot \left(\frac{n+q}{n}\right) < \frac{14}{3}$$

$$9 - \frac{14}{3} < \left(\frac{m+q}{m}\right)^2 + \left(\frac{n+q}{n}\right)^2$$

$$\frac{13}{3} < (2 - (k\epsilon' + \epsilon)^2 + (1 + (k\epsilon' + \epsilon)^2$$

$$\frac{13}{3} < 4 - 4(k\epsilon' + \epsilon) + (k\epsilon' + \epsilon)^2 + 1 + 2(k\epsilon' + \epsilon) + (k\epsilon' + \epsilon)^2$$

$$\frac{13}{3} < 5 - 2(k\epsilon' + \epsilon) + 2(k\epsilon' + \epsilon)^2$$

$$2(k\epsilon' + \epsilon) - 2(k\epsilon' + \epsilon)^2 < 5 - \frac{13}{3}$$

$$1 + 2(k\epsilon' + \epsilon) - 2(k\epsilon' + \epsilon)^2 < \frac{5}{3}$$

$$\mathbf{q} = \left(\frac{\mathbf{n}\{\mathbf{2} + (\mathbf{2k} - \mathbf{1})\mathbf{k}\epsilon' + (\mathbf{2k} + \mathbf{1})\epsilon\}}{(\mathbf{4k} + \mathbf{3})}\right)$$

$$\mathbf{q} = \left(\frac{\mathbf{n}\{\mathbf{2} + (\mathbf{2k} - \mathbf{1})\mathbf{k}\epsilon' + (\mathbf{2k} + \mathbf{1})\epsilon\}}{(\mathbf{4k} + \mathbf{3})}\right)$$

$$\mathbf{q} = \left(\frac{\mathbf{n}\{\mathbf{3} - (\mathbf{5k}\epsilon' + \mathbf{2}\epsilon)\}}{(\mathbf{3k} + \mathbf{1})}\right)$$

$$\mathbf{2mn + (m+q)(\epsilon) = 2mq\left(k + 2 - \frac{\epsilon}{2}\right)}$$

$$\mathbf{2mn + (m+q)(\epsilon) = 2mq\left(k + 2 - \frac{\epsilon}{2}\right)}$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{2k+3-\epsilon}{k+2}\right) = \left(\frac{q(2k+3-\epsilon)}{q(k+2)}\right)$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{q(2k+3-\epsilon)}{n(1+\epsilon)}\right)$$

$$n(1+\epsilon)(m+q) = mq(2k+3-\epsilon)$$

$$m(1+\epsilon)(m+q) = m(1+\epsilon)(m+q)$$

$$(m+n)(1+\epsilon)(m+q) = m\{q(2k+3-\epsilon) + (1+\epsilon)(m+q)\}$$

$$m(k+2)(m+q) = m\{q(2k+4) + m(1+\epsilon)\}$$

$$(k+2)(m+q) = \{q(2k+4) + m(1+\epsilon)\}$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{2q}{m}\right) + \left(\frac{(1+\epsilon)}{(k+2)}\right)$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{2q}{m}\right) + \left(\frac{(1+\epsilon)}{(k+2)}\right) \leq \left(\frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}\right)$$

$$m(1+\epsilon)(m+q) = m(1+\epsilon)(m+q)$$

$$(m+n)(1+\epsilon)(m+q) = m\{q(2k+3-\epsilon) + (1+\epsilon)(m+q)\}$$

$$n(1+\epsilon)(m+q) = mq(2k+3-\epsilon)$$

$$mn + nq + mq + (\epsilon)(m+q) = mq + mq(2k+3-\epsilon)$$

$$2mn + (\epsilon)(m+q) = 2mq(k + 2 + \frac{\epsilon}{2})$$

$$\mathbf{2mn \leqslant 2(m+q)\left(k + 2 - \frac{3\epsilon}{2}\right)}$$

$$\mathbf{n(1+\epsilon)(m+q) = mq(2k+3-\epsilon)}$$

$$n(1+\epsilon) = q(k+2)$$

$$2n(1+\epsilon) = 2q(k+2)$$

$$2n(1+\epsilon) = q(2k+3-\epsilon+1+\epsilon)$$

$$2n(1+\epsilon) - q(1+\epsilon) = q(2k+3-\epsilon)$$

$$2n(1+\epsilon) - n(k\epsilon'+\epsilon)(1+\epsilon) = q(2k+3-\epsilon)$$

$$n(1+\epsilon)(2-(k\epsilon'+\epsilon) = q(2k+3-\epsilon)$$

$$n(1+\epsilon)\left(\frac{m+q}{m}\right) = q(2k+3-\epsilon)$$

$$\mathbf{n(1+\epsilon)(m+q) = mq(2k+3-\epsilon)}$$

Derivation 4

$$\therefore \mathbf{1} + \left(\frac{\mathbf{k+1}}{\mathbf{k}}\right)\left\{\mathbf{k}\epsilon' + \left(\frac{\mathbf{k+1}}{\mathbf{k+2}}\right)\epsilon\right\} \leqslant \left(\frac{\mathbf{5}}{\mathbf{3}}\right)$$

$$\because \; q = mk\{(k+1)\epsilon' + \epsilon\} \qquad and \qquad q(k+2) = m(k+1-\epsilon)$$

$$\therefore \; \frac{q}{m} = k\{(k+1)\epsilon' + \epsilon\} \qquad and \qquad \frac{q}{m} = \frac{(k+1-\epsilon)}{(k+2)}$$

$$\therefore (k+2)k\{(k+1)\epsilon' + \epsilon\} = (k+1-\epsilon)$$

$$\therefore (k+1)\epsilon' + \epsilon + \frac{\epsilon}{(k+2)k} = \frac{(k+1)}{(k+2)k}$$

$$\therefore (k+1)\epsilon' + \left(1 + \frac{1}{(k+2)k}\right)\epsilon = \frac{(k+1)}{(k+2)k}$$

$$\therefore (k+1)\epsilon' + \left(\frac{(k^2+2k+1)\epsilon}{k(k+2)}\right) = \left(\frac{(k+1)}{k(k+2)}\right)$$

$$\therefore (k+1)\epsilon' + \left(\frac{(k+1)^2\epsilon}{k(k+2)}\right) \leqslant \left(\frac{2}{(k+2)}\right) \leqslant \left(\frac{2}{3}\right) \qquad \because \left(\frac{(k+1)}{k}\right) \leqslant 2$$

$$\therefore \left(\frac{k+1}{k}\right)\left\{k\epsilon' + \left(\frac{k+1}{k+2}\right)\epsilon\right\} \leqslant \left(\frac{2}{3}\right)$$

$$\therefore 1 + \left(\frac{k+1}{k}\right)\left\{k\epsilon' + \left(\frac{k+1}{k+2}\right)\epsilon\right\} \leqslant \left(\frac{5}{3}\right)$$

$$\therefore (k+2)\left\{1 + \left(\frac{k+1}{k}\right)\left\{k\epsilon' + \left(\frac{k+1}{k+2}\right)\epsilon\right\}\right\} \leqslant (k+2)\left(\frac{5}{3}\right) \qquad \because n < q(k+2)$$

$$\therefore q \approx 1 + \left(\frac{k+1}{k}\right)\left\{k\epsilon' + \left(\frac{k+1}{k+2}\right)\epsilon\right\} \leqslant q \leqslant \left(\frac{5}{3}\right)$$

$$\boxed{q = n(1+\epsilon) - mk(1+\epsilon')}$$

$$\boxed{q = k\{n(1+\epsilon)(\epsilon') + m(1+\epsilon')(\epsilon)\}}$$

Greatest lower bound for q

$$\because m + m\{k\epsilon' + \epsilon)\} = 2m - q$$
$$m\{1 + k\epsilon' + \epsilon)\} \approx 3q - q$$
$$m\{1 + k\epsilon' + \epsilon)\} \approx 2q$$
$$\left(\frac{\{3 + k\epsilon' + \epsilon)\}}{2}\right) \approx \left(\frac{m+q}{m}\right)$$
$$\frac{3}{2} < \left(\frac{\{3 + k\epsilon' + \epsilon)\}}{2}\right) \approx q \leqslant \frac{5}{3} \leqslant \left(\frac{m+q}{m}\right)$$

$$\because m + m\{k\epsilon' + \epsilon)\} = 2m - q$$
$$m\{1 + k\epsilon' + \epsilon)\} \approx 3q - q$$
$$m\{1 + k\epsilon' + \epsilon)\} \approx 2q$$
$$\because mk\{k\epsilon' + \epsilon\} < q$$
$$\left(\frac{4 + (k^2+k)\epsilon' + (k+1)\epsilon)}{3}\right) \approx \left(\frac{m+q}{m}\right)$$
$$\left(\frac{4 + (k^2+k)\epsilon' + (k+1)\epsilon)}{3}\right) \approx q \leqslant \frac{5}{3} \leqslant \left(\frac{m+q}{m}\right)$$

Connection with π

$$\sum_{k=1}^{n} x_k = 1.666666666.... + 0.875000000000005..... + 0.5999999999999....... = 3.141666666666.....$$

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n}\left(\frac{(2+\epsilon-k\epsilon')}{k(1+\epsilon)}\right) = \pi$$

$$\sum_{k=1}^{n}\left(\frac{(2+\epsilon-k\epsilon')}{k(1+\epsilon)}\right) = \pi = \sum_{k=1}^{n} = \left(\frac{(2+\epsilon-k\epsilon')}{k(1+\epsilon)}\right) = \pi = x_1 + x_2 + \cdots + x_{n-1} + x_n$$

$x_1 = \left(\frac{(2+\epsilon-\epsilon')}{(1+\epsilon)}\right) = 1.666666666......$**always for any/all abc triple with k=1 only**

$x_2 = \left(\frac{(2+\epsilon-2\epsilon')}{2(1+\epsilon)}\right) = 0.875000000000005......$**always for any/all abc triple with k=2 only**

$x_3 = \left(\frac{(2+\epsilon-3\epsilon')}{3(1+\epsilon)}\right) = 0.5999999999999......$**always for any/all abc triple with k=3 only**

$$\pi = x_1 + x_2 + x_3 = \left(\frac{(2+\epsilon-\epsilon')}{(1+\epsilon)}\right) + \left(\frac{(2+\epsilon-2\epsilon')}{2(1+\epsilon)}\right) + \left(\frac{(2+\epsilon-3\epsilon')}{3(1+\epsilon)}\right) = \pi$$

$$\sum_{k=1}^{n} x_k = 1.666666666....+0.875000000000005.....+0.5999999999999....... = 3.141666666666..... = \pi$$

It is not a mere coincidence, mathematics will be developed to that extent when it will be established that π is really connected to natural numbers

Connection with Golden Ratio ϕ

$$\because m\{1-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}=q(1+\epsilon)$$

$$\therefore m\{1-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}\leqslant q^2$$

$$\therefore \frac{q+1-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}}{q}\leqslant\left(\frac{m+q}{m}\right)$$

$$\left(\frac{m+q}{m}\right)=\frac{2+\epsilon-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}}{(1+\epsilon)}$$

$$\therefore \frac{q+1-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}}{q}\approx q\approx\frac{(m+q)}{m}$$

$$\therefore\ q+1-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}\approx q^2$$

$$\therefore\ q^2-q-1+\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}\approx 0$$

$$\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}\approx 0$$

$$\therefore\ q^2-q-1\approx 0$$

$$q\approx\phi=\frac{1+\sqrt{5}}{2}$$

$$q<\left(\frac{5}{3}\right)\leqslant(2k+1)\{(k+1)\epsilon'+\epsilon\}=\frac{m+2n}{(m+n)}+\epsilon'=\left(\frac{m+q}{m}+\epsilon'\right)$$

$$\left(\frac{5}{3}\right)-\left\{\frac{(\epsilon\cdot(k\epsilon'+\epsilon))}{(1+\epsilon)}\right\}\leqslant\left(\frac{m+q}{m}\right)=\left\{\frac{2+\epsilon-\{k\epsilon'+\epsilon\cdot(k\epsilon'+\epsilon)\}}{(1+\epsilon)}\right\}$$

$$\left(\frac{5}{3}\right)-\left\{\frac{(\epsilon\cdot(k\epsilon'+\epsilon))}{(1+\epsilon)}\right\}\leqslant\left(\frac{m+q}{m}\right)=(2k+1)[k\epsilon'+\epsilon]+2k\epsilon'$$

$$\therefore q\leqslant\left(\frac{5}{3}\right)\leqslant\left(\frac{m+q}{m}\right)+\left\{\frac{\epsilon(k\epsilon'+\epsilon)}{(1+\epsilon)}\right\}=(2k+1)[k\epsilon'+\epsilon]+2k\epsilon'+\left\{\frac{\epsilon(k\epsilon'+\epsilon)}{1+\epsilon}\right\}$$

$$q<\left(\frac{5}{3}\right)\leqslant(2k+1)\{(k+1)\epsilon'+\epsilon\}=\frac{m+2n}{(m+n)}+\epsilon'=\left(\frac{m+q}{m}+\epsilon'\right)$$

$$\because q=k\left\{n(1+\epsilon)\epsilon'+m(1+\epsilon')\epsilon\right\}=\left(\frac{n(1-k\epsilon')}{(k+1)}\right)=q$$

$$\therefore nk(k+1)(1+\epsilon)\epsilon'+mk(k+1)(1+\epsilon')\epsilon=n(1-k\epsilon')$$

$$\therefore mk(k+1)(1+\epsilon')\epsilon=n(1-k\epsilon')-n(k+1)(1+\epsilon)k\epsilon'$$

$$\therefore mk(k+1)(1+\epsilon')\epsilon=n\left\{1-k\epsilon'\left\{1+(k+1)(1+\epsilon)\right\}\right\}$$

$$\because mk(1+\epsilon')=n\left\{1-k\epsilon'\right\}$$

$$\therefore mk(k+1)(1+\epsilon')\epsilon=n\left\{1-k\epsilon'\left\{1+(k+1)(1+\epsilon)\right\}\right\}$$

$$\therefore mk(1+\epsilon')\left\{1-(k+1)\epsilon\right\}=n\left\{k\epsilon'\left\{(k+1)(1+\epsilon)\right\}\right\}$$

$$\therefore m(1+\epsilon')\left\{1-(k+1)\epsilon\right\}=n\epsilon'(k+1)(1+\epsilon)$$

$$\left(\frac{m+q}{m}\right)=\left(\frac{2+\epsilon-k\epsilon'-\epsilon(k\epsilon'+\epsilon)}{(1+\epsilon)}\right)$$

$$q=m\{1-(k\epsilon'+\epsilon\})$$

$$q(\epsilon)=m(\epsilon)\{1-(k\epsilon'+\epsilon\})$$

$$q(1+\epsilon)=m\{1-(k\epsilon'+\epsilon)+(\epsilon)\{1-(k\epsilon'+\epsilon)$$

$$q(1+\epsilon)=m\{1-k\epsilon'-\epsilon)+(\epsilon)-\epsilon.(k\epsilon'+\epsilon)$$

$$q(1+\epsilon)=m\{1-k\epsilon'+\cancel{(-\epsilon)}+\cancel{(\epsilon)}-\epsilon.(k\epsilon'+\epsilon)$$

$$q(1+\epsilon)=m\{1-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}\}$$

$$\left(\frac{m+q}{m}\right)=\left(\frac{(1+\epsilon)+\{1-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}}{(1+\epsilon)}\right)$$

$$\left(\frac{m+q}{m}\right)=\left(\frac{(2+\epsilon-k\epsilon'-\epsilon(k\epsilon'+\epsilon)\}}{(1+\epsilon)}\right)$$

$$q(k+2)=m(k+1-\epsilon)$$

$$q(k+3+\epsilon)=m\{k+2-\{(k\epsilon'+\epsilon)+\epsilon(k\epsilon'+\epsilon)\}\}$$

$$q(k+3+\epsilon)=m\{k+2-\{(k\epsilon'+\epsilon)(1+\epsilon)\}\}$$

$$\left(\frac{m+q}{m}\right)=\left(\frac{\{2k+5-\{(k\epsilon'+\epsilon)(1+\epsilon)\}\}}{k+3+\epsilon}\right)=\left(\frac{(2k+3-\epsilon)}{k+2}\right)$$

$$\implies(\epsilon)^2=0$$

$$\left(\frac{m+q}{m}\right)=\left(\frac{(1+\epsilon)+\{1-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}}{(1+\epsilon)}\right)$$

$$\left(\frac{m+q}{m}\right)=\left(\frac{(2+\epsilon-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}}{(1+\epsilon)}\right)$$

$$\left\{\frac{3+k\epsilon'+\left(\frac{(k-1)\epsilon}{(k+1)}\right)}{2}\right\}\leqslant\left(\frac{m+q}{m}\right)=\left\{\frac{5+(k-1)k\epsilon'+(k-2)\epsilon}{3}\right\}=\left\{\frac{5+(k-1)\left\{k\epsilon'+\left(\frac{(k-2)\epsilon}{(k-1)}\right)\right\}}{3}\right\}$$

$$\left(\frac{3}{2}\right)\leq\boxed{q\leqslant\left(\frac{m+q}{m}\right)=\left\{\frac{5+(k-1)k\epsilon'+(k-2)\epsilon}{3}\right\}\approx\left\{\frac{5+(k-2)(k\epsilon'+\epsilon)}{3}\right\}}$$

$$\left(\frac{3}{2}\right)\leq q<\left\{\frac{5+(k-2)(k\epsilon'+\epsilon)}{3}\right\}\leqslant\left(\frac{m+q}{m}\right)$$

$$\left(\frac{3}{2}\right)\leq q<\left(\frac{m+q}{m}\right)=\left\{\frac{5+(k-1)k\epsilon'+(k-2)\epsilon}{3}\right\}$$

$$\because \left(\frac{n+q}{n}\right) \leqslant \left\{\frac{4}{3}\right\} + \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\}$$

$$\therefore 1 + k\epsilon' + \epsilon \leqslant \left\{\frac{4}{3}\right\} + \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\}$$

$$\therefore 3 + 3(k\epsilon' + \epsilon) \leqslant 4 + (k-1)\epsilon' + \epsilon)$$

$$(3k - k + 1)\epsilon' + (3-1)\epsilon) \leqslant 1$$

$$(2k+1)\epsilon' + 2\epsilon \leqslant 1$$

$$\left\{\frac{5}{3}\right\} - \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\} \leqslant \left(\frac{m+q}{m}\right)$$

$$\left\{\frac{5}{3}\right\} - \left\{\frac{(k-1)\epsilon' + \epsilon)}{3}\right\} \leqslant 2 - (k\epsilon' + \epsilon)$$

$$5 - ((k-1)\epsilon' + \epsilon) \leqslant 6 - 3(k\epsilon' + \epsilon)$$

$$(3k - k + 1)\epsilon' + (3-1)\epsilon) \leqslant 1$$

$$(2k+1)\epsilon' + 2\epsilon \leqslant 1$$

$$m(1+\epsilon') = m(k+1)\{(k+1)\epsilon' + \epsilon)\}$$

$$q + m\epsilon = m(k+1)\{k\epsilon' + \epsilon)\}$$

$$m + q + m(\epsilon' + \epsilon) = m(k+1)\{(2k+1)\epsilon' + 2\epsilon)\}$$

$$\left(\frac{m+q}{m}\right) + (\epsilon' + \epsilon) = (k+1)\{(2k+1)\epsilon' + 2\epsilon\}$$

$$n\left\{\left(\frac{m+q}{m}\right) + (\epsilon' + \epsilon)\right\} = 2n(k+1)\left\{\left(\frac{2k+1}{2k}\right)k\epsilon' + \epsilon\right\}$$

$$\left(\frac{n}{2(k+1)}\right)\left\{\left(\frac{m+q}{m}\right) + (\epsilon' + \epsilon)\right\} = n\left\{\left(\frac{2k+1}{2k}\right)k\epsilon' + \epsilon\right\}$$

$$\left(\frac{3}{2}\right) \leq \boxed{\left(\frac{n}{2(k+1)}\right)\left\{\left(\frac{m+q}{m}\right) + (\epsilon' + \epsilon)\right\} = n\left\{\left(\frac{2k+1}{2k}\right)k\epsilon' + \epsilon\right\}}$$

$$\left(\frac{3}{2}\right) \leq q < \left\{\frac{5 + (k-2)(k\epsilon' + \epsilon)}{3}\right\} \leqslant \left(\frac{m+q}{m}\right)$$

$$(m+n)(1+\epsilon) = m(k+2)$$

$$(m+n)(1+\epsilon) < (m + (1+\epsilon))(k+2)$$

$$n(1+\epsilon) = q(k+2)$$

$$\because m(1+\epsilon') \leqslant 2q$$

$$(m+n)(1+\epsilon) \leqslant q(k+4)$$

$$m(k+2) \leqslant q(k+4)$$

$$mn \leqslant q^2(k+4)$$

$$(m+n)q < (m+q)(k+2)$$

$$mn < (m+q)(k+2)$$

$$\left(\frac{n}{k+2}\right) \approx q \leqslant \frac{(m+q)}{m}$$

$$\because m \leqslant 2q$$

$$\because mn \leqslant 2nq$$

$$\therefore mn \leqslant nq + nq$$

$$\because n(1+\epsilon) = q(k+2)$$

$$nq \approx q(k+2)$$

$$mq < m(k+2)$$

$$mq + nq \leqslant (m+q)(k+2)$$

$$q(m+n) \leqslant (m+q)(k+2)$$

$$mn < (m+q)(k+2)$$

$$\left(\frac{n}{k+2}\right) \approx q \leqslant \frac{(m+q)}{m}$$

Derivation 5

$$1 + k\epsilon' + \epsilon \leqslant 1 + k^2\epsilon' + k\epsilon = (2k+1)\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\}$$

$$\because (k^2 + 2k)\epsilon' + (k+1)\epsilon = 1$$

$$k^2\epsilon' + k\epsilon = k^2\epsilon' + k\epsilon$$

$$\left\{(2k^2 + 2k)\epsilon' + (2k+1)\epsilon)\right\} = 1 + k^2\epsilon' + k\epsilon$$

$$1 + k^2\epsilon' + k\epsilon = \left\{(2k^2 + 2k)\epsilon' + (2k+1)\epsilon)\right\}$$

$$1 + k^2\epsilon' + k\epsilon = (2k+1)\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\}$$

$$1 + k\epsilon' + \epsilon \leqslant 1 + k^2\epsilon' + k\epsilon = (2k+1)\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\}$$

$$1 + k\epsilon' + \epsilon \approx 1 + (k^2 - k)\epsilon' + k\epsilon \leqslant 1 + k^2\epsilon' + k\epsilon = (2k+1)\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\}$$

$$\left(\frac{n(1 + k\epsilon' + \epsilon)}{(2k+1)}\right) \approx \left(\frac{n(1 + (k^2 - k)\epsilon' + k\epsilon)}{(2k+1)}\right) = q \leqslant \left(\frac{n(1 + k^2\epsilon' + k\epsilon)}{(2k+1)}\right) = n\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\}$$

$$\left(\frac{n(1 + k\epsilon' + \epsilon)}{(2k+1)}\right) \approx \left(\frac{n(1 + (k^2 - k)\epsilon' + k\epsilon)}{(2k+1)}\right) = q \leqslant \left(\frac{n(1 + k^2\epsilon' + k\epsilon)}{(2k+1)}\right) = n\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\}$$

$$\left(\frac{n(1 + (k^2 - k)\epsilon' + k\epsilon)}{(2k+1)}\right) = q \leqslant n\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\} < \left(\frac{5}{3}\right)$$

$$q = n((k\epsilon' + \epsilon)$$

$$q(k+2) = n(1 + \epsilon)$$

$$q(k+3) = n(1 + k\epsilon' + 2\epsilon)$$

$$\therefore \left(\frac{mn}{m+n}\right) = \left(\frac{n(1 + k\epsilon' + 2\epsilon)}{k+3}\right) \leqslant \left(\frac{m+q}{m}\right)$$

$$mn(1 + k\epsilon' + 2\epsilon) \leqslant (m+q)(k+3)$$

$$2q(k+1) \leqslant (m+q)(k+1) < mn$$

$$mq(k+3) = mn\left\{1 + k\epsilon' + 2\epsilon\right\}$$

$$mq(k+3) = mn\left\{1 + k\epsilon' + 2\epsilon\right\} < 2mq(k+1) \leqslant 2(m+q)(k+1) < 2mn \leqslant 2(m+q)(k+2)$$

$$\frac{n(1+\epsilon)}{(k+2)} \leqslant 1+k\{(k+1)\epsilon'+\epsilon\}\} = \left(\frac{m+q}{m}\right)$$

$$\frac{n(1+\epsilon)}{(k+2)} = q \leqslant 1+k\{(k+1)\epsilon'+\epsilon\}$$

$$n(1+\epsilon) \leqslant (k+2)\left\{1+k\{(k+1)\epsilon'+\epsilon\}\right\}$$

$$mn(1+\epsilon) \leqslant m(k+2)\left\{1+k\{(k+1)\epsilon'+\epsilon\}\right\}$$

$$mn\cancel{(1+\epsilon)} \leqslant (m+n)\cancel{(1+\epsilon)}\left\{1+k\{(k+1)\epsilon'+\epsilon\}\right\}$$

$$\left(\frac{mn}{m+n}\right) = q \leqslant \left\{1+k\{(k+1)\epsilon'+\epsilon\}\right\}$$

$$\left(\frac{k+3+\epsilon}{k+2}\right) = \{1+k\epsilon'+\epsilon\} = \left(\frac{n+q}{n}\right)$$

$$\because \left(k^2\epsilon'+(k-1)\epsilon\right) = \left(\frac{n-m}{m+n}\right)$$

$$2 = \left(\frac{1+k^2\epsilon'+(k-1)\epsilon}{k\left((k+1)\epsilon'+\epsilon\right)}\right)$$

$$(k+1)\epsilon'+\epsilon = \left(\frac{1+k^2\epsilon'+(k-1)\epsilon}{2k}\right)$$

$$(k+1)\epsilon'+\epsilon = \left(\frac{(2k^2+2k)\epsilon'+(2k-1)\epsilon}{2k}\right)$$

$$(k+1)\epsilon'+\epsilon = \left(\frac{(2k-1)}{2k}\right)\left(\left(\frac{(2k+1)}{2k-1}\right)k\epsilon'+\epsilon\right)$$

$$(k+1)\epsilon'+\epsilon = \left(\left(\frac{(2k+1)}{2k}\right)k\epsilon' + \left(\frac{(2k-1)}{2k}\right)\epsilon\right)$$

$$1+(k+1)\epsilon'+\epsilon < 1+\left(\left(\frac{(2k+1)}{2k}\right)k\epsilon'+\epsilon\right) \leqslant \left(\frac{3}{2}\right) < \left(\frac{5}{3}\right)$$

$$(m+q)\{2k+1+k\epsilon'-\epsilon\} = n(4-4k\epsilon'+(k\epsilon')^2-(\epsilon)^2)$$

$$9q^2 = mn\left\{2+(2k+1)k\epsilon'+(2k-1)\epsilon-\{(2k+1)k\epsilon'+(2k-1)\epsilon\}^2\right\}$$

$$4q^2 = mn\left\{1-\left\{k^2\epsilon'+(k-1)\epsilon\right\}^2\right\}$$

$$q < \left(\frac{5}{3}\right) \leqslant (2k+1)\{(k+1)\epsilon'+\epsilon\} = \left(\frac{m+2n}{m+n}\right)+\epsilon' = \left(\frac{m+q}{m}+\epsilon'\right)$$

Derivation 6

$$\because n\left\{k+k\epsilon'+(k+1)\epsilon\right\}=q(k+1)^2$$
$$nk<q(k+1)^2$$
$$n<q(k+1)\left(\frac{k+1}{k}\right)$$
$$n<2\left(\frac{k+1}{k}\right)(k+1)$$
$$n<2\left(\frac{3}{2}\right)(k+1)$$
$$n<3(k+1)$$
$$n(1+\epsilon)=q(k+2)$$
$$\because q(k+2)(k+3)=n(1+\epsilon)(k+3)$$
$$\therefore q(k^2+5k+6)=n(k+(1+\epsilon)$$
$$n<q\left\{\left(\frac{(k^2+5k+6)}{k+3}\right)\right\}$$
$$n\approx 4.5\leqslant\left(\frac{36}{8}\right)\text{ when k=1}$$
$$\because n\left\{k+1+k\epsilon'+(k+2)\epsilon\right\}=q(k^2+3k+3)$$
$$n<q\left\{\left(\frac{(k^2+3k+3)}{k+1}\right)\right\}$$
$$\therefore n\left\{k+1+k\epsilon'+(k+2)\epsilon\right\}=q(k^2+3k+3)$$
$$n<\left(\frac{3}{2}\right)\left\{\left(\frac{k^2+3k+3}{k+1}\right)\right\}$$
$$n<\left(\frac{3}{2}\right)\left\{\left(\frac{7}{2}\right)\right\}$$
$$n<5\leqslant\left(\frac{21}{4}\right)\text{ when k=1}$$

$$\mathbf{2mn+(m+q)(\epsilon)=2mq\left(k+2-\frac{\epsilon}{2}\right)}$$
$$\left(\frac{m+q}{m}\right)=\left(\frac{2k+3-\epsilon}{k+2}\right)=\left(\frac{q(2k+3-\epsilon)}{q(k+2)}\right)$$
$$\left(\frac{m+q}{m}\right)=\left(\frac{q(2k+3-\epsilon)}{n(1+\epsilon)}\right)$$
$$n(1+\epsilon)(m+q)=mq(2k+3-\epsilon)$$
$$m(1+\epsilon)(m+q)=m(1+\epsilon)(m+q)$$
$$(m+n)(1+\epsilon)(m+q)=m\{q(2k+3-\epsilon)+(1+\epsilon)(m+q)\}$$
$$m(k+2)(m+q)=m\{q(2k+4)+m(1+\epsilon)\}$$
$$(k+2)(m+q)=\{q(2k+4)+m(1+\epsilon)\}$$
$$\left(\frac{m+q}{m}\right)=\left(\frac{2q}{m}\right)+\left(\frac{(1+\epsilon)}{(k+2)}\right)$$
$$\left(\frac{m+q}{m}\right)=\left(\frac{2q}{m}\right)+\left(\frac{(1+\epsilon)}{(k+2)}\right)\leq\left(\frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}\right)$$
$$m(1+\epsilon)(m+q)=m(1+\epsilon)(m+q)$$
$$(m+n)(1+\epsilon)(m+q)=m\{q(2k+3-\epsilon)+(1+\epsilon)(m+q)\}$$
$$n(1+\epsilon)(m+q)=mq(2k+3-\epsilon)$$
$$mn+nq+mq+(\epsilon)(m+q)=mq+mq(2k+3-\epsilon)$$
$$2mn+(\epsilon)(m+q)=2mq\left(k+2+\frac{\epsilon}{2}\right)$$
$$\mathbf{2mn\leqslant 2(m+q)\left(k+2-\frac{3\epsilon}{2}\right)}$$
$$\mathbf{mn<(m+q)(k+2)}$$
$$\left(\frac{n}{k+2}\right)\leqslant\left(\frac{3}{2}\right)\approx q<\left(\frac{5}{3}\right)\leqslant\left(\frac{m+q}{m}\right)$$

7 Least Upper bound for q

$$(1+\epsilon) = \left(\frac{(k+2)\log\operatorname{rad}(c)}{\log(\operatorname{rad}(abc))}\right)$$

$$(1+\epsilon) = (k+2)\{k\epsilon' + \epsilon\}$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\cdot\left(\frac{\log\operatorname{rad}(ab)}{\log(\operatorname{rad}(abc))}\right)$$

$$(1+\epsilon') = (k+1)\{(k+1)\epsilon' + \epsilon\}$$

$$\because (\operatorname{rad}(abc))^{(1+\epsilon)} = (\operatorname{rad}(c))^{(k+2)}$$

$$\because (\operatorname{rad}(c))^{n} = \operatorname{rad}(c)^{n-(k+2)}\cdot(\operatorname{rad}(abc))^{1+\epsilon}$$

$$\textit{if } (\operatorname{rad}(c)^{n-(k+2)} < (\operatorname{rad}(ab)) \textit{ then}$$

abc -Conjecture is proved here itself because

$$(\operatorname{rad}(c)^{n-(k+2)} < \operatorname{rad}(ab) < (\operatorname{rad}(c))^{(k+1)}$$

$$n < (2k+3)$$

therefore we have to assume that

$$(\operatorname{rad}(ab)) < \operatorname{rad}(c)^{n-(k+2)}$$

$$(\operatorname{rad}(ab)) < \operatorname{rad}(c)^{(k+1)} \implies 2(k+1) \approx n \textit{ when } k=1, \therefore n \leqslant 2(k+1)$$

again abc -Conjecture is proved here itself

$$(\operatorname{rad}(ab))^2 < \operatorname{rad}(c)^{(n-1)} < (\operatorname{rad}(ab))^m = \operatorname{rad}(c)^n$$

$$\because (n-1) \approx n \therefore 2 \approx\leqslant m$$

$$(\operatorname{rad}(ab))^2 < \operatorname{rad}(c)^{2(n-(k+2))}$$

$$(\operatorname{rad}(abc))^{1+\epsilon'} < \operatorname{rad}(c)^{2n-2(k+2)} \qquad \because (\operatorname{rad}(abc))^{1+\epsilon'} \leqslant (\operatorname{rad}(ab))^2$$

$$(\operatorname{rad}(abc))^{1+\epsilon'}(\operatorname{rad}(abc))^{2+2\epsilon} < \operatorname{rad}(c)^{2n}$$

$$(\operatorname{rad}(abc))^{3+2\epsilon+\epsilon'} < \operatorname{rad}(c)^{2n}$$

$$(\operatorname{rad}(abc))^{\left(\frac{3+2\epsilon+\epsilon'}{2}\right)} < \operatorname{rad}(c)^n = \operatorname{rad}(abc)^q$$

$$\left(\frac{3+2\epsilon+\epsilon'}{2}\right) \leqslant q$$

$$(m-q)(k+1) = q + m\epsilon$$
$$(m-q)(k+1-\epsilon) = q(1+\epsilon)$$
$$(m-q)\{2(k+1)-\epsilon\} = 2q + (m+q)\epsilon$$
$$(m-q)\{2(k+1)-\epsilon\} < (m+q) + (m+q)\epsilon$$
$$(m-q)\{2(k+1)-\epsilon\} < (m+q)(1+\epsilon)$$
$$(m-q) < \left\{\frac{(m+q)(1+\epsilon)}{\{2(k+1)-\epsilon\}}\right\} < q$$
$$(m-q) < \left\{\frac{(m+q)(1+\epsilon)}{\{2(k+1)-\epsilon\}}\right\} \leqslant (1+\epsilon)$$
$$(m+q)(k+2) < 2(k+1)(k+2)$$
$$m(2k+3-\epsilon) < 2(k+1)(k+2)$$
$$m < \left\{\frac{2(k+1)(k+2)}{(2k+3-\epsilon)}\right\}$$
$$m \leqslant \frac{12}{5} \leqslant \left\{\frac{2(k+1)(k+2)}{(2k+3-\epsilon)}\right\}$$

$$(m-q)(k+1) = q + m\epsilon$$
$$(m-q)(k+1-\epsilon) = q(1+\epsilon)$$
$$(m-q)\{2(k+1)-\epsilon\} = 2q + (m+q)\epsilon$$
$$(m-q)\{2(k+1)-\epsilon\} < (m+q) + (m+q)\epsilon$$
$$(m-q)\{2(k+1)-\epsilon\} < (m+q)(1+\epsilon)$$
$$(m-q) < \left\{\frac{(m+q)(1+\epsilon)}{\{2(k+1)-\epsilon\}}\right\} < q$$
$$(m-q) < \left\{\frac{(m+q)(1+\epsilon)}{\{2(k+1)-\epsilon\}}\right\} \leqslant (1+\epsilon)$$
$$(m+q)(k+2) < 2(k+1)(k+2)$$
$$m(2k+3-\epsilon) < 2(k+1)(k+2)$$
$$m < \left\{\frac{2(k+1)(k+2)}{(2k+3-\epsilon)}\right\}$$
$$m \leqslant \frac{12}{5} \leqslant \left\{\frac{2(k+1)(k+2)}{(2k+3-\epsilon)}\right\}$$

$$\left(\frac{m+q}{m}\right) - \left(\frac{n+q}{n}\right) = \left(k^2\epsilon' + (k-1)\epsilon\right) = \left(\frac{n-m}{m+n}\right)$$
$$(m+q)\cdot(n+q) = 2mn + q^2$$
$$(m+q)\cdot(n+m) = 2mn + m^2$$
$$(m+q)\cdot(n+m) < 2(m+q)(k+2) + 2(m+q)$$
$$(m+q)\cdot(n+m) < 2(m+q)(k+3)$$
$$\cdot(n+m) < 2(k+3)$$
$$\cdot m(k+2) < 2(k+3)$$
$$m < \left\{\frac{2(k+3)}{(k+2)}\right\} \leqslant \frac{8}{3}$$

$$m(1+k\epsilon') \leqslant 2q$$
$$n(1+\epsilon) = q(k+2)$$
$$(m+n)(1+\epsilon) \leqslant q(k+4)$$
$$m(k+2) \leqslant q(k+4) \implies m \leqslant \frac{5}{2}$$
$$m(k+2) \leqslant q(k+4)$$
$$m + q \leqslant \frac{3}{2}\left(\frac{k+4}{k+2}\right) + \frac{3}{2}$$
$$(m+q)(k+2) \leqslant \frac{3}{2}\left(\frac{(2k+6)(k+2)}{k+2}\right)$$
$$mn < 3(k+3)$$

$$\because q = n(k\epsilon' + \epsilon)$$
$$m = (m+n)(k\epsilon' + \epsilon)$$
$$(m+q) = (m+2n)(k\epsilon' + \epsilon)$$
$$(m+q)(k+2) = (m+2n)(k+2)(k\epsilon' + \epsilon)$$
$$(m+2n)(1+\epsilon) = (m+q)(k+2)$$

$$\because (m+n)(1+\epsilon') \leqslant 2n$$
$$(m+n)(k\epsilon' + \epsilon) = m$$
$$(m+n)(1+(k+1)\epsilon' + \epsilon) \leqslant m + 2n$$
$$(m+n)(1+\epsilon)(1+(k+1)\epsilon' + \epsilon) \leqslant (m+2n)(1+\epsilon)$$
$$m(k+2)(1+(k+1)\epsilon' + \epsilon) \leqslant (m+2n)(1+\epsilon)$$

$$mn < m(k+2)(1+(k+1)\epsilon' + \epsilon) \leqslant (m+2n)(1+\epsilon)$$
$$mn < m(k+2)(1+(k+1)\epsilon' + \epsilon) \leqslant (m+q)(k+2)$$
$$\left(\frac{n}{k+2}\right) < (1+(k+1)\epsilon' + \epsilon) \leqslant \left(\frac{5}{3}\right) \approx\leqslant \left(\frac{m+q}{m}\right)$$

$$\because 1 = (2+\epsilon)k\epsilon' + \{k + (k+1)\epsilon\}(k\epsilon' + \epsilon)$$
$$\because (1 - k\epsilon') = (k+1)(k\epsilon' + \epsilon)$$
$$\therefore 2 - (3+\epsilon)k\epsilon' = \{(2k+1) + (k+1)\epsilon\}(k\epsilon' + \epsilon)$$
$$\therefore \left(\frac{n(2-(3+\epsilon)k\epsilon')}{(2k+1)+(k+1)\epsilon}\right) = n(k\epsilon' + \epsilon) = q$$

$$\because (k^2 + 2k)\epsilon' + (k+1)\epsilon = 1$$
$$\therefore (k+1)(k\epsilon' + \epsilon) = 1 - k\epsilon'$$
$$\therefore (k+2)(k\epsilon' + \epsilon) = 1 + \epsilon$$
$$\therefore k(k\epsilon' + \epsilon) = 1 - (2k\epsilon' + \epsilon)$$
$$\therefore (2k+1)(k\epsilon' + \epsilon) = 2 - (3k\epsilon' + \epsilon)$$
$$\therefore (2k+1)(k\epsilon' + \epsilon) = 1 + k\{(k-1)\epsilon' + \epsilon\}$$
$$(k+3)(1 - k\epsilon') = (k+1)(1 + k\epsilon' + 2\epsilon)$$
$$(k+2)(1 - k\epsilon') = (k+1)(1+\epsilon)$$
$$(2k+5)(1 - k\epsilon') = (k+1)(2 + k\epsilon' + 3\epsilon)$$
$$\frac{n(1-k\epsilon')}{(k+1)} = q = \frac{n(2 + k\epsilon' + 3\epsilon)}{(2k+5)}$$

$$q < \left(\frac{5}{3}\right) \leqslant (2k+1)\{(k+1)\epsilon' + \epsilon\} = \left(\frac{m+2n}{m+n}\right) + \epsilon' = \left(\frac{m+q}{m} + \epsilon'\right)$$

Some more Derivations

$$n(1+\epsilon) = q(k+2)$$
$$\because (\text{rad}(abc))^{(1+\epsilon)} = (\text{rad}(c))^{(k+2)} \leqslant (\text{rad}(c))^{(2k+1)}$$
$$\because (\text{rad}(ab))^{(1+\epsilon)} \leqslant (\text{rad}(c))^{(2k-\epsilon)}$$
$$n(1+\epsilon) \leqslant m(2k-\epsilon)$$
$$n(1+\epsilon) = m(k+1-\epsilon)$$
$$2n(1+\epsilon)(1+\epsilon') \leqslant m(1+\epsilon')(3k+1-2\epsilon)$$
$$2n(1+\epsilon)(1+\epsilon') \leqslant 2q(3k+1-2\epsilon)$$
$$n(1+\epsilon)(1+\epsilon') \leqslant q(3k+1)$$
$$n < \frac{3}{2}(3k+1) \because \frac{3}{2} \leqslant q$$

$$(\text{rad}(abc)^q = (\text{rad}(ab)^m$$
$$(\text{rad}(c)^q = (\text{rad}(ab)^{(m-q)}$$
$$(\text{rad}(c)^{(k+1-\epsilon)} = (\text{rad}(ab)^{(1+\epsilon)}$$
$$(\text{rad}(c)^{q+(k+1-\epsilon)} = (\text{rad}(ab)^{m-q+(1+\epsilon)}$$
$$(\text{rad}(c)^{q+(k+1-\epsilon)} \leqslant (\text{rad}(ab)^m$$
$$\therefore mq + m(k+1-\epsilon) \leqslant mn$$
$$\therefore mn \approx (m+q) + m(k+1-\epsilon)$$
$$\therefore mn < (m+q) + (m+q)(k+1-\epsilon)$$
$$\therefore mn < (m+q)(k+2-\epsilon)$$
$$\left\{\frac{n}{(k+2)}\right\} < \left\{\frac{m+q}{(m)}\right\}$$

$$(m-q)(k+1) = q + m\epsilon$$
$$(m-q)(k+1-\epsilon) = q + q\epsilon$$
$$(m-q)\{2(k+1)-\epsilon\} = 2q + (m+q)\epsilon$$
$$(m-q)\{2(k+1)-\epsilon\} < (m+q) + (m+q)\epsilon$$
$$(m-q)\{2(k+1)-\epsilon\} < (m+q)(1+\epsilon)$$
$$(m-q) < \left\{\frac{(m+q)(1+\epsilon)}{\{2(k+1)-\epsilon\}}\right\} < q$$
$$(m-q) < \left\{\frac{(m+q)(1+\epsilon)}{\{2(k+1)-\epsilon\}}\right\} \leqslant (1+\epsilon)$$
$$(m+q)(k+2) < 2(k+1)(k+2)$$
$$m(2k+3-\epsilon) < 2(k+1)(k+2)$$
$$m < \frac{2(k+1)(k+2)}{(2k+3-\epsilon)}$$
$$m \leqslant \frac{12}{5} \leqslant \frac{2(k+1)(k+2)}{(2k+3-\epsilon)}$$

$$m(k-1+(2k\epsilon'+\epsilon)) = n\{1-((2k\epsilon'+\epsilon))$$
$$m(1+\epsilon') \leqslant n(1-(\epsilon'))$$
$$m(k+((2k+1)\epsilon'+\epsilon)) \leqslant n\{2-((2k+1)\epsilon'+\epsilon))$$
$$m\{2-((2k+1)\epsilon'+\epsilon)) = m\{2-((2k+1)\epsilon'+\epsilon))$$
$$m\{(k+2) \leqslant (m+n)\{2-((2k+1)\epsilon'+\epsilon))$$
$$\left(\frac{m(k+2)}{m+n}\right) \leqslant \{2-((2k+1)\epsilon'+\epsilon))$$
$$1+\epsilon \leqslant \{2-((2k+1)\epsilon'+\epsilon))$$
$$(2k+1)\epsilon' + 2\epsilon \leqslant 1$$
$$(k+1)\epsilon' + \epsilon \leqslant \frac{2}{3}$$
$$(3k+2)\epsilon' + 3\epsilon \leqslant \frac{5}{3}$$
$$\left(\left(\frac{(3k+2)}{3k}\right)k\epsilon' + \epsilon\right) \leqslant \frac{5}{9}$$
$$\left(\left(\frac{(4k+1)}{3k}\right)k\epsilon' + \epsilon\right) \leqslant \frac{5}{9}$$
$$\left(\left(\frac{(7k+3)}{3k}\right)k\epsilon' + 2\epsilon\right) \leqslant \frac{10}{9}$$
$$\left(\left(\frac{(7k+3)}{6k}\right)k\epsilon' + \epsilon\right) \leqslant \frac{5}{9}$$
$$1 + \left(\left(\frac{(7k+3)}{6k}\right)k\epsilon' + \epsilon\right) \leqslant \frac{14}{9}$$

$$\because (2k+1)\left\{\left(\frac{(2k+3)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\} = \left(\frac{m+q}{m}\right)$$

$$\therefore 3-(2k+1)\left\{\left(\frac{(2k+3)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\} = \left(\frac{n+q}{n}\right)$$

$$\therefore 3-(2k+1)\left\{\left(\frac{(2k+3)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\} = 1+(k\epsilon'+\epsilon)$$

$$\therefore 2 = (2k+1)\left\{\left(\frac{(2k+3)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\} + (k\epsilon'+\epsilon)$$

$$\therefore \left(\frac{2}{(2k+1)}\right) = \left\{\left(\frac{(2k+3)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\} + \left(\frac{(k\epsilon'+\epsilon)}{(2k+1)}\right)$$

$$\therefore \left\{\left(\frac{(2k+3)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\} + \left(\frac{(k\epsilon'+\epsilon)}{(2k+1)}\right) = \left(\frac{2}{(2k+1)}\right) \leqslant \left(\frac{2}{3}\right)$$

$$\therefore 1+\left\{\left(\frac{(2k+3)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\} + \left(\frac{(k\epsilon'+\epsilon)}{(2k+1)}\right) = 1+\left(\frac{2}{(2k+1)}\right) \leqslant 1+\left(\frac{2}{3}\right)$$

$$\therefore q < 1+\left\{\left(\frac{(2k+3)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\} + \left(\frac{(k\epsilon'+\epsilon)}{(2k+1)}\right) = 1+\left(\frac{2}{(2k+1)}\right) \leqslant \left(\frac{5}{3}\right)$$

$$\because n(1+k\epsilon'+2\epsilon) = q(k+3)$$

$$\therefore (\text{rad}(abc))^{(1+k\epsilon'+2\epsilon)} = (\text{rad}(c))^{(k+3)} \leqslant (\text{rad}(c))^{2(k+1)}$$

$$\therefore (\text{rad}(abc))^{(1+k\epsilon'+2\epsilon)} \leqslant (\text{rad}(c))^{2(k+1)}$$

$$\therefore (\text{rad}(ab))^{(1+k\epsilon'+2\epsilon)} \leqslant (\text{rad}(c))^{(2k+2-(1+k\epsilon'+2\epsilon))}$$

$$\therefore (\text{rad}(ab))^{(1+k\epsilon'+2\epsilon)} \leqslant (\text{rad}(c))^{(2k+1-(k\epsilon'+2\epsilon))}$$

$$n(1+k\epsilon'+2\epsilon) \leqslant m(2k+1-(k\epsilon'+2\epsilon))$$

$$m(1+k\epsilon'+2\epsilon) = m(1+k\epsilon'+2\epsilon)$$

$$(m+n)(1+k\epsilon'+2\epsilon) \leqslant 2m(k+1)$$

$$\left(\frac{n(1+k\epsilon'+2\epsilon)}{2(k+1)}\right) \leqslant \left(\frac{mn}{m+n}\right) = q$$

$$\because m\{1-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\} = q(1+\epsilon)$$

$$\therefore m\{1-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\} \leqslant q^2$$

$$\therefore \frac{q+1-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}}{q} \leqslant \left(\frac{m+q}{m}\right)$$

$$\left(\frac{m+q}{m}\right) = \frac{2+\epsilon-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}}{(1+\epsilon)}$$

$$\therefore \frac{q+1-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\}}{q} \approx q \approx \frac{(m+q)}{m}$$

$$\therefore\ q+1-\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\} \approx q^2$$

$$\therefore\ q^2-q-1+\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\} \approx 0$$

$$\{k\epsilon'+\epsilon(k\epsilon'+\epsilon)\} \approx 0$$

$$\therefore\ q^2-q-1 \approx 0$$

$$q \approx \phi = \frac{1+\sqrt{5}}{2}$$

$$(k+3)(1-k\epsilon') = (k+1)(1+k\epsilon'+2\epsilon)$$

$$(k+2)(1-k\epsilon') = (k+1)(1+\epsilon)$$

$$(2k+5)(1-k\epsilon') = (k+1)(2+k\epsilon'+3\epsilon)$$

$$\left(\frac{n(1-k\epsilon')}{(k+1)}\right) = q = \left(\frac{n(2+k\epsilon'+3\epsilon)}{(2k+5)}\right)$$

$$\because (m-q)(k+2) = m(1+\epsilon)$$
$$\because (m-q)(k+1-\epsilon) = q(1+\epsilon)$$
$$(m-q)\{2k+3)-\epsilon\} = (m+q)(1+\epsilon)$$
$$(m-q)(k+2)\{2k+3)-\epsilon\} = (m+q)(k+2)(1+\epsilon)$$
$$m(1+\epsilon)\left(\frac{2k+3-\epsilon}{k+2}\right) - \epsilon\} = (m+q)(1+\epsilon)$$

$$\because m(1+\epsilon') \leqslant 2q$$
$$q(1+\epsilon') = q(1+\epsilon')$$
$$(m+q)(1+\epsilon'\} = q(3+\epsilon')$$
$$m(\not{q})(1+\epsilon'\} \approx (\not{q})(3+\epsilon')$$
$$m(1+\epsilon'\} \leqslant (3+\epsilon')$$
$$m \leqslant 3$$

$$\therefore (m-q)\{2k+3)-\epsilon\} = (m+q)(1+\epsilon) \approx (m+q)(1+\epsilon') \leqslant q(3+\epsilon')$$
$$\therefore (m-q)\{2k+3)-\epsilon\} \leqslant q(3+\epsilon')$$
$$\therefore (m-q) \leqslant \left(\frac{q(3+\epsilon')}{(2k+3)-\epsilon)}\right)$$
$$\therefore (m-q) \leqslant \left(\frac{3q}{5}\right)$$
$$5m \leqslant 8q$$
$$\because 2m \approx 3q \qquad \because m(k+1-\epsilon) = q(k+2)$$
$$7m \leqslant 11q$$
$$\therefore (q) \leqslant \left(\frac{18}{11}\right) < \left(\frac{5}{3}\right) \leqslant \left(\frac{m+q}{m}\right)$$

$$qk = n(1-(2k\epsilon'+\epsilon))$$
$$q(k+1) = n(1-k\epsilon')$$
$$q(2k+1) = n(2-(3k\epsilon'+\epsilon))$$
$$q = n(k\epsilon'+\epsilon)(k\epsilon'-\epsilon)$$
$$q(k\epsilon'-\epsilon) = n(k\epsilon'+\epsilon)(k\epsilon'-\epsilon)$$
$$q(k\epsilon'-\epsilon) = n((k\epsilon')^2-(\epsilon)^2$$
$$q(2k+1) = n(2-(3k\epsilon'+\epsilon))$$
$$q(2k+1+k\epsilon'-\epsilon) = n(2-3k\epsilon'-\epsilon+(k\epsilon')^2-(\epsilon)^2)$$

$$m(k+k\epsilon') = n(1-k\epsilon')$$
$$m(k+1-\epsilon) = n(1+\epsilon)$$
$$m(2k+1+k\epsilon'-\epsilon) = n(2+\epsilon-k\epsilon')$$
$$q(2k+1+k\epsilon'-\epsilon) = n(2-3k\epsilon'-\epsilon+(k\epsilon')^2-(\epsilon)^2)$$
$$(m+q)\left\{2k+1+k\epsilon'-\epsilon\right\} = n\left\{4-4k\epsilon'+(k\epsilon')^2-(\epsilon)^2\right\}$$
$$(m+q)\left\{2k+1\right\} \approx\leqslant 4n$$
$$\left(\frac{m+q}{2}\right) \approx q \leqslant \left(\frac{2n}{2k+1}\right)$$

$$(m+q)\left\{2k+1+k\epsilon'-\epsilon\right\} = n\left\{4-4k\epsilon'+(k\epsilon')^2-(\epsilon)^2\right\}$$

$$4n \approx (m+q)(2k+1)$$
$$4n \leqslant 4(2k+1)$$

$$\left(\frac{3}{2}\right) \leq q < \left\{\frac{5+(k-2)(k\epsilon'+\epsilon)}{3}\right\} \leqslant \left(\frac{m+q}{m}\right)$$

$$\left(\frac{m+q}{m}\right)\cdot\left(\frac{n+q}{n}\right) = \left(\frac{3}{2}\right)^2 - \left(\frac{k^2\epsilon'+(k-1)\epsilon}{2}\right)^2$$

$$q = n\left\{(k\epsilon' + \epsilon\right\}$$
$$q(k+2) = n\left\{1+\epsilon\right\}$$
$$q(k+3) = n\left\{1+k\epsilon' + 2\epsilon\right\}$$
$$mq(k+3) = mn\left\{1+k\epsilon' + 2\epsilon\right\}$$
$$mn(1+k\epsilon' + 2\epsilon) \leqslant (m+q)(k+3)$$
$$mq(k+3) \leqslant (m+q)(k+3)$$
$$mq \leqslant (m+q))$$

$$q = n((k\epsilon' + \epsilon)$$
$$q(k+2) = n(1+\epsilon)$$
$$q(k+3) = n(1+k\epsilon' + 2\epsilon)$$
$$\therefore \left(\frac{mn}{m+n}\right) = \left(\frac{n(1+k\epsilon'+2\epsilon)}{k+3}\right) \leqslant \left(\frac{m+q}{m}\right)$$
$$mn(1+k\epsilon' + 2\epsilon) \leqslant (m+q)(k+3)$$

$$q(k+1) = n\left\{1-k\epsilon'\right\}$$
$$q(k+2) = n\left\{1+\epsilon\right\}$$
$$q(2k+3) = n\left\{2+\epsilon-k\epsilon'\right\}$$
$$mq(2k+3) = mn\left\{2+\epsilon-k\epsilon'\right\}$$
$$mn\left\{2+\epsilon-k\epsilon'\right\} = mq(2k+3)$$
$$\frac{mn}{mq} = \frac{(2k+3)}{\{2+\epsilon-k\epsilon'\}} < (k+2)$$
$$mn\left\{2+\epsilon-k\epsilon'\right\} = mq(2k+3)$$
$$mn\left\{2+\epsilon-k\epsilon'\right\} = mq(2k+3) \approx (m+q)(2k+3) < 2(m+q)(k+2)$$
$$mn\left\{2+\epsilon-k\epsilon'\right\} < 2(m+q)(k+2)$$
$$mn \leqslant (m+q)(k+2)$$
$$\left(\frac{n}{k+2}\right) \leqslant q < \frac{(m+q)}{m}$$

$$q = n\left\{(k\epsilon' + \epsilon\right\}$$
$$q(k+2) = n\left\{1+\epsilon\right\}$$
$$q(k+3) = n\left\{1+k\epsilon' + 2\epsilon\right\}$$
$$mq(k+3) = mn\left\{1+k\epsilon' + 2\epsilon\right\}$$
$$mn(1+k\epsilon' + 2\epsilon) \leqslant (m+q)(k+3)$$
$$mq(k+3) \leqslant (m+q)(k+3)$$
$$mq \leqslant (m+q))$$
$$(1+k\epsilon' + 2\epsilon) \approx q$$

$$q = n((k\epsilon' + \epsilon)$$
$$q(k+2) = n(1+\epsilon)$$
$$q(k+3) = n(1+k\epsilon' + 2\epsilon)$$
$$\therefore \left(\frac{mn}{m+n}\right) = \left(\frac{n(1+k\epsilon'+2\epsilon)}{k+3}\right) \leqslant \left(\frac{m+q}{m}\right)$$
$$mn(1+k\epsilon' + 2\epsilon) \leqslant (m+q)(k+3)$$
$$mn(1+k\epsilon' + 2\epsilon) \leqslant 3q(k+3)$$
$$(1+k\epsilon' + 2\epsilon) \approx q$$

$$q = \left(\frac{n\{1+k\epsilon'+2\epsilon\}}{(k+3)}\right) = q = \left(\frac{n\{1+(k^2-k)\epsilon'+k\epsilon\}}{(2k+1)}\right)$$

$$mk\{(k+1)\epsilon' + \epsilon\} = n(1 - k(k+1)\epsilon' + \epsilon)$$
$$m(1+\epsilon') \leqslant n(1-\epsilon')$$
$$m\{1+(k^2+k+1)\epsilon'+k\epsilon\} \leqslant n\{2-(k^2+k+1)\epsilon'+k\epsilon\}$$
$$\frac{m+n}{n} \leqslant \frac{3}{1+(k^2+k+1)\epsilon'+\epsilon}$$
$$\frac{1+(k^2+k+1)\epsilon'+\epsilon}{3} \leqslant \frac{n}{m+n}$$
$$\frac{1+(k^2+2k)\epsilon'+(k+1)\epsilon+(1-k)(\epsilon'+\epsilon)}{3} \leqslant \frac{n}{m+n}$$
$$\frac{2+(1-k)(\epsilon'+\epsilon)}{3} \leqslant \frac{n}{m+n}$$
$$\left\{\frac{5+(1-k)(\epsilon'+\epsilon)}{3}\right\} \leqslant \frac{m+2n}{m+n} = \left(\frac{m+q}{m}\right)$$

$$\because 2q = m\{1+\{k^2\epsilon'+(k-1)\epsilon\}\}$$
$$2q = m\{1-\epsilon+k\{k\epsilon'+\epsilon\}\}$$
$$\implies 2q < m\left\{1+k\{k\epsilon'+\epsilon\}\right\}$$
$$\because 2q = n\{1-\{k^2\epsilon'+(k-1)\epsilon\}\}$$
$$2q = n\{1+\epsilon-k\{k\epsilon'+(k)\epsilon\}\}$$
$$n\{1-k\{k\epsilon'+\epsilon\} < 2q < m\left\{1+k\{k\epsilon'+\epsilon\right\}$$
$$n\{1-k\{k\epsilon'+\epsilon\} < m\left\{1+k\{k\epsilon'+\epsilon\}\right\}$$
$$\frac{2}{\{1+k\{k\epsilon'+\epsilon\}} < \frac{n}{m+n}$$
$$\frac{n}{m+n} < \frac{\{1+k\{k\epsilon'+\epsilon\}}{2}$$
$$\frac{m+2n}{m+n} = \left(\frac{m+q}{m}\right) < \left\{\frac{3+k\{k\epsilon'+\epsilon}{2}\right\}$$

$$2q = m\left(1+\left(\frac{k-2\epsilon}{k+2}\right)\right) \quad and \quad 2q = n\left(1-\left(\frac{k-2\epsilon}{k+2}\right)\right)$$
$$4q^2 = mn\left(1-\left(\frac{(k-2\epsilon)}{k+2}\right)^2\right)$$
$$2q = m\{1+(k^2\epsilon'+(k-1)\epsilon)\}\ and\ 2q = n\{1-(k^2\epsilon'+(k-1)\epsilon)\}$$
$$4q^2 = mn\{1-[k^2\epsilon'+(k-1)\epsilon]^2\}$$
$$8q^2 = mn\left\{2-\left[\left(\frac{(k-2\epsilon)}{(k+2)}\right)^2+\left(k^2\epsilon'+(k-1)\epsilon\right)^2\right]\right\}$$

$$(m+n)(1+\epsilon) = m(k+2)$$
$$(m+n)(1+\epsilon) < (m+(1+\epsilon))(k+2)$$
$$(m+n)q < (m+q)(k+2)$$
$$mn < (m+q)(k+2)$$
$$\left(\frac{n}{k+2}\right) \approx q \leqslant \left(\frac{m+q}{m}\right)$$
$$\left(\frac{m+q}{m}\right) = \left\{\frac{5+(k-1)k\epsilon'+(k-2)\epsilon}{3}\right\}$$

$$\left\{\frac{5+(1-k)(\epsilon'+\epsilon)}{3}\right\} \leqslant \left(\frac{m+q}{m}\right) = \left\{\frac{5+(k-1)k\epsilon'+(k-2)\epsilon}{3}\right\} < \left\{\frac{3+k\{k\epsilon'+\epsilon\}}{2}\right\} < \left(\frac{5}{3}\right)$$

$$\left\{\frac{5+(1-k)(\epsilon'+\epsilon)}{3}\right\} \leqslant \left(\frac{m+q}{m}\right) = \left\{\frac{5+(k-1)\left\{k\epsilon'+\frac{(k-2)\epsilon}{(k-1)}\right\}}{3}\right\} < \left\{\frac{3+k\{k\epsilon'+\epsilon\}}{2}\right\} < \left(\frac{5}{3}\right)$$

$$\left(\frac{m+q}{m}\right) = \left\{\frac{5+(k-1)k\epsilon'+(k-2)\epsilon}{3}\right\}$$

Formulae without Derivation

$$1 + k\epsilon' + \epsilon \leqslant 1 + k^2\epsilon' + k\epsilon = (2k+1)\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon' + \epsilon)\right\} < \left(\frac{5}{3}\right)$$

$$1 + k\epsilon' + \epsilon \leqslant 1 + k^2\epsilon' + k\epsilon = (2k+1)\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon' + \epsilon\right\} < \left(\frac{5}{3}\right)$$

$$1 + k\epsilon' + \epsilon \leqslant 1 + (k^2 - k)\epsilon' + k\epsilon = (2k+1)(k\epsilon' + k\epsilon) < (2k+1)\left\{\left(\frac{2(k+1)}{(2k+1)}\right)k\epsilon' + \epsilon\right\} < \left(\frac{5}{3}\right)$$

$$1 + \left\{\left(\frac{2k+1}{2k}\right)k\epsilon' + \epsilon\right\} = \left\{\left(\frac{k+2}{k+1}\right) + \frac{(1-k)\epsilon'}{2(k+1)}\right\} \leqslant \left(\frac{3}{2}\right)$$

$$\left\{\frac{5 + (1-k)(\epsilon' + \epsilon)}{3}\right\} \leqslant \left(\frac{m+q}{m}\right) = \left\{\frac{5 + (k-1)\left\{k\epsilon' + \frac{(k-2)\epsilon}{(k-1)}\right\}}{3}\right\} < \left\{\frac{3 + k\{k\epsilon' + \epsilon\}}{2}\right\}$$

$$n(1+\epsilon)(1+k\epsilon') < q\left\{k + 2 + (1+\epsilon)(1+k\epsilon')\right\}$$

$$n(1+\epsilon)(1+k\epsilon') < \left(\frac{3}{2}\right)\left\{k + 2 + (1+\epsilon)(1+k\epsilon')\right\}$$

$$9q^2 = mn\left\{2 + (2k+1)k\epsilon' + (2k-1)\epsilon - \{(2k+1)k\epsilon' + (2k-1)\epsilon\}^2\right\}$$

$$4q^2 = mn\left\{1 - \{k^2\epsilon' + (k-1)\epsilon\}^2\right\}$$

$$\because 2q = m\{1 + [k^2\epsilon' + (k-1)\epsilon]\}$$

$$\therefore 2q + m\epsilon = m\{1 + k[k\epsilon' + \epsilon]\}$$

$$\because q + m\epsilon = m(k+1)[k\epsilon' + \epsilon]$$

$$\therefore 3q + 2m\epsilon = m\{1 + (2k+1)[k\epsilon' + \epsilon]\}$$

$$\therefore \frac{3q}{3m} + \frac{2m\epsilon}{3m} = \frac{m}{3m}\{1 + (2k+1)[k\epsilon' + \epsilon]\}$$

$$\therefore \frac{q}{m} + \frac{2\epsilon}{3} = \frac{1}{3}\{1 + (2k+1)[k\epsilon' + \epsilon]\}$$

$$\therefore \left(\frac{m+q}{m}\right) + \left(\frac{2\epsilon}{3}\right) = \frac{4}{3} + \left\{\frac{(2k+1)\{k\epsilon' + \epsilon\}}{3}\right\}$$

$$\therefore q < \frac{5}{3} < \left\{\frac{m+q}{m} + \frac{2\epsilon}{3}\right\} = \frac{4}{3} + \left\{\left(\frac{(2k+1)}{3}\right)\{k\epsilon' + \epsilon\}\right\}$$

$$\left(\frac{m+q}{m}\right)+\left(\frac{n+q}{n}\right)=3$$

$$\left(\frac{m+q}{m}\right)+\left(\frac{m}{m+n}\right)=2$$

$$\left(\frac{n+q}{n}\right)+\left(\frac{n}{m+n}\right)=2$$

$$\left\{\left(\frac{m+q}{m}\right)+\left(\frac{m}{m+n}\right)\right\}-\left\{\left(\frac{n+q}{n}\right)+\left(\frac{n}{m+n}\right)\right\}=0$$

$$\left(\frac{m+q}{m}\right)-\left(\frac{n+q}{n}\right)=\left(\frac{n-m}{m+n}\right)$$

$$\left(\frac{m+q}{m}\right)-\left(\frac{n+q}{n}\right)=\left(k^2\epsilon'+(k-1)\epsilon\right)$$

$$\left(\frac{n-m}{m+n}\right)=\left(k^2\epsilon'+(k-1)\epsilon\right)$$

$$\left(\frac{m+q}{m}\right)=\left(\frac{n+q}{n}\right)+\left(\frac{n-m}{m+n}\right)$$

$$\left(\frac{m+q}{m}\right)=1+k\epsilon'+\epsilon+\left(k^2\epsilon'+(k-1)\epsilon\right)$$

$$\left(\frac{m+q}{m}\right)=1+k\left((k+1)\epsilon'+\epsilon\right)$$

$$\because\left(\frac{n-m}{m+n}\right)=\left(k^2\epsilon'+(k-1)\epsilon\right)$$

$$\because\left(\frac{n}{m+n}\right)=k\left((k+1)\epsilon'+\epsilon\right)$$

$$\therefore\left(\frac{2n-m}{m+n}\right)=\left((k^2+k^2+k)\epsilon'+(k+k-1)\epsilon\right)$$

$$\therefore\left(\frac{2n-m}{m+n}\right)=(2k+1)\left\{k\epsilon'+\left(\frac{(2k-1))\epsilon}{(2k+1)}\right)\right\}$$

$$\because\left(\frac{2n-m}{m+n}\right)<\left(\frac{2n}{m+n}\right)<\left(\frac{2n+m}{m+n}\right)=\left(\frac{m+q}{m}\right)$$

$$\because 2q = m\{1 + \{k^2\epsilon' + (k-1)\epsilon\}\}$$

$$2q = m\{1 - \epsilon + k\{k\epsilon' + \epsilon\}\}$$

$$\implies 2q < m\,\{1 + k\{k\epsilon' + \epsilon\}\}$$

$$\because 2q = n\{1 - \{k^2\epsilon' + (k-1)\epsilon\}\}$$

$$2q = n\{1 + \epsilon - k\{k\epsilon' + (k)\epsilon\}\}$$

$$\implies n\{1 - k\{k\epsilon' + \epsilon\} < 2q$$

$$n\{1 - k\{k\epsilon' + \epsilon\}\} < 2q < m\{1 + k\{k\epsilon' + \epsilon\}\}$$

$$n\{1 - k\{k\epsilon' + \epsilon\}\} < m\{1 + k\{k\epsilon' + \epsilon\}\}$$

$$n(1 - \epsilon') \approx m(1 + \epsilon')$$

$$n\{2 - \{(k^2+1)\epsilon' + (k)\epsilon\}\} \approx m\{2 + \{(k^2+1)\epsilon' + k\epsilon\}\}$$

$$\left(\frac{\{2 - \{(k^2+1)\epsilon' + (k)\epsilon\}}{\{2 + \{(k^2+1)\epsilon' + k\epsilon\}}\right) \approx \left(\frac{m}{n}\right)$$

$$\left(\frac{4}{\{2 + \{(k^2+1)\epsilon' + k\epsilon\}}\right) \approx \left(\frac{m+n}{n}\right)$$

$$\left(\frac{n}{m+n}\right) \approx \left(\frac{\{2 + \{(k^2+1)\epsilon' + k\epsilon\}}{4}\right)$$

$$q < \left(\frac{5}{3}\right) \approx \left(\frac{6 + \{(k^2+1)\epsilon' + k\epsilon\}}{4}\right) < \left(\frac{5}{3}\right) \approx \left(\frac{m+q}{m}\right) = \left(\frac{m+2n}{m+n}\right) \leqslant \left(\frac{5}{3}\right)$$

$$q \leqslant \left(\frac{5}{3}\right) \leqslant \left(\frac{m+q}{m}\right) + \left\{\frac{\epsilon(k\epsilon' + \epsilon)}{(1+\epsilon)}\right\} = (2k+1)[k\epsilon' + \epsilon] + 2k\epsilon' + \left\{\frac{\epsilon(k\epsilon' + \epsilon)}{1+\epsilon}\right\}$$

$$\because n(1+\epsilon) = q(k+2)$$

$$n \approx (k+2)(1+\epsilon)$$

$$m \leqslant 2q$$

$$mn \approx 2q(k+2)(1+\epsilon)$$

$$mn \leqslant (q+q)(1+\epsilon))(k+2)$$

$$mn < (m+q)(k+2) \because q(1+\epsilon) \leqslant m$$

$$\because q = n(k\epsilon' + \epsilon)$$
$$m = (m+n)(k\epsilon' + \epsilon)$$
$$(m+q) = (m+2n)(k\epsilon' + \epsilon)$$
$$(m+q)(k+2) = (m+2n)(k+2)(k\epsilon' + \epsilon)$$
$$(m+2n)(1+\epsilon) = (m+q)(k+2)$$

$$\because (m+n)(1+\epsilon') \leqslant 2n$$
$$(m+n)(k\epsilon' + \epsilon) = m$$
$$(m+n)(1+(k+1)\epsilon' + \epsilon) = m + 2n$$
$$(m+n)(1+\epsilon)(1+(k+1)\epsilon' + \epsilon) = (m+2n)(1+\epsilon)$$
$$m(k+2)(1+(k+1)\epsilon' + \epsilon) = (m+2n)(1+\epsilon)$$

$$mn < m(k+2)(1+(k+1)\epsilon' + \epsilon) = (m+2n)(1+\epsilon)$$
$$mn < m(k+2)(1+(k+1)\epsilon' + \epsilon) = (m+q)(k+2)$$
$$\left(\frac{n}{k+2}\right) < (1+(k+1)\epsilon' + \epsilon) \leqslant \left(\frac{5}{3}\right) \approx\leqslant \left(\frac{m+q}{m}\right)$$

$$(k^2+2k)\epsilon' + (k+1)\epsilon = 1$$
$$(k+1)\{(k+1)\epsilon' + \epsilon\} = 1 + \epsilon'$$
$$(k+1)\{k\epsilon' + \epsilon\} = 1 - k\epsilon'$$
$$(k+1)\{(2k+1)\epsilon' + 2\epsilon = 2 + (1-k)\epsilon')$$
$$2(k+1)\left\{\left(\frac{2k+1}{2k}\right)k\epsilon' + \epsilon\right\} = 2 + (1-k)\epsilon'$$
$$\left\{\left(\frac{2k+1}{2k}\right)k\epsilon' + \epsilon\right\} = \frac{2}{2(k+1)} + \frac{(1-k)\epsilon'}{2(k+1)}$$
$$q = n(k\epsilon' + \epsilon) < n\left\{\left(\frac{2k+1}{2k}\right)k\epsilon' + \epsilon\right\} = \frac{n}{(k+1)} + \frac{n(1-k)\epsilon'}{2(k+1)}$$
$$1 + \left\{\left(\frac{2k+1}{2k}\right)k\epsilon' + \epsilon\right\} = \left\{\frac{k+2}{(k+1)} + \frac{(1-k)\epsilon'}{2(k+1)}\right\} \leqslant \left(\frac{3}{2}\right)$$

$$\because q = n(k\epsilon' + \epsilon)$$
$$m = (m+n)(k\epsilon' + \epsilon)$$
$$(m+q) = (m+2n)(k\epsilon' + \epsilon)$$
$$(m+q)(k+2) = (m+2n)(k+2)(k\epsilon' + \epsilon)$$
$$(m+2n)(1+\epsilon) = (m+q)(k+2)$$

$$\because (m+n)(1+\epsilon') \leqslant 2n$$
$$(m+n)(k\epsilon' + \epsilon) \leqslant m$$
$$(m+n)(1+(k+1)\epsilon' + \epsilon) \leqslant (m+2n)$$
$$(m+n)(1+\epsilon)(1+(k+1)\epsilon' + \epsilon) \leqslant (m+2n)(1+\epsilon)$$
$$m(k+2)(1+(k+1)\epsilon' + \epsilon) \leqslant (m+2n)(1+\epsilon)$$

$$mn < m(k+2)(1+(k+1)\epsilon' + \epsilon) \leqslant (m+2n)(1+\epsilon)$$
$$mn < m(k+2)(1+(k+1)\epsilon' + \epsilon) \leqslant (m+q)(k+2)$$
$$\left(\frac{n}{k+2}\right) < (1+(k+1)\epsilon' + \epsilon) \leqslant \left(\frac{5}{3}\right) \approx\leqslant \left(\frac{m+q}{m}\right)$$

$$qk = m\{k - 1 + 2k\epsilon' + \epsilon)\}$$
$$q(k+1) = m\{k + k\epsilon')$$
$$q(2k+1) = m\{2k - 1 + [3k\epsilon' + \epsilon]\}$$

$$qk = n\{1 - (2k\epsilon' + \epsilon)\}$$
$$q(k+1) = n(1 - k\epsilon')$$
$$q(2k+1) = n\{2 - [3k\epsilon' + \epsilon]\}$$

$$m\{(2k-1) + [3k\epsilon' + \epsilon]\} = n\{2 - [3k\epsilon' + \epsilon]\}$$
$$\frac{m}{n} = \frac{\{(2 - [3k\epsilon' + \epsilon]\}}{\{(2k-1) + [3k\epsilon' + \epsilon]\}}$$
$$\frac{m+n}{n} = \frac{2k+1}{\{(2k-1) + [3k\epsilon' + \epsilon]\}}$$
$$\frac{n}{m+n} = \frac{\{(2k-1) + [3k\epsilon' + \epsilon]\}}{2k+1}$$
$$\left(\frac{m+q}{m}\right) = \left(\frac{m+2n}{m+n}\right) = \frac{\{4k + [3k\epsilon' + \epsilon]\}}{2k+1}$$

$$\left(\frac{n(1+\epsilon)}{k+2}\right) \leqslant \left(\frac{m+q}{m}\right) < \left(\frac{(m+2n)(1+\epsilon)}{m+n}\right)$$
$$mn(1+\epsilon) + n^2(1+\epsilon) < m(k+2) + 2n(k+2)(1+\epsilon)$$
$$(m+n)n(1+\epsilon) < (m+m)(1+\epsilon) + 2n(k+2)(1+\epsilon)$$
$$(m+n)(1+\epsilon)(n-1) < 2n(k+2)(1+\epsilon)$$
$$m(k+2)(n-1) < 2n(k+2)(1+\epsilon)$$
$$\cancel{(k+2)(n-1)}m < \cancel{n(k+2)}2(1+\epsilon) \qquad \because (n-1) \approx n$$
$$m \leqslant 2(1+\epsilon)$$

$$(m-q)(k+1) = q + m\epsilon$$
$$(m-q)(k+1-\epsilon) = q + q\epsilon$$
$$(m-q)\{2(k+1-\epsilon)\} = 2q + (m+q)\epsilon$$
$$(m-q)\{2(k+1-\epsilon) < (m+q)(1+\epsilon)$$
$$(m-q) < \frac{(m+q)(1+\epsilon)}{\{2(k+1-\epsilon)}$$
$$(m-q) < \frac{5(1+\epsilon)}{\{2(k+1-\epsilon)} < (1+\epsilon)$$
$$\because \left(\frac{q}{m} + \epsilon\right) . \left(\frac{k+2}{k+1}\right) = (1+\epsilon)$$

$$\left(\frac{m+q}{m}\right) + \{\epsilon' + \epsilon\} = (k+1)\{(2k+1)\epsilon' + 2\epsilon\}$$
$$\left(\frac{m+q}{m}\right) + \{\epsilon' + \epsilon\} = (2k+1)\left\{(k+1)\epsilon' + \left(\frac{2k+2}{2k+1}\right)\epsilon\right\}$$

$$\because q = k\left\{n(1+\epsilon)\epsilon' + m(1+\epsilon')\epsilon\right\}$$
$$q = m(1 - \{k\epsilon' + \epsilon)\}$$
$$m(1 - \{k\epsilon' + \epsilon)\} = k\left\{n(1+\epsilon)\epsilon' + m(1+\epsilon')\epsilon\right\}$$
$$m(1 - \{k\epsilon' + \epsilon)\} - mk\left\{(1+\epsilon')\epsilon\right\} = n(1+\epsilon)k\epsilon'$$
$$m\left\{1 - \{(k\epsilon')(1+\epsilon) + (k+1)\epsilon\}\right\} = n(1+\epsilon)k\epsilon'$$
$$m\left\{k + 1 - \epsilon\right\} = n(1+\epsilon)$$
$$m\left\{k + 2 - \{(k\epsilon')(1+\epsilon) + (k+2)\epsilon\}\right\} = n(1+\epsilon)(1+k\epsilon')$$
$$m(1+\epsilon)(1+k\epsilon') = m(1+\epsilon)(1+k\epsilon')$$
$$m\left\{k + 2 + (1+\epsilon)(1+k\epsilon') - \{(k\epsilon')(1+\epsilon) + (k+2)\epsilon\}\right\} = (m+n)(1+\epsilon)(1+k\epsilon')$$
$$\left(\frac{m}{m+n}\right) = \left(\frac{(1+\epsilon)(1+k\epsilon')}{\{k + 2 + (1+\epsilon)(1+k\epsilon') - \{(k\epsilon')(1+\epsilon) + (k+2)\epsilon\}\}}\right)$$
$$\left(\frac{mn}{m+n}\right) = q = \left(\frac{n(1+\epsilon)(1+k\epsilon')}{\{k + 2 + (1+\epsilon)(1+k\epsilon') - \{(k\epsilon')(1+\epsilon) + (k+2)\epsilon\}\}}\right)$$
$$n(1+\epsilon)(1+k\epsilon') < q\left\{k + 2 + (1+\epsilon)(1+k\epsilon')\right\}$$
$$n(1+\epsilon)(1+k\epsilon') < \left(\frac{3}{2}\right)\left\{k + 2 + (1+\epsilon)(1+k\epsilon')\right\}$$

$$\because m\{1 - \{k\epsilon' + \epsilon(k\epsilon' + \epsilon)\} = q(1+\epsilon)$$
$$\therefore m\{1 - \{k\epsilon' + \epsilon(k\epsilon' + \epsilon)\} \leqslant q^2$$
$$\therefore \frac{q + 1 - \{k\epsilon' + \epsilon(k\epsilon' + \epsilon)\}}{q} \leqslant \left(\frac{m+q}{m}\right)$$
$$\left(\frac{m+q}{m}\right) = \frac{2 + \epsilon - \{k\epsilon' + \epsilon(k\epsilon' + \epsilon)\}}{(1+\epsilon)}$$
$$\therefore \frac{q + 1 - \{k\epsilon' + \epsilon(k\epsilon' + \epsilon)\}}{q} \approx q \approx \frac{(m+q)}{m}$$
$$\therefore \; q + 1 - \{k\epsilon' + \epsilon(k\epsilon' + \epsilon)\} \approx q^2$$
$$\therefore \; q^2 - q - 1 + \{k\epsilon' + \epsilon(k\epsilon' + \epsilon)\} \approx 0$$
$$\{k\epsilon' + \epsilon(k\epsilon' + \epsilon)\} \approx 0$$
$$\therefore \; q^2 - q - 1 \approx 0$$
$$q \approx \phi = \frac{1+\sqrt{5}}{2}$$

$$\because \left(\frac{m+q}{m}\right)(k+3+\epsilon) = \left(\frac{n+q}{n}\right)(2k+3-\epsilon)$$

$$(k+3+\epsilon) < (2k+3-\epsilon)$$

$$\left(\frac{n+q}{n}\right) < \left(\frac{m+q}{m}\right)$$

$$\{1+(k\epsilon'+\epsilon)\} < \{2-(k\epsilon'+\epsilon)\}$$

$$\{2(k\epsilon'+\epsilon)\} < 1$$

$$\because \{(k+1)\epsilon'+\epsilon\} < \frac{2}{3}$$

$$\therefore \{(3k+1)\epsilon'+3\epsilon\} < \frac{5}{3}$$

$$\therefore 3\left\{\left(\frac{(3k+1)}{3}\right)\epsilon'+\epsilon\right\} < \frac{5}{3}$$

$$\therefore 3\left\{\left(\frac{(3k+1)}{3k}\right)k\epsilon'+\epsilon\right\} < \frac{5}{3}$$

$$\therefore (k\epsilon'+\epsilon) < \left\{\left(\frac{(3k+1)}{3k}\right)k\epsilon'+\epsilon\right\} < \frac{5}{9}$$

$$\therefore 1+\left\{\left(\frac{(3k+1)}{3k}\right)k\epsilon'+\epsilon\right\} < \frac{14}{9}$$

$$(1+\epsilon) < \{1+(k\epsilon'+\epsilon)\} < \frac{3}{2} \approx q \approx 1+\left\{\left(\frac{(3k+1)}{3k}\right)k\epsilon'+\epsilon\right\} \leqslant \frac{14}{9} < \frac{5}{3}$$

$$\because \left(k^2\epsilon'+(k-1)\epsilon\right) = \left(\frac{n-m}{m+n}\right)$$

$$2 = \left(\frac{1+k^2\epsilon'+(k-1)\epsilon}{k\left((k+1)\epsilon'+\epsilon\right)}\right)$$

$$(k+1)\epsilon'+\epsilon = \left(\frac{1+k^2\epsilon'+(k-1)\epsilon}{2k}\right)$$

$$(k+1)\epsilon'+\epsilon = \left(\frac{(2k^2+2k)\epsilon'+(2k-1)\epsilon}{2k}\right)$$

$$(k+1)\epsilon'+\epsilon = \left(\frac{(2k-1)}{2k}\right)\left(\left(\frac{(2k+1)}{2k-1}\right)k\epsilon'+\epsilon\right)$$

$$(k+1)\epsilon'+\epsilon = \left(\left(\frac{(2k+1)}{2k}\right)k\epsilon'+\left(\frac{(2k-1)}{2k}\right)\epsilon\right)$$

$$1+(k+1)\epsilon'+\epsilon < 1+\left(\left(\frac{(2k+1)}{2k}\right)k\epsilon'+\epsilon\right) \leqslant \left(\frac{3}{2}\right) < \left(\frac{5}{3}\right)$$

Computational Verification

7.1 Example 1. a + b = c

$\boxed{4+121=125 \Rightarrow 2^2+11^2=5^3}$

$$\operatorname{rad} c^1 < (\operatorname{rad}(ab) < \operatorname{rad} c^2 \because 5 < 22 < 5^2 \gcd(a,b)=1 \implies k=1$$

$$a=4 \Rightarrow (\operatorname{rad}(a))=2 \Rightarrow b=121 \Rightarrow (\operatorname{rad}(b))=11 \Rightarrow (\operatorname{rad}(ab))=22$$

$$c=125 \Rightarrow \operatorname{rad}(c))=5 \Rightarrow (\operatorname{rad}(abc))=110 \quad \& \quad k=1$$

$$\operatorname{rad} c^k < (\operatorname{rad}(ab) < \operatorname{rad} c^{k+1}$$

$$(1+\epsilon)=\frac{(k+2)\log\operatorname{rad}(c)}{\log\operatorname{rad}(abc)} \Rightarrow (1+\epsilon)=\frac{(1+2)\log\operatorname{rad}(5)}{\log\operatorname{rad}(110)}$$

$$(1+\epsilon)=1.02719581012192.....$$

$$(1+\epsilon')=\left(\frac{k+1}{k}\right)\left(\frac{\log\operatorname{rad}(ab)}{\log(\operatorname{rad}(abc))}\right) \Rightarrow (1+\epsilon')=\left(\frac{1+1}{1}\right)\left(\frac{\log\operatorname{rad}(22)}{\log(\operatorname{rad}(110))}\right)$$

$$(1+\epsilon')=1.31520279325206......$$

$$q=\frac{\log(c)}{\log(\operatorname{rad}(abc))} \Rightarrow q=\frac{\log(125)}{\log(\operatorname{rad}(110))}=q=1.02719581012192.......$$

$$m=\frac{\log(c)}{\log(\operatorname{rad}(ab))} \Rightarrow m=\frac{\log(125)}{\log(\operatorname{rad}(22))}=m=1.56203410666731.......$$

$$n=\frac{\log(c)}{\log(\operatorname{rad}(c))} \Rightarrow n=\frac{\log(125)}{\log(\operatorname{rad}(5)}=n=3.00.......$$

$$q \le \frac{5}{3} \le \left(\frac{m+q}{m}\right) \le \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

$$\left(\frac{m+q}{m}\right)=\frac{1.56203410666731+1.02719581012192}{1.56203410666731}=1.65760139662603$$

$$\left(\frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}\right)=\frac{(2+0.02719581012192-0.31520279325206}{1.02719581012192}=\frac{5}{3}$$

$$q \le \left(\frac{m+q}{m}\right) \le \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}=\frac{5}{3}$$

Verified

7.2 Example 2. a + b = c

This abc triple is my invention	$33 + 2000000 = 2000033 \Rightarrow 3 \times 11 + 2^7 \times 5^6 = 7^6 \times 17$

$$\text{rad}\, c < (\text{rad}(ab) \because 119 < 330 < 119^2 \implies k = 1 \; gcd(a,b) = 1$$

$$a = 33 \Rightarrow (\text{rad}(a)) = 33 \Rightarrow b = 2000000 \Rightarrow (\text{rad}(b)) = 10$$

$$(\text{rad}(ab)) = 330$$

$$c = 2000033 \Rightarrow \text{rad}(c)) = 119$$

$$\Rightarrow (\text{rad}(abc)) = 39270$$

$$\text{rad}\, c^k < (\text{rad}(ab) < \text{rad}\, c^{k+1}$$

$$k = 1$$

$$(1+\epsilon) = \frac{(k+2)\log \text{rad}(c)}{\log \text{rad}(abc)} \Rightarrow (1+\epsilon) = \frac{(1+2)\log \text{rad}(119)}{\log \text{rad}(39270)}$$

$$(1+\epsilon) = 1.35536750992041.....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \text{rad}(ab)}{\log(\text{rad}(abc))}\right) \Rightarrow (1+\epsilon') = \left(\frac{1+1}{1}\right)\left(\frac{\log \text{rad}(330)}{\log(\text{rad}(39270))}\right)$$

$$(1+\epsilon') = 1.09642166005306......$$

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} \Rightarrow q = \frac{\log(2000033)}{\log(\text{rad}(39270))} = q = 1.3715615219035.......$$

$$m = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow m = \frac{\log(2000033)}{\log(\text{rad}(330))} = m = 2.50188695075054.......$$

$$n = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow n = \frac{\log(2000033)}{\log(\text{rad}(119))} = n = 3.03584417923086.......$$

$$q \leq \frac{5}{3} \leq \left(\frac{m+q}{m}\right) \leq \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

$$\left(\frac{m+q}{m}\right) = \frac{2.50188695075054 + 1.3715615219035}{2.50188695075054} = 1.54821083002653$$

$$\left(\frac{2+\epsilon-k\epsilon'}{1+\epsilon}\right) = \frac{(2+0.35536750992041 - 0.09642166005306}{1.35536750992041} = \frac{5}{3}$$

$$q \leqslant \left(\frac{m+q}{m}\right) \leq \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} = \frac{5}{3}$$

Verified

7.3 Example 3. a + b = c

$$\boxed{121 + 48234375 = 48234496 \Rightarrow 11^2 + 3^2 \times 5^6 \times 7^3 = 2^{21} \times 23}$$

$$\text{rad}(c) < \text{rad}(ab) \because 46 < 1155 < 46^2 \Rightarrow k = 1 \quad \gcd(a, b) = 1$$

$$a = 121 \Rightarrow (\text{rad}(a)) = 11 \Rightarrow b = 48234375 \Rightarrow (\text{rad}(b)) = 105$$

$$\Rightarrow (\text{rad}(ab)) = 1155$$

$$c = 48234496 \Rightarrow \text{rad}(c)) = 46 \Rightarrow \text{rad}(abc)) = 53130 \quad \& \quad k = 1$$

$$\text{rad}\, c^k < (\text{rad}(ab) < \text{rad}\, c^{k+1}$$

$$(1+\epsilon) = \frac{(k+2)\log \text{rad}(c)}{\log \text{rad}(abc)} \Rightarrow (1+\epsilon) = \frac{(1+2)\log \text{rad}(46)}{\log \text{rad}(53130)}$$

$$(1+\epsilon) = 1.05564333770234.....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \text{rad}(ab)}{\log(\text{rad}(abc))}\right) \Rightarrow (1+\epsilon') = \left(\frac{1+1}{1}\right)\left(\frac{\log \text{rad}(1155)}{\log(\text{rad}(53130))}\right)$$

$$(1+\epsilon') = 1.2962377748651......$$

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} \Rightarrow q = \frac{\log(48234496)}{\log(\text{rad}(53130))} = q = 1.62599052010865.......$$

$$m = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow m = \frac{\log(48234496)}{\log(\text{rad}(1155))} = m = 2.50878434749807.......$$

$$n = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow n = \frac{\log(48234496)}{\log(\text{rad}(46))} = n = 4.62085193560082.......$$

$$q \leq \frac{5}{3} \leq \left(\frac{m+q}{m}\right) \leq \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

$$\left(\frac{m+q}{m}\right) = \frac{2.50878434749807 + 1.62599052010865}{2.50878434749807} = 1.64811888743255$$

$$\left(\frac{2+\epsilon-k\epsilon'}{1+\epsilon}\right) = \frac{(2+0.05564333770234 - 0.2962377748651}{1.05564333770234} = \frac{5}{3}$$

$$q \leq \left(\frac{m+q}{m}\right) \leq \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} = \frac{5}{3}$$

Verified

7.4 Example 4. $\mathbf{a+b=c}$

$\boxed{1+2400=2401 \Rightarrow 1+2^5\times 3\times 5^2=7^4}$

$$\operatorname{rad} c < (\operatorname{rad}(ab) \because 7 < 30 < 7^2 \implies k=1 \quad \gcd(a,b)=1$$

$$a=1 \Rightarrow (\operatorname{rad}(a))=1 \quad b=2400 \Rightarrow (\operatorname{rad}(b))=30 \Rightarrow (\operatorname{rad}(ab))=30$$

$$c=2401 \implies \operatorname{rad}(c))=7 \Rightarrow (\operatorname{rad}(abc))=210 \quad \& \quad k=1$$

$$\operatorname{rad} c^k < (\operatorname{rad}(ab) < \operatorname{rad} c^{k+1}$$

$$(1+\epsilon)=\frac{(k+2)\log\operatorname{rad}(c)}{\log\operatorname{rad}(abc)} \Rightarrow (1+\epsilon)=\frac{(1+2)\log\operatorname{rad}(7)}{\log\operatorname{rad}(210)}$$

$$(1+\epsilon)=1.09175482513303.....$$

$$(1+\epsilon')=\left(\frac{k+1}{k}\right)\left(\frac{\log\operatorname{rad}(ab)}{\log(\operatorname{rad}(abc))}\right) \Rightarrow (1+\epsilon')=\left(\frac{1+1}{1}\right)\left(\frac{\log\operatorname{rad}(30)}{\log(\operatorname{rad}(210))}\right)$$

$$(1+\epsilon')=1.27216344991131......$$

$$q=\frac{\log(c)}{\log(\operatorname{rad}(abc))} \Rightarrow q=\frac{\log(2401)}{\log(\operatorname{rad}(210))}=q=1.45567310017737.......$$

$$m=\frac{\log(c)}{\log(\operatorname{rad}(c))} \Rightarrow m=\frac{\log(2401)}{\log(\operatorname{rad}(30))}=m=2.28850011416198.......$$

$$n=\frac{\log(c)}{\log(\operatorname{rad}(c))} \Rightarrow n=\frac{\log(2401)}{\log(\operatorname{rad}(7))}=n=4.0......$$

$$q \le \frac{5}{3} \le \left(\frac{m+q}{m}\right) \le \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

$$\left(\frac{m+q}{m}\right)=\frac{2.28850011416198+1.45567310017737}{2.28850011416198}=1.63608172495566$$

$$\left(\frac{2+\epsilon-k\epsilon'}{1+\epsilon}\right)=\frac{(2+0.09175482513303-0.27216344991131}{1.09175482513303}=\frac{5}{3}$$

$$q \le \left(\frac{m+q}{m}\right) < \frac{5}{3}=\frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}=\frac{5}{3}$$

Verified

7.5 Example 5:a + b = c

$$\boxed{1 + 17903468100000 = 17903468100001 \implies 1 + 2^5 \times 3 \times 5^5 \times 7^3 \times 257 \times 677 = 11^8 \times 17^4}$$

$$\text{rad}(c)^3 < (\text{rad}(ab) < \text{rad}(c)^4 \because 187^3 < 36537690 < 187^4 \implies k = 3 \ gcd(a,b) = 1$$

$$a = 1 \implies \text{rad}(a) = 1$$

$$b = 17903468100000 \implies \text{rad}(b)) = 36537690 \therefore (\text{rad}(ab)) = 36537690$$

$$c = 17903468100001 \implies \text{rad}(c)) = 187 \therefore (\text{rad}(abc)) = 6832548030$$

$$\therefore \text{rad}(c)^k < (\text{rad}(ab) < \text{rad}(c)^{k+1} \implies k = 3$$

$$(1+\epsilon) = \frac{(k+2)\log \text{rad}(c)}{\log \text{rad}(abc)}$$

$$\Rightarrow (1+\epsilon) = \frac{(3+2)\log \text{rad}(187)}{\log \text{rad}(6832548030)}$$

$$(1+\epsilon) = 1.1550269479595.....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \text{rad}(ab)}{\log(\text{rad}(abc))}\right)$$

$$\Rightarrow (1+\epsilon') = \left(\frac{3+1}{3}\right)\left(\frac{\log \text{rad}(36537690)}{\log(\text{rad}(6832548030))}\right)$$

$$(1+\epsilon') = 1.0253261472108.......$$

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} \Rightarrow q = \frac{\log((17903468100001)}{\log(\text{rad}(1974))} = q = 1.02580760323428.......$$

$$m = \frac{\log(c)}{\log(\text{rad}(ab))} \Rightarrow m = \frac{\log(17903468100001)}{\log(\text{rad}(36537690))} = m = 1.37961162747883.......$$

$$n = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow n = \frac{\log(17903468100001)}{\log(\text{rad}(187))} = n = 5.83356565380606.00........$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{1.37961162747883 + 1.02580760323428}{1.37961162747883}\right) = \frac{5}{3}$$

$$q \leq \frac{5}{3} = \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} == \frac{(2+0.1550269479595-0.0.0759784416324)}{(1.02580760323428)} = 1.7999999....$$

$$q \leq \frac{5}{3} = \left(\frac{m+q}{m}\right) = \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} = \frac{5}{3}$$

$$q \leq \left(\frac{m+q}{m}\right) \leq \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} = \frac{5}{3}$$

Verified

7.6 Example 6. a + b = c

This abc triple is my invention | $263 + 3442688 = 3442951 \Rightarrow 263 + 2^{11} \times 41^2 = 151^3$

$$\operatorname{rad} c < (\operatorname{rad}(ab) < (\operatorname{rad} c)^2 \because\ 151 < 21566 < 151^2 \gcd(a, b) = 1 \implies k = 1$$

$$a = 263 \implies (\operatorname{rad}(a)) = 263 \qquad b = 3442688 \implies (\operatorname{rad}(b)) = 82$$

$$\implies (\operatorname{rad}(ab)) = 21566$$

$$c = 3442951 \Rightarrow \operatorname{rad}(c)) = 151$$

$$\operatorname{rad} c^k < (\operatorname{rad}(ab) < \operatorname{rad} c^{k+1}$$

$$(\operatorname{rad}(abc)) = 3256466$$

$$(1+\epsilon) = \frac{(k+2)\log\operatorname{rad}(c)}{\log\operatorname{rad}(abc)} \Rightarrow (1+\epsilon) = \frac{(1+2)\log\operatorname{rad}(151)}{\log\operatorname{rad}(3256466)}$$

$$(1+\epsilon) = 1.00371337860824.....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log\operatorname{rad}(ab)}{\log(\operatorname{rad}(abc))}\right)$$

$$\implies (1+\epsilon') = \left(\frac{1+1}{1}\right)\left(\frac{\log\operatorname{rad}(21566)}{\log(\operatorname{rad}(3256466))}\right)$$

$$(1+\epsilon') = 1.33085774759451......$$

$$q = \frac{\log(c)}{\log(\operatorname{rad}(abc))} \Rightarrow q = \frac{\log(3442951)}{\log(\operatorname{rad}(3256466))} = q = 1.00371337860824.......$$

$$m = \frac{\log(c)}{\log(\operatorname{rad}(ab))} \Rightarrow m = \frac{\log(3442951)}{\log(\operatorname{rad}(21566))} = m = 1.5083706435529.......$$

$$n = \frac{\log(c)}{\log(\operatorname{rad}(c))} \Rightarrow m = \frac{\log(3442951)}{\log(\operatorname{rad}(151))} = n = 3.00.......$$

$$q \le \frac{5}{3} \le \left(\frac{m+q}{m}\right) \le \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

$$\left(\frac{m+q}{m}\right) = \frac{1.5083706435529 + 1.00371337860824}{1.5083706435529.} = 1.66542887379725$$

$$\left(\frac{2+\epsilon-k\epsilon'}{1+\epsilon}\right) = \frac{(2+0.00371337860824 - 0.2962377748651)}{1.00371337860824} = \frac{5}{3}$$

$$q \le \left(\frac{m+q}{m}\right) < \frac{5}{3} = \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} = \frac{5}{3}$$

Verified

7.7 Example 7. a + b = c

$$\boxed{282475249 + 1067649858 = 1350125107 \implies 7^{10} + 2 \times 3^2 \times 11 \times 31^3 \times 181 = 67^5}$$

$$a = 282475249 \implies \operatorname{rad} a = 7$$

$$b = 1067649858 \therefore \operatorname{rad}(b) = 370326 \implies \operatorname{rad}(ab) = 2592282$$

$$c = 1350125107 \implies \operatorname{rad}(c) = 67 \implies \operatorname{rad}(abc) = 173682894$$

$$\operatorname{rad}(c)^k < \operatorname{rad}(ab) < \operatorname{rad}(c)^{k+1} \because 67^3 < 2592282 < 67^4 \Rightarrow k = 3 \because \gcd(a,b) = 1$$

$$(1+\epsilon) = \frac{(k+2)\log \operatorname{rad}(c)}{\log \operatorname{rad}(abc)} \Rightarrow (1+\epsilon) = \frac{(3+2)\log \operatorname{rad}(67)}{\log \operatorname{rad}(173682894)}$$

$$(1+\epsilon) = 1.1080877702422385915448490418757.....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \operatorname{rad}(ab)}{\log(\operatorname{rad}(abc))}\right) \Rightarrow (1+\epsilon') = \left(\frac{3+1}{3}\right)\left(\frac{\log \operatorname{rad}(2592282)}{\log(\operatorname{rad}(173682894)}\right)$$

$$(1+\epsilon') = 1.0378432612687363755880402554998......$$

$$q = \frac{\log(c)}{\log(\operatorname{rad}(abc))} \Rightarrow q = \frac{\log(1350125107)}{\log(\operatorname{rad}(173682894))} = q = 1.1080877702422385915448490418757.......$$

$$m = \frac{\log(c)}{\log(\operatorname{rad}(c))} \Rightarrow m = \frac{\log(1350125107)}{\log(\operatorname{rad}(2592282)} = m = 1.4235775434113626906727775568362.......$$

$$n = \frac{\log(c)}{\log(\operatorname{rad}(c))} \Rightarrow n = \frac{\log(1350125107)}{\log(\operatorname{rad}(67)} = n = 5.......$$

$$q \leq \frac{5}{3} \leq \left(\frac{m+q}{m}\right) \leq \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

$$\left(\frac{m+q}{m}\right) = \frac{1.42357754341136269069...... + 1.10808777024223....}{1.423577543411362690.....} = 1.77838244595155.....$$

$$\left(\frac{2+\epsilon-k\epsilon'}{1+\epsilon}\right) = \frac{(2 + 0.1080877702422385.... - 3(0.0378432612687363755 88)}{1.108087770242238.....}$$

$$\frac{5}{3} < 1.778382445951552...... < 1.800$$

$$q \leqslant \frac{5}{3} \leq \left(\frac{m+q}{m}\right) \leq \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

Verified

7.8 Example 8. a + b = c, 1+3=4

$$\boxed{1+3=2^2}$$

$$a=1 \quad \text{rad}(a)=1 \quad b=3 \quad \text{rad}(b)=3 \quad \text{rad}(ab)=3 \quad \text{rad}(abc)=6$$

$$\Rightarrow \text{rad}(c)^1 < \text{rad}(ab) < \text{rad}(c)^2 \because 2 < 3 < 2^2 \implies k=1$$

$$c=4 \implies \text{rad}\, c = 2 \implies 2^1 < 3 < 2^2$$

$$\text{rad}(c)^k < \text{rad}(ab) < \text{rad}(c)^{k+1} \therefore k=1 \because \gcd(a,b)=1$$

$$(1+\epsilon) = \frac{(k+2)\log \text{rad}(c)}{\log \text{rad}(abc)} = \frac{(1+2)\log \text{rad}(2)}{\log \text{rad}(6)}$$

$$(1+\epsilon) = 1.16055842170362....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \text{rad}(ab)}{\log(\text{rad}(abc))}\right) = \left(\frac{1+1}{1}\right)\left(\frac{\log \text{rad}(3)}{\log(\text{rad}(6))}\right)$$

$$(1+\epsilon') = 1.22629438553092......$$

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} \Rightarrow q = \frac{\log(4)}{\log(\text{rad}(6))} = q = 0.773705614469083.......$$

$$m = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow m = \frac{\log(4)}{\log(\text{rad}(3))} = m = 1.26185950714292.......$$

$$n = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow n = \frac{\log(4)}{\log(\text{rad}(2))} = n = 2.00.......$$

$$\left(\frac{m+q}{m}\right) = \frac{1.26185950714292... + 0.773705614469083}{1.26185950714292...} = 1.61314719276545$$

$$\left(\frac{2+\epsilon-k\epsilon'}{1+\epsilon}\right) = \frac{(2+0.16055842170362.... - 0.22629438553092..)}{1.05564333770234} = \frac{5}{3}$$

$$q \le \left(\frac{m+q}{m}\right) \le \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} = \frac{5}{3}$$

Verified

7.9 Example 9: a + b = c 1+65024=65025

$$1 + 2^9 \times 127 = (3 \times 5 \times 17)^2$$

$$(\text{rad}(ab) < \text{rad}(c) < (\text{rad}(ab)^2 \ \because 254 < 255 < 254^2 \because gcd(a,b) = 1$$

$$\text{rad}(ab)^k < (\text{rad}(c) < \text{rad}(ab)^{k+1} \implies k = 1$$

$$a = 1 \therefore \text{rad}(a)) = 1 \quad , b = 65024 \quad \therefore \text{rad}(b)) = 254 \implies (\text{rad}(ab)) = 254$$

$$\text{rad}(c)) = 255 \implies \text{rad}(abc)) = 64770$$

$$\text{rad}(ab)^k < (\text{rad}(c) < \text{rad}(ab)^{k+1} \implies k = 1$$

$$(1+\epsilon) = \frac{(k+2)\log \text{rad}(ab)}{\log \text{rad}(abc)} = \frac{(1+2)\log \text{rad}(254)}{\log \text{rad}(64770)}$$

$$(1+\epsilon) = 1.49946799068711.....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \text{rad}(c)}{\log(\text{rad}(abc))}\right) = \left(\frac{1+1}{1}\right)\left(\frac{\log \text{rad}(255)}{\log(\text{rad}(64770))}\right)$$

$$(1+\epsilon') = 1.00035467287526......$$

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} \Rightarrow q = \frac{\log((65025)}{\log(\text{rad}(64770))} = q = 1.00035467287526.......$$

$$m = \frac{\log(c)}{\log(\text{rad}(ab))} \Rightarrow m = \frac{\log(65025)}{\log(\text{rad}(254))} = m = 2.00141641755451.......$$

$$n = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow n = \frac{\log(65025)}{\log(\text{rad}(255))} = n = 2.00........$$

$$q \leq \frac{5}{3} \leq \left(\frac{m+q}{m}\right) \leq \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

$$q \leq \frac{5}{3} \leq \left(\frac{m+q}{m}\right) \leq \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

$$\left(\frac{m+q}{m}\right) = \frac{2.50878434749807 + 1.62599052010865}{2.50878434749807} = \frac{5}{3}$$

$$\left(\frac{2+\epsilon-k\epsilon'}{1+\epsilon}\right) = \frac{(2+0.05564333770234 - 0.2962377748651}{1.05564333770234} = \frac{5}{3}$$

$$q \leq \frac{5}{3} = \left(\frac{m+q}{m}\right) = \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} = \frac{5}{3}$$

Verified

7.10 Example 10.a+b=c 7890481+144=7890625

$\boxed{7890481 + 144 = 7890625 \Rightarrow 53^4 + 12^2 = 5^7 \times 101}$

$$a = 7890481 \Rightarrow \operatorname{rad} a = 53 \Rightarrow b = 144 \Rightarrow \operatorname{rad}(b) = 6 \implies \operatorname{rad}(ab) = 318$$

$$\Rightarrow \operatorname{rad}(ab) < \operatorname{rad}(c) < \operatorname{rad}(ab)^2 \because 318 < 505 < 318^2$$

$$c = 7890625 \implies \operatorname{rad} c = 505 \implies 318 < 505 < 318^2 \implies k = 1$$

$$\operatorname{rad}(abc) = 160590$$

$$\operatorname{rad}(ab) < \operatorname{rad}(c) \because 318 < 505 < 318^2 \Rightarrow k = 1 \quad \gcd(a, b) = 1$$

$$\operatorname{rad}(ab)^k < (\operatorname{rad}(c) < \operatorname{rad}(ab)^{k+1}$$

$$(1+\epsilon) = \frac{(k+2)\log \operatorname{rad}(ab)}{\log \operatorname{rad}(abc)}$$

$$\Rightarrow (1+\epsilon) = \frac{(1+2)\log \operatorname{rad}(318)}{\log \operatorname{rad}(160590)} = 1.44212203612025$$

$$(1+\epsilon) = 1.44212203612025.....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \operatorname{rad}(c)}{\log(\operatorname{rad}(abc))}\right)$$

$$\Rightarrow (1+\epsilon') = \left(\frac{1+1}{1}\right)\left(\frac{\log \operatorname{rad}(505)}{\log(\operatorname{rad}(160590))}\right) = 1.03858530925317$$

$$(1+\epsilon') = 1.03858530925317......$$

$$q = \frac{\log(c)}{\log(\operatorname{rad}(abc))} \Rightarrow q = \frac{\log(7890625)}{\log(\operatorname{rad}(160590))} = q = 1.32491055877263.......$$

$$m = \frac{\log(c)}{\log(\operatorname{rad}(ab))} \Rightarrow m = \frac{\log(7890625)}{\log(\operatorname{rad}(318))} = m = 2.75616874076144.......$$

$$n = \frac{\log(c)}{\log(\operatorname{rad}(c))} \Rightarrow n = \frac{\log(7890625)}{\log(\operatorname{rad}(505))} = n = 2.55137550467637.......$$

$$\left(\frac{n+q}{n}\right) = \left(\frac{2.55137550467637 + 1.32491055877263}{2.55137550467637}\right) = 1.51929265462658....$$

$$\left(\frac{n+q}{n}\right) = 1.51929265462658....$$

$$\left(\frac{2+\epsilon-k\epsilon'}{1+\epsilon}\right) = \left(\frac{2 + 0.44212203612025 - 0.03858530925317}{1.44212203612025}\right) = \frac{5}{3}$$

Verified

7.11 Example 11.a+b=c 2048+625=2673

$\boxed{2048 + 625 = 2673 \Rightarrow 2^{11} + 5^4 = 3^5 \times 11}$

$$a = 2^{11} \quad \text{rad}(a) = 2 \quad b = 5^4 \quad \text{rad}(b) = 5 \quad \text{rad}(ab) = 10$$

$$\because c = 2673 \therefore \text{rad}(c) = 33 \implies \text{rad}(abc) = 330$$

$$\Rightarrow \text{rad}(ab)^1 < \text{rad}(c) < \text{rad}(ab)^2 \because 10 < 33 < 10^2 \implies k = 1$$

$$c = 2673 \implies \text{rad}\, c = 33 \implies 10^1 < 33 < 10^2$$

$$\text{rad}(ab) < \text{rad}(c) \because 10 < 33 < 10^2 \Rightarrow k = 1 \quad \gcd(a, b) = 1$$

$$\text{rad}(ab^k < (\text{rad}(c) < \text{rad}(ab)^{k+1}$$

$$(1 + \epsilon) = \frac{(k+2)\log \text{rad}(ab)}{\log \text{rad}(abc)}$$

$$\Rightarrow (1 + \epsilon) = \frac{(1+2)\log \text{rad}(10)}{\log \text{rad}(330)} = 1.19117863613868.......$$

$$(1 + \epsilon) = 1.19117863613868.....$$

$$(1 + \epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \text{rad}(c)}{\log(\text{rad}(abc))}\right) = \left(\frac{1+1}{1}\right)\left(\frac{\log \text{rad}(33)}{\log(\text{rad}(330))}\right)$$

$$(1 + \epsilon') = 1.20588090924088......$$

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} \Rightarrow q = \frac{\log(2673)}{\log(\text{rad}(330))} = q = 1.3607226485801......$$

$$m = \frac{\log(c)}{\log(\text{rad}(ab))} \Rightarrow m = \frac{\log(2673)}{\log(\text{rad}(10))} = m = 3.42699895875654.......$$

$$n = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow n = \frac{\log(2673)}{\log(\text{rad}(33))} = n = 2.25681099709372.......$$

$$\left(\frac{n+q}{n}\right) = \left(\frac{2.25681099709372 + 1.3607226485801}{2.25681099709372}\right) = 1.60294045462044....$$

$$\left(\frac{n+q}{n}\right) = 1.60294045462044..........$$

$$\left(\frac{2 + \epsilon - k\epsilon'}{1 + \epsilon}\right) = \left(\frac{2 + 0.19117863613868 - 0.20588090924088}{1.19117863613868}\right) = \frac{5}{3}$$

Verified

7.12 Example 12: a + b = c

$$\boxed{1 + 1046528 = 1046529 \implies 1 + 2^{11} \times 7 \times 73 = 3^2 \times 11^2 \times 31^2}$$

$$\because a = 1 \implies (\text{rad}(a)) = 1 \quad , \because b = 1046528 \implies (\text{rad}(b)) = 1022$$

$$\text{rad}(ab)) = 1022 \quad , \because c = 1046529 \implies \text{rad}(c)) = 1023 \quad \therefore \text{rad}(abc)) = 1045506 \implies k = 1$$

$$\text{rad}(ab)^k < (\text{rad}(c) < \text{rad}(ab)^{k+1} \because 1022^1 < 1023 < 1022^2 \; gcd(a,b) = 1 \implies k = 1$$

$$(1+\epsilon) = \frac{(k+2)\log \text{rad}(ab)}{\log \text{rad}(abc)} = \frac{(1+2)\log \text{rad}(1022)}{\log \text{rad}(1045506)}$$

$$(1+\epsilon) = 1.49989415645303908791171021984 85.....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \text{rad}(c)}{\log(\text{rad}(abc))}\right) = \left(\frac{1+1}{1}\right)\left(\frac{\log \text{rad}(1023)}{\log(\text{rad}(1045506))}\right)$$

$$(1+\epsilon') = 1.00007056236464060805885985343 43.......$$

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} \Rightarrow q = \frac{\log((1023)^2}{\log(\text{rad}(1045506))} = q = 1.00007056236464060805885985343 43.......$$

$$m = \frac{\log(c)}{\log(\text{rad}(ab))} \Rightarrow m = \frac{\log((1023)^2}{\log(\text{rad}(1022))} = m = 2.00028226937615707950824198681 79.......$$

$$n = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow n = \frac{\log((1023)^2}{\log(\text{rad}(1023))} = n = 2.00........$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{2.00028226937615707950824198681 79 + 1.00007056236464060805885985343 4}{2.00028226937615707......}\right)$$

$$\left(\frac{m+q}{m}\right) == 1.49996471881767969 5970...$$

$$q \leq \frac{5}{3} = \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} = \frac{(2+0.499894156453..... - 0.0000705623646406......}{(1.4998941564530390 8791.....)}$$

$$q \leq \frac{5}{3} = \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} == 1.6667607564587190.....$$

$$q \leq \frac{5}{3} = \left(\frac{m+q}{m}\right) = 1.49996471881767969 59705... < \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} = 1.66676075645871902.....$$

$$q < \frac{5}{3} \leqslant \left(\frac{m+q}{m}\right) \leqslant \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

Verified

7.13 Example 13.a+b=c 1+1046528=1046529

$$\boxed{1 + 1046528 = 1046529 \Rightarrow 1 + 2^{11} \times 7 \times 73 = 1023^2}$$

$$a = 1 \Rightarrow \operatorname{rad} a = 1 \Rightarrow b = 1046528 \Rightarrow \operatorname{rad}(b) = 1022 \implies \operatorname{rad}(ab) = 1022$$

$$\Rightarrow \operatorname{rad}(ab) < \operatorname{rad}(c) < \operatorname{rad}(ab)^2 \because 1022 < 1023 < 1022^2$$

$$c = 1023^2 \implies \operatorname{rad} c = 1023 \implies 1022 < 1023 < 1022^2 \implies k = 1$$

$$\operatorname{rad}(abc) = 1045506$$

$$\operatorname{rad}(ab)^k < \operatorname{rad}(c) < \operatorname{rad}(ab)^{k+1} \because 1022 < 1023 < 1022^2 \Rightarrow k = 1 \quad \gcd(a, b) = 1$$

$$(1 + \epsilon) = \frac{(k+2)\log \operatorname{rad}(ab)}{\log \operatorname{rad}(abc)}$$

$$\Rightarrow (1 + \epsilon) = \frac{(1+2)\log \operatorname{rad}(1022)}{\log \operatorname{rad}(1045506)} = 1.49989415645304$$

$$(1 + \epsilon) = 1.49989415645304....$$

$$(1 + \epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \operatorname{rad}(c)}{\log(\operatorname{rad}(abc))}\right)$$

$$\Rightarrow (1 + \epsilon') = \left(\frac{1+1}{1}\right)\left(\frac{\log \operatorname{rad}(1023)}{\log(\operatorname{rad}(1045506))}\right) = 1.00007056236464$$

$$(1 + \epsilon') = 1.00007056236464......$$

$$q = \frac{\log(c)}{\log(\operatorname{rad}(abc))} \Rightarrow q = \frac{\log(1023^2)}{\log(\operatorname{rad}(1045506)} = q = 1.00007056236464.......$$

$$m = \frac{\log(c)}{\log(\operatorname{rad}(ab))} \Rightarrow m = \frac{\log(1045506)}{\log(\operatorname{rad}(1022))} = m = 2.00028226937616......$$

$$n = \frac{\log(c)}{\log(\operatorname{rad}(c))} \Rightarrow n = \frac{\log(1023^2)}{\log(\operatorname{rad}(1023)} = n = 2.000000.......$$

$$\left(\frac{n+q}{n}\right) = \left(\frac{2.000 + 1.00007056236464.}{2.0000}\right) = 1.50003528118232....$$

$$\left(\frac{n+q}{n}\right) = 1.50003528118232....$$

$$\left(\frac{2 + \epsilon - k\epsilon'}{1 + \epsilon}\right) = \left(\frac{2 + 0.49989415645304.. - 0.00007056236464..}{1.49989415645304..}\right) = \frac{5}{3}$$

Verified

7.14 Example 14 :a + b = c

$$\boxed{28 + 50625 = 50653 \implies 2^2 \times 7 + (3 \times 5)^4 = 37^3}$$

$$a = 28 \implies \text{rad}(a)) = 14 \quad and \quad b = 50625 \implies \text{rad}(b)) = 15$$

$$\therefore ab = 1417500 \implies \text{rad}(ab)) = 210 \quad \text{rad}(c)) = 37$$

$$\text{rad}(c)^1 < \text{rad}(ab) < \text{rad}(c)^2 \implies k = 1$$

$$\text{rad}(abc)) = 7770$$

$$\because 37^1 < 105 < 37^2 \because \gcd(a, b) = 1 \implies k = 1$$

$$\text{rad}\, c^k < (\text{rad}(ab) < \text{rad}\, c^{k+1}$$

$$(1+\epsilon) = \frac{(k+2)\log\text{rad}(c)}{\log\text{rad}(abc)} = \frac{(1+2)\log\text{rad}(37)}{\log\text{rad}(7770)}$$

$$\text{rad}(c)^k < \text{rad}(ab) < \text{rad}(c)^{k+1} \implies k = 1$$

$$(1+\epsilon) = 1.20927918841314.....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log\text{rad}(ab)}{\log(\text{rad}(abc))}\right) = \left(\frac{1+1}{1}\right)\left(\frac{\log\text{rad}(210)}{\log(\text{rad}(7770))}\right)$$

$$(1+\epsilon') = 1.19381387439124......$$

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} \Rightarrow q = \frac{\log(50653)}{\log(\text{rad}(7770))} = q = 1.20927918841314.......$$

$$m = \frac{\log(c)}{\log((\text{rad}(ab))} \Rightarrow m = \frac{\log(50653)}{\log(\text{rad}(210))} = m = 2.02590908742752.......$$

$$n = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow n = \frac{\log(50653)}{\log(\text{rad}(37))} = n = 3.000.......$$

$$q \leq \frac{5}{3} \leq \left(\frac{m+q}{m}\right) \leq \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

$$\left(\frac{m+q}{m}\right) = \frac{2.02590908742752 + 1.20927918841314}{2.02590908742752} = \frac{5}{3}$$

$$\left(\frac{2+\epsilon-k\epsilon'}{1+\epsilon}\right) = \frac{(2+0.20927918841314 - 0.19381387439124}{1.20927918841314} = \frac{5}{3}$$

$$q \leq \frac{5}{3} = \left(\frac{m+q}{m}\right) = \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} = \frac{5}{3}$$

Verified

7.15 Example 15: a + b = c

$$\boxed{192 + 2209 = 2401 \implies 2^6 \times 3 + 47^2 = 7^4}$$

$$\because a = 192 \implies (\text{rad}(a)) = 6 \quad , \because b = 2209 \implies (\text{rad}(b)) = 47 \quad , \quad , \text{rad}(c) = 7$$

$$(\text{rad}(ab)) = 282 \quad , \because c = 2401 \implies \text{rad}(c)) = 7$$

$$(\text{rad}(abc)) = 1974$$

$$\text{rad}(c)^k < (\text{rad}(ab) < \text{rad}(c)^{k+1} \because 7^2 < 282 < 7^3 \; gcd(a,b) = 1 \implies k = 2$$

$$(1+\epsilon) = \frac{(k+2)\log \text{rad}(c)}{\log \text{rad}(abc)} = \frac{(2+2)\log \text{rad}(7)}{\log \text{rad}(1974)}$$

$$(1+\epsilon) = 1.0258076032342798004070588447275 8.....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \text{rad}(ab)}{\log(\text{rad}(abc))}\right) = \left(\frac{2+1}{2}\right)\left(\frac{\log \text{rad}(282)}{\log(\text{rad}(1974))}\right)$$

$$(1+\epsilon') = 1.1153221487871450748473529332272.......$$

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} \Rightarrow q = \frac{\log((2401)}{\log(\text{rad}(1974))} = q = 1.025807603234279800407058844727 5.......$$

$$m = \frac{\log(c)}{\log(\text{rad}(ab))} \Rightarrow m = \frac{\log(2401)}{\log(\text{rad}(282))} = m = 1.3796116274788306300526238075803.......$$

$$n = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow n = \frac{\log(2401)}{\log(\text{rad}(7))} = n = 4.00........$$

$$\left(\frac{m+q}{m}\right) = \left(\frac{1.37961162747883 + 1.02580760323428}{1.37961162747883}\right) = 1.7435480991914300498982352888181$$

$$\frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} = \frac{(2+0.02580760323428 - 2(0.11532214878714)}{(1.02580760323428)} = 1.75$$

$$q \leq \frac{5}{3} = \left(\frac{m+q}{m}\right) = 1.74354809919143004 < \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} = 1.75$$

$$q < \frac{5}{3} \leqslant \left(\frac{m+q}{m}\right) \leqslant \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

Verified

7.16 Example 16. a + b = c

$$\boxed{1 + 2825760 = 2825761 \Rightarrow 1 + 2^5 \times 3 \times 5 \times 7 \times 29^2 = 41^4}$$

$$\operatorname{rad} c^2 < (\operatorname{rad}(ab) < \operatorname{rad} c^3 \because 41^2 < 6090 < 41^3 \implies k = 2 \quad \gcd(a, b) = 1$$

$$a = 1 \Rightarrow (\operatorname{rad}(a)) = 1 \quad b = 2825760 \Rightarrow (\operatorname{rad}(b)) = 6090 \Rightarrow (\operatorname{rad}(ab)) = 6090$$

$$c = 2825761 \implies \operatorname{rad}(c)) = 41 \Rightarrow (\operatorname{rad}(abc)) = 249690 \quad \& \quad k = 2$$

$$\operatorname{rad} c^k < (\operatorname{rad}(ab) < \operatorname{rad} c^{k+1}$$

$$(1+\epsilon) = \frac{(k+2)\log \operatorname{rad}(c)}{\log \operatorname{rad}(abc)} \Rightarrow (1+\epsilon) = \frac{(2+2)\log \operatorname{rad}(41)}{\log \operatorname{rad}(249690)}$$

$$(1+\epsilon) = 1.1952299353648599697148576448318.....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \operatorname{rad}(ab)}{\log(\operatorname{rad}(abc))}\right) \Rightarrow (1+\epsilon') = \left(\frac{2+1}{2}\right)\left(\frac{\log \operatorname{rad}(6090)}{\log(\operatorname{rad}(249690))}\right)$$

$$(1+\epsilon') = 1.0517887742381775113569283831881 5......$$

$$q = \frac{\log(c)}{\log(\operatorname{rad}(abc))} \Rightarrow q = \frac{\log(2825761)}{\log(\operatorname{rad}(249690))} = q = 1.19522993536485.......$$

$$m = \frac{\log(c)}{\log(\operatorname{rad}(ab))} \Rightarrow m = \frac{\log(2825761)}{\log(\operatorname{rad}(6090))} = m = 1.70456744449080830.......$$

$$n = \frac{\log(c)}{\log(\operatorname{rad}(c))} \Rightarrow n = \frac{\log(2825761)}{\log(\operatorname{rad}(41))} = n = 4.0......$$

$$q < \frac{5}{3} \leq \left(\frac{m+q}{m}\right) \leq \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

$$\left(\frac{m+q}{m}\right) = \frac{1.70456744449080830 + 1.19522993536485}{1.70456744449080830} = 1.70119251615877915$$

$$\frac{5}{3} < \left(\frac{2+\epsilon-k\epsilon'}{1+\epsilon}\right) = \frac{(2 + 0.195229935364859969 - 2(0..051788774238177 51)}{1.195229935364859969} = 1.75$$

$$q < \frac{5}{3} < \left(\frac{m+q}{m}\right) < \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)} = 1.75$$

Verified

7.17 Example 17. a + b = c, 1+63=64

$$\boxed{1 + 3^2 \times 7 = 2^6}$$

$$a = 1 \quad \text{rad}(a) = 1 \quad b = 63 \quad \text{rad}(b) = 21 \quad \text{rad}(ab) = 21 \quad \text{rad}(c) = 2 \quad \text{rad}(abc) = 42$$

$$\Rightarrow \text{rad}(c)^4 < \text{rad}(ab) < \text{rad}(c)^5 \because 2 < 3 < 2^2 \implies k = 4$$

$$c = 64 \implies \text{rad}\, c = 2 \implies 2^4 < 21 < 2^5$$

$$\text{rad}(c)^k < \text{rad}(ab) < \text{rad}(c)^{k+1} \therefore k = 4 \because \gcd(a, b) = 1$$

$$(1 + \epsilon) = \frac{(k+2)\log \text{rad}(c)}{\log \text{rad}(abc)} = \frac{(4+2)\log \text{rad}(2)}{\log \text{rad}(42)}$$

$$(1 + \epsilon) = 1.112694140492213403 2....$$

$$(1 + \epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \text{rad}(ab)}{\log(\text{rad}(abc))}\right) = \left(\frac{4+1}{4}\right)\left(\frac{\log \text{rad}(21)}{\log(\text{rad}(42))}\right)$$

$$(1 + \epsilon') = 1.01818872073078887......$$

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} \Rightarrow q = \frac{\log(64)}{\log(\text{rad}(42))} = q = 1.1126941404922134.......$$

$$m = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow m = \frac{\log(64)}{\log(\text{rad}(21))} = m = 1.366021492181717987.......$$

$$n = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow n = \frac{\log(4)}{\log(\text{rad}(2))} = n = 6.00.......$$

$$\left(\frac{m+q}{m}\right) = \frac{1.366021492181717987... + 1.112694140492213....}{1.366021492181717987...} = 1.8145509765846310...$$

$$\left(\frac{2 + \epsilon - k\epsilon'}{1 + \epsilon}\right) = \frac{(2 + 0.112694140492.... - 0.018188720730..)}{1.1126941404922134032..} = 1.882373011180678369 44 < \frac{5}{3}$$

$$q \leq \left(\frac{m+q}{m}\right) \leq \frac{(2 + \epsilon - k\epsilon')}{(1 + \epsilon)} < \frac{5}{3}$$

Verified

7.18 Example 18. a + b = c, 1+48=49

$\boxed{1+48=49 \Rightarrow 1+2^4.3=7^2}$

$$\text{rad}(ab) < (\text{rad}(c) \because 6 < 7 < 6^2 \; gcd(a,b)=1$$

$$\text{rad}(ab)^k < (\text{rad}(c) < \text{rad}(ab)^{k+1} \implies k=1$$

$$\text{rad}(ab) < (\text{rad}(c) < \text{rad}(ab)^2$$

$$a=1 \Rightarrow (\text{rad}(a))=1$$

$$b=48 \Rightarrow (\text{rad}(b))=6$$

$$(\text{rad}(ab))=6$$

$$c=49 \Rightarrow \text{rad}(c))=7$$

$$(\text{rad}(abc))=42$$

$$k=1$$

$$(1+\epsilon)=\frac{(k+2)\log\text{rad}(ab)}{\log\text{rad}(abc)} \Rightarrow (1+\epsilon)=\frac{(1+2)\log\text{rad}(6)}{\log\text{rad}(42)}$$

$$(1+\epsilon)=1.43813631397226.....$$

$$(1+\epsilon')=\left(\frac{k+1}{k}\right)\left(\frac{\log\text{rad}(c)}{\log(\text{rad}(abc))}\right) \Rightarrow (1+\epsilon')=\left(\frac{1+1}{1}\right)\left(\frac{\log\text{rad}(7)}{\log(\text{rad}(42))}\right)$$

$$(1+\epsilon')=1.04124245735182.......$$

$$q=\frac{\log(c)}{\log(\text{rad}(abc))} \Rightarrow q=\frac{\log((49)}{\log(\text{rad}(42))}=q=1.04124245735182......$$

$$m=\frac{\log(c)}{\log(\text{rad}(ab))} \Rightarrow m=\frac{\log(49)}{\log(\text{rad}(6))}=m=2.17206626500338......$$

$$n=\frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow n=\frac{\log(49)}{\log(\text{rad}(7))}=n=2.00........$$

$$q \le \frac{5}{3} \le \left(\frac{m+q}{m}\right) \le \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

$$\left(\frac{m+q}{m}\right)=\frac{2.50878434749807+1.62599052010865}{2.50878434749807}=\frac{5}{3}$$

$$\left(\frac{2+\epsilon-k\epsilon'}{1+\epsilon}\right)=\frac{(2+0.05564333770234-0.2962377748651}{1.05564333770234}=\frac{5}{3}$$

$$q \le \frac{5}{3}=\left(\frac{m+q}{m}\right)=\frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}=\frac{5}{3}$$

Verified

7.19 Example 19. a + b = c, 14641+ 282460608= 282475249

This abc triple is my invention

$$14641 + 282460608 = 282475249 \Rightarrow 11^4 + 2^6.3^4.23^2.103 = 7^{10}$$

$$\text{rad}(c)^k < (\text{rad}(ab) < \text{rad}(c)^{k+1} \because 7^6 < 156354 < 7^7 \ gcd(a,b) = 1$$

$$a = 14641 \Rightarrow (\text{rad}(a)) = 11$$

$$b = 282460608 \Rightarrow (\text{rad}(b)) = 14214$$

$$(\text{rad}(ab)) = 156354$$

$$c = 282475249 \implies \text{rad}(c)) = 7$$

$$(\text{rad}(abc)) = 1094478$$

$$c = 282475249 \implies \text{rad}(c)) = 7 \because 7^6 < 156354 < 7^7 \ gcd(a,b) = 1$$

$$\text{rad}(c)^k < (\text{rad}(ab) < \text{rad}(c)^{k+1} \implies k = 6$$

$$(1+\epsilon) = \frac{(k+2)\log \text{rad}(c)}{\log \text{rad}(abc)} \Rightarrow (1+\epsilon) = \frac{(6+2)\log \text{rad}(7)}{\log \text{rad}(1094478)}$$

$$(1+\epsilon) = 1.11948212397762.....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \text{rad}(ab)}{\log(\text{rad}(abc))}\right) \Rightarrow (1+\epsilon') = \left(\frac{6+1}{6}\right)\left(\frac{\log \text{rad}(156354)}{\log(\text{rad}(1094478))}\right)$$

$$(1+\epsilon') = 1.00340885691993.......$$

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} \Rightarrow q = \frac{\log(282475249)}{\log(\text{rad}(42))} = q = 1.39935265497202......$$

$$m = \frac{\log(c)}{\log(\text{rad}(ab))} \Rightarrow m = \frac{\log 282475249}{\log(\text{rad}(6))} = m = 1.62703177892881......$$

$$n = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow n = \frac{\log 282475249}{\log(\text{rad}(7))} = n = 10........$$

$$q \le \frac{5}{3} \le \left(\frac{m+q}{m}\right) \le \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

$$\left(\frac{m+q}{m}\right) = \frac{1.62703177892881 + 1.39935265497202}{1.62703177892881} = 1.8600647345028$$

$$\left(\frac{2+\epsilon-k\epsilon'}{1+\epsilon}\right) = \frac{(2+0.05564333770234 - 0.2962377748651}{1.05564333770234} = 1.875$$

$$q \le \frac{5}{3} \le \left(\frac{m+q}{m}\right) \le \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

Verified

7.20 Example 20. a + b = c, 243+13=256

$\boxed{243 + 13 = 256 \Rightarrow 3^5 + 13 = 2^8}$

$$\text{rad}(c)^k < (\text{rad}(ab) < \text{rad}(c)^{k+1} \because 2^5 < 39 < 2^6 \; gcd(a,b) = 1$$

$$a = 243 \Rightarrow (\text{rad}(a)) = 3$$

$$b = 13 \Rightarrow (\text{rad}(b)) = 13$$

$$(\text{rad}(ab)) = 39$$

$$c = 256 \implies \text{rad}(c)) = 2$$

$$(\text{rad}(abc)) = 78$$

$$c = 256 \implies \text{rad}(c)) = 2 \because 2^5 < 39 < 2^6 \; gcd(a,b) = 1$$

$$\text{rad}(c)^k < (\text{rad}(ab) < \text{rad}(c)^{k+1} \implies k = 5$$

$$(1+\epsilon) = \frac{(k+2)\log \text{rad}(c)}{\log \text{rad}(abc)} \Rightarrow (1+\epsilon) = \frac{(5+2)\log \text{rad}(2)}{\log \text{rad}(78)}$$

$$(1+\epsilon) = 1.11369165508505906.....$$

$$(1+\epsilon') = \left(\frac{k+1}{k}\right)\left(\frac{\log \text{rad}(c)}{\log(\text{rad}(abc))}\right) \Rightarrow (1+\epsilon') = \left(\frac{5+1}{5}\right)\left(\frac{\log \text{rad}(39)}{\log(\text{rad}(78))}\right)$$

$$(1+\epsilon') = 1.009081430556847 01766.......$$

$$q = \frac{\log(c)}{\log(\text{rad}(abc))} \Rightarrow q = \frac{\log(256)}{\log(\text{rad}(78))} = q = 1.27279046295435321555334......$$

$$m = \frac{\log(c)}{\log(\text{rad}(ab))} \Rightarrow m = \frac{\log 256}{\log(\text{rad}(39))} = m = 1.513602876134960298......$$

$$n = \frac{\log(c)}{\log(\text{rad}(c))} \Rightarrow n = \frac{\log 256}{\log(\text{rad}(2))} = n = 8........$$

$$q \leq \frac{5}{3} \leq \left(\frac{m+q}{m}\right) \leq \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

$$\left(\frac{m+q}{m}\right) = \frac{1.513602876134 + 1.272790462954353}{1.5136028.........} = 1.84090119213070.....$$

$$\left(\frac{2+\epsilon-k\epsilon'}{1+\epsilon}\right) = \frac{(2+0.113691655085... - 0.2962377748651}{1.1136916550850....} = 1.8571428571428571428.......$$

$$q \leq \frac{5}{3} \leq \left(\frac{m+q}{m}\right) \leq \frac{(2+\epsilon-k\epsilon')}{(1+\epsilon)}$$

Verified