New Tricks For Memorizing Trigonometric Identities

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Abstract

The product to sum (P2S) and sum to product (S2P) trigonometric identities are generally not memorized, but puzzled out using the sum of two angles for the P2S and then P2S for S2P. We show here a faster way to recall these using what might be called heuristic generalizations. We touch on all 27 of the standard identities.

Introduction

DeMoivre's Theorem implies

$$(\cos a/2 + i \sin a/2)^2 = \cos^2 a/2 - \sin^2 a/2 + 2i \cos a/2 \sin a/2$$

$$\cos^2 a/2 - \sin^2 a/2 + 2i \cos a/2 \sin a/2 = \cos a + i \sin a \qquad (*)$$

and we claim this motivates and suggests many of the 27 standard trigonometric identities (listed in section 3), including the allusive power to sum (P2S) and sum to power ones (S2P). You need to generalize function arguments to see the various identities. So, for example, the double angle formula for $\cos$ and $\sin$ can be seen with function argument adjustments:

$$\cos 2a = \cos^2 a - \sin^2 a \qquad (7)$$

$$\sin 2a = 2 \sin a \cos a \qquad (10).$$

Equation references are to the list of identities.

P2S

From (10) comes

$$\sin a \cos a = \frac{1}{2} \left(\sin(a + a) + \sin(a - a) \right), \tag{1}$$

where we use the simple fact that $\sin 0 = 0$. If you can write and easily believe (1), you can scratch out the second a argument and replace it with a b:

$$\sin a \cos b = \frac{1}{2} \left(\sin(a + b) + \sin(a - b) \right)$$

and you have a P2S identity (22). Test this with $a = 0$ and $b = 90$ degrees. As

$$\cos a \sin b = \sin b \cos a = \frac{1}{2} \left(\sin(b + a) + \sin(b - a) \right),$$

we get a second P2S:

$$\cos a \sin b = \sin b \cos a = \frac{1}{2} \left(\sin(a + b) - \sin(a - b) \right).$$

That's (23).

One can remember the reduction (R) formulas using

$$\cos^2 a + \sin^2 a = \frac{1 + \cos 2a}{2} + \frac{1 - \cos 2a}{2}$$

implies (or suggests)

$$\cos^2 a = \frac{1 + \cos 2a}{2} \text{ and } \sin^2 a = \frac{1 - \cos 2a}{2}$$

From these using the same heuristic generalization idea, we have

$$\cos a \cos a = \frac{1}{2} \left(\cos(a - a) + \cos(a + a) \right),$$

using $\cos 0 = 1$. Thence a simple scratch out the second a and replace it with a b for two more P2S identities:

$$\cos a \cos b = \frac{1}{2} \left(\cos(a - b) + \cos(a + b) \right)$$

and

$$\sin a \sin b = \frac{1}{2} \left(\cos(a - b) - \cos(a + b) \right)$$

That's (21) and (20).

S2P

Some sum to product identities are even easier to recall. For example, just add and recall that $\cos 0 = 1$, for

$$\sin a + \sin a = 2\sin\left(\frac{a}{2}+\frac{a}{2}\right)\cos\left(\frac{a}{2}-\frac{a}{2}\right).$$

Working with pencil or pen, just scratch out the second as and replace them with bs for

$$\sin a + \sin b = 2\sin\left(\frac{a}{2}+\frac{b}{2}\right)\cos\left(\frac{a}{2}-\frac{b}{2}\right).$$

That's (24). This heuristic generalization step at some point might not be necessary. I suspect

$$\cos a + \cos b = 2\cos\left(\frac{a}{2}+\frac{b}{2}\right)\cos\left(\frac{a}{2}-\frac{b}{2}\right)$$

and, indeed, that's (26).

One subtraction identity is easy enough. Just using $\sin$ is odd we have

$$\sin a - \sin b = \sin a + \sin(-b) = 2\sin\left(\frac{a}{2}-\frac{b}{2}\right)\cos\left(\frac{a}{2}+\frac{b}{2}\right),$$

using (24): (25).

The difference in cosines is a slight stretch. We need a couple of minor identities:

$$\sin\left(a \pm \frac{\pi}{2}\right) = \pm\cos a \text{ and } \cos\left(a \pm \frac{\pi}{2}\right) = \mp\sin a.$$

Then, using (25),

$$\cos a - \cos b = \sin(a+\pi/2) - \sin(b+\pi/2) = \sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}+\frac{\pi}{2}\right),$$

giving

$$\cos a - \cos b = -2\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right),$$

for (27).

Basic Identities (27)

The 27 identities can be categorized with symbols (number in each category): $\pm$ (6); $2a$ (5); R (3); $a/2$ (5); $P2S$ (4); and $S2P$ (4). We'll show how (*) so motivates these six categories: $trig(a \pm b, 2a, a/2)$; (R)eduction: square to 2a; power to sum (P2S), sum to power (S2P). Here they are.

$$= trig(a \pm b(6), 2a(3+2\cos=5), R(3), a/2(3+2\tan=5)) + P2S(4) + S2P(4)$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \tag{1}$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b \tag{2}$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \tag{3}$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b \tag{4}$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \tag{5}$$

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} \tag{6}$$

$$\cos(2a) = \cos^2 a - \sin^2 a \tag{7}$$

$$\cos(2a) = 2\cos^2 a - 1 \tag{8}$$

$$\cos(2a) = 1 - 2\sin^2 a \tag{9}$$

$$\sin(2a) = 2 \sin a \cos a \tag{10}$$

$$\tan(2a) = \frac{2\tan a}{1 - \tan^2 a} \tag{11}$$

$$\cos^2 a = \frac{1 + \cos 2a}{2} \tag{12}$$

$$\sin^2 a = \frac{1 - \cos 2a}{2} \tag{13}$$

$$\tan^2 a = \frac{1 - \cos 2a}{1 + \cos 2a} \tag{14}$$

$$\cos a/2 = \pm\sqrt{\frac{1 + \cos a}{2}} \tag{15}$$

$$\sin a/2 = \pm\sqrt{\frac{1 - \cos a}{2}} \tag{16}$$

$$\tan a/2 = \pm\sqrt{\frac{1 - \cos a}{1 + \cos a}} \tag{17}$$

$$\tan a/2 = \frac{\sin a}{1 + \cos a} \tag{18}$$

$$\tan a/2 = \frac{1 - \cos a}{\sin a} \tag{19}$$

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] \tag{20}$$

$$\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right] \tag{21}$$

$$\sin a \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right] \tag{22}$$

$$\cos a \sin b = \frac{1}{2}\left[\sin(a+b) - \sin(a-b)\right] \tag{23}$$

$$\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2} \tag{24}$$

$$\sin a - \sin b = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2} \tag{25}$$

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2} \tag{26}$$

$$\cos a - \cos b = -2\sin\frac{a+b}{2}\sin\frac{a-b}{2} \tag{27}$$

$\pm$

One can read (*) as

$$\cos(a+a) = \cos a \cos a - \sin a \sin a$$

and using our heuristic generalization we arrive at (1):

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

Similarly, $\sin(2a) = 2 \sin a \cos a$ can be re-written as

$$\sin(a+a) = 2 \sin a \cos a = \sin a \cos a + \cos a \sin a$$

and with a heuristic generalization this is

$$\sin(a+b) = \sin a \cos b + \cos a \sin b,$$

(3).

Knowing that $\sin$ is an odd function and $\cos$ is even, one can further infer:

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

and

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b.$$

2a

A handy mnemonic to from $\cos 2a = \cos^2 a - \sin^2 a$ to

$$\cos 2a = 2\cos^2 a - 1$$

and

$$\cos 2a = 1 - 2\sin^2 a$$

is to remember the 2 goes in front of the one present and a 1 replaces the one not present. Its another scratch out and replace idea, especially easy with pen or pencil.

R

The R stands for reduction; as in, reducing a square to a double angle. These identities are used in integration as substitutions are easy, squares of functions less so.

As was mentioned, one can remember the reduction (R) formulas using

$$1 = \cos^2 a + \sin^2 a = \frac{1 + \cos 2a}{2} + \frac{1 - \cos 2a}{2}$$

implies (or heuristically suggests)

$$\cos^2 a = \frac{1 + \cos 2a}{2} \text{ and } \sin^2 a = \frac{1 - \cos 2a}{2}.$$

Curiously $\cos(a+a) = \cos(a-a) - 2\sin a \sin a$ can be re-written as

$$\cos(a+a) - \cos(a-a) = -2\sin a \sin a,$$

reminiscent of (27).

a/2

In addition to jogging into memory the reduction (R) formulas using

$$1 = \cos^2 a + \sin^2 a = \frac{1 + \cos 2a}{2} + \frac{1 - \cos 2a}{2},$$

this one equation works for half angle formulas. Starting with a change in arguments giving

$$\cos^2 a/2 = \frac{1 + \cos a}{2} \text{ and } \sin^2 a/2 = \frac{1 - \cos a}{2}$$

and then taking square roots:

$$\cos a/2 = \pm\sqrt{\frac{1 + \cos a}{2}} \text{ and } \sin a/2 = \pm\sqrt{\frac{1 - \cos a}{2}}.$$

References

[1] Blitzer, R. (2010). *Algebra and Trignometry*, 3rd ed., Pearson.