Signal Transmission in the Schwarzschild Metric: The Case of Forced Motion at a Constant Fraction of the Speed of Light

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Abstract

In this article, we examine the case of a body following a nongeodesic centripetal trajectory in the Schwarzschild metric while maintaining a constant fraction of the speed of light. At any moment, light signals can be sent to the body and recovered after reflecting off its surface. Nevertheless, in its own comoving reference frame, the body would require only a finite amount of proper time to reach the Schwarzschild radius. This raises a paradox: the ability to send and recover signals implies that the body remains outside the event horizon, contradicting the usual interpretation of gravitational collapse based on finite proper time. As the body follows a nongeodesic trajectory, we compute the four-force necessary to sustain its motion.

1 Introduction

In a previous study, we analyzed the transmission of light signals between a freely falling body following a geodesic trajectory and a fixed point in the Schwarzschild metric [1]. We showed that if the body begins its motion at $t_0 = 0$, there exists a critical time $\Delta t_{\max}$ beyond which no light signal emitted from a fixed point at $r = r_0$ can ever reach the freely falling body. This observation might seem consistent with the common interpretation that a freely falling body reaches the event horizon in a finite proper time, while a distant observer perceives the process as infinite due to the increasing delay of light signals emitted near the Schwarzschild radius.

Yet, we previously argued against this explanation by drawing an analogy with special relativity: A particle asymptotically approaching the speed of light also ceases to receive signals from a fixed point after a certain time. While it would reach the speed of light in its own reference frame in a finite proper time, it distinctly never attains that speed. Thus, the requirement of a finite proper time to complete a process does not necessarily imply that the process is ever fully completed, as the system's "clocks" may slow down to the extent that the finite proper time interval never effectively elapses.

To reinforce this argument, we now examine the motion of a body moving at a constant fraction β of the speed of light. Since the trajectory is non-geodesic, an external force is required to sustain it, such as a hypothetical tether to a continuously operating rocket. Although maintaining such a force indefinitely is practically unfeasible, no fundamental physical law forbids it. We show that even though the body would reach the Schwarzschild radius in a finite proper time, light signals can still be sent to it at arbitrarily late times and recovered. This proves that the body remains outside the event horizon not merely in appearance but as a physical reality.

2 Mathematical formulation and analysis

As is well known, the Schwarzschild metric can be written as follows [1]:

$$ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{1}$$

Since we consider purely radial trajectories, with $d\theta = d\phi = 0$, the motion satisfies the following condition:

$$ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2. \tag{2}$$

2.1 Motion of a body at a constant fraction β of the speed of light

Light trajectories satisfy the condition

$$ds^2 = 0,$$

which implies that

$$\left|\frac{dr}{dt}\right| = c\left(1 - \frac{2GM}{c^2 r}\right). \tag{3}$$

Now, consider a body descending from an initial radius r_0 to the event horizon at $r_S = \frac{2GM}{c^2}$, while maintaining a constant fraction β of the speed of light:

$$\frac{dr}{dt} = -\beta c\left(1 - \frac{2GM}{c^2 r}\right) = -\beta c \frac{r - \frac{2GM}{c^2}}{r}, \tag{4}$$

where the negative sign takes into account that the motion is directed inward.

To solve equation (4), we introduce the following change of variables:

$$r^* = r - \frac{2GM}{c^2}, \quad dr = dr^*, \tag{5}$$

so that

$$\frac{dr^*}{dt} = -\beta c \frac{r^*}{r^* + \frac{2GM}{c^2}} \implies \left(1 + \frac{2GM}{c^2 r^*}\right) dr^* = -\beta c dt. \tag{6}$$

Integrating both sides, we obtain:

$$\left[r^* + \frac{2GM}{c^2}\ln r^*\right]_{{r^*}_0}^{r^*(t)} = \left[-\beta c t\right]_{t_0}^{t}, \tag{7}$$

which yields

$$r^* - {r^*}_0 + \frac{2GM}{c^2}\ln\left(\frac{r^*}{{r^*}_0}\right) = -\beta c(t - t_0), \tag{8}$$

or, in terms of the original coordinates,

$$r - r_0 + \frac{2GM}{c^2}\ln\left(\frac{r - \frac{2GM}{c^2}}{r_0 - \frac{2GM}{c^2}}\right) = -\beta c(t - t_0). \tag{9}$$

We set $t_0 = 0$ as the moment when the body departs from $r = r_0$, so that

$$r^* - {r^*}_0 + \frac{2GM}{c^2}\ln\left(\frac{r^*}{{r^*}_0}\right) = -\beta c t, \tag{10}$$

$$r - r_0 + \frac{2GM}{c^2}\ln\left(\frac{r - \frac{2GM}{c^2}}{r_0 - \frac{2GM}{c^2}}\right) = -\beta c t. \tag{11}$$

For $t \gg t_0$, where $r^* \approx 0$, $r_0^* \approx 0$, and $r \approx \frac{2GM}{c^2}$, the equations (10) and (11) can be approximated as follows:

$$r^* \approx {r^*}_0 e^{\frac{c^2}{2GM}({r^*}_0 - \beta c t)}, \tag{12}$$

$$r \approx \frac{2GM}{c^2} + \left(r_0 - \frac{2GM}{c^2}\right) e^{\frac{c^2 r_0}{2GM} - 1} e^{-\frac{c^2}{2GM}\beta c t}. \tag{13}$$

2.2 Signal transmission

Assume that at $t_0 = t_1$, we send a light signal to the falling body from r_0. Since for light $\beta = 1$, its trajectory, according to equation (9), is given by:

$$r_L - r_0 + \frac{2GM}{c^2} \ln \left(\frac{r_L - \frac{2GM}{c^2}}{r_0 - \frac{2GM}{c^2}} \right) = -\beta c(t - t_1). \tag{14}$$

The light signal reaches the body at $t = t_a$, when $r_L = r$. From equations (11) and (14), this occurs at

$$c(t_a - t_1) = c\beta t_a \implies t_a = \frac{t_1}{1-\beta}. \tag{15}$$

Therefore, the time interval Δt required for the light to reach the body is

$$\Delta t = t_a - t_1 = \frac{\beta}{1-\beta} t_1. \tag{16}$$

Since the Schwarzschild metric is time-independent, if the light signal is reflected by the body and returns to $r = r_0$, it will take the same time interval Δt to travel back. Consequently, it will reach the original point at

$$t_b = t_a + \Delta t = \frac{1+\beta}{1-\beta} t_1. \tag{17}$$

Thus, at any time t_1, we can always send a light signal to the body and receive it after reflection. We must therefore conclude that, in our reference frame, the body is not merely *apparently* outside the event horizon, it is actually outside it.

2.3 Motion of the body in proper time

Since $d\tau = \frac{ds}{c}$, using equations (2) and (4), and dividing (2) by dt^2, we obtain:

$$\left(\frac{d\tau}{dt}\right)^2 = \left(1 - \frac{2GM}{c^2 r}\right) - \frac{1}{c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}\left(\frac{dr}{dt}\right)^2 = (1-\beta^2)\left(1 - \frac{2GM}{c^2 r}\right). \tag{18}$$

$$\frac{d\tau}{dt} = \sqrt{1-\beta^2}\sqrt{1 - \frac{2GM}{c^2 r}}. \tag{19}$$

Alternatively, dividing (2) by $d\tau^2$ instead of dt^2, and using $\frac{dt}{d\tau} = \left(\frac{d\tau}{dt}\right)^{-1}$, we obtain:

$$1 = \left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2. \tag{20}$$

$$\left(\frac{dr}{d\tau}\right)^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)^2\left(\frac{dt}{d\tau}\right)^2 - c^2\left(1 - \frac{2GM}{c^2 r}\right) = c^2\frac{\beta^2}{1-\beta^2}\left(1 - \frac{2GM}{c^2 r}\right). \tag{21}$$

$$\frac{dr}{d\tau} = -\frac{c\beta}{\sqrt{1-\beta^2}}\sqrt{1 - \frac{2GM}{c^2 r}}, \tag{22}$$

where we take into account that we are considering a centripetal motion, so that $\frac{dr}{d\tau} < 0$.
To solve equation (22), we apply again the change of variables given by equation (5):

$$\frac{dr^*}{d\tau} = -\frac{c\beta}{\sqrt{1-\beta^2}}\sqrt{\frac{r^*}{r^* + \frac{2GM}{c^2}}}. \tag{23}$$

Proceeding by separation of variables, we obtain:

$$d\tau = -\frac{\sqrt{1-\beta^2}}{c\beta}\sqrt{\frac{r^* + \frac{2GM}{c^2}}{r^*}}\,dr^*. \tag{24}$$

By integration ([4], pages 63 and 64),

$$[d\tau]_0^{\Delta\tau} = -\frac{\sqrt{1-\beta^2}}{c\beta}\left[\sqrt{r^*\left(r^*+\frac{2GM}{c^2}\right)} + \frac{2GM}{c^2}\ln\left(\sqrt{1+\frac{c^2r^*}{2GM}} + \sqrt{\frac{c^2r^*}{2GM}}\right)\right]_{r^*_0}^{r^*(t)}. \quad (25)$$

$$\Delta\tau = K_0 - \frac{\sqrt{1-\beta^2}}{c\beta}\left[\sqrt{r^*\left(r^*+\frac{2GM}{c^2}\right)} + \frac{2GM}{c^2}\ln\left(\sqrt{1+\frac{c^2r^*}{2GM}} + \sqrt{\frac{c^2r^*}{2GM}}\right)\right], \quad (26)$$

where

$$K_0 = \frac{\sqrt{1-\beta^2}}{c\beta}\left[\sqrt{r^*_0\left(r^*_0+\frac{2GM}{c^2}\right)} + \frac{2GM}{c^2}\ln\left(\sqrt{1+\frac{c^2r^*_0}{2GM}} + \sqrt{\frac{c^2r^*_0}{2GM}}\right)\right]. \quad (27)$$

For large times, when $r^* \sim 0$, we use the approximations:

$$\sqrt{r^*\left(r^*+\frac{2GM}{c^2}\right)} \approx \sqrt{\frac{2GM}{c^2}r^*}, \quad (28)$$

$$\ln\left(\sqrt{1+\frac{c^2r^*_0}{2GM}} + \sqrt{\frac{c^2r^*_0}{2GM}}\right) \approx \ln\left(1+\sqrt{\frac{c^2r^*_0}{2GM}}\right) \approx \sqrt{\frac{c^2r^*_0}{2GM}}. \quad (29)$$

Thus, for large times:

$$\Delta\tau \approx K_0 - \frac{\sqrt{1-\beta^2}}{c\beta}\sqrt{\frac{8GM}{c^2}r^*}. \quad (30)$$

We note that an equivalent expression could be obtained by simplifying equation (24) as follows:

$$d\tau = -\frac{\sqrt{1-\beta^2}}{c\beta}\sqrt{1+\frac{2GM}{c^2r^*}}dr^* \approx -\frac{\sqrt{1-\beta^2}}{c\beta}\sqrt{\frac{2GM}{c^2r^*}} \implies$$

$$\implies \Delta\tau \approx -\frac{\sqrt{1-\beta^2}}{c\beta}\sqrt{\frac{8GM}{c^2}r^*} + C. \quad (31)$$

Expressing $\Delta\tau$ in terms of t instead of r using equation (12) in (30):

$$\Delta\tau \approx K_0 - \frac{\sqrt{1-\beta^2}}{c\beta}\sqrt{\frac{8GM}{c^2}r^*_0 e^{\frac{c^2}{4GM}(r^*_0-\beta ct)}}. \quad (32)$$

The proper time $\Delta\tau_{r_S}$ at which the body reaches the event horizon in its own reference frame occurs at $t \to \infty$, yielding:

$$\Delta\tau_{r_S} = \lim_{t\to\infty} \Delta\tau = K_0. \quad (33)$$

However, we can send and recover signals from the body at any time t, indicating that it has not crossed the event horizon: the finite proper time $\Delta\tau_{r_S}$ has not *yet* elapsed for it.

2.4 Calculation of the Four-Force

The equation for the four-force in general relativity is given by:

$$f^\mu = m\left(\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda\right), \quad (34)$$

where m is the mass of the body, u^μ represents the four-velocity components, and $\Gamma^\mu_{\nu\lambda}$ are the Christoffel symbols.

The Christoffel symbols for the Schwarzschild metric are given by the following set of equations:

$$\Gamma^t_{tr} = \frac{\frac{GM}{c^2}}{r\left(r - \frac{2GM}{c^2}\right)}, \quad \Gamma^r_{tt} = \frac{GM}{r^3}\left(r - \frac{2GM}{c^2}\right), \quad \Gamma^r_{rr} = -\frac{\frac{GM}{c^2}}{r\left(r - \frac{2GM}{c^2}\right)}, \tag{35}$$

$$\Gamma^r_{\theta\theta} = -\left(r - \frac{2GM}{c^2}\right), \quad \Gamma^r_{\phi\phi} = -\left(r - \frac{2GM}{c^2}\right)\sin^2\theta, \quad \Gamma^\theta_{r\theta} = \frac{1}{r}, \tag{36}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^\phi_{\theta\phi} = \cot\theta. \tag{37}$$

We restrict our analysis to late times, when $r^* \ll \frac{2GM}{c^2}$. Under this assumption, the Christoffel symbols simplify to:

$$\Gamma^t_{tr} = \frac{\frac{GM}{c^2}}{\left(r^* + \frac{2GM}{c^2}\right)r^*} \approx \frac{1}{2r^*}, \tag{38}$$

$$\Gamma^r_{tt} = \frac{GM}{\left(r^* + \frac{2GM}{c^2}\right)^3}r^* \approx \frac{c^6 r^*}{8G^2M^2}, \tag{39}$$

$$\Gamma^r_{rr} = -\frac{\frac{GM}{c^2}}{\left(r^* + \frac{2GM}{c^2}\right)r^*} \approx -\frac{1}{2r^*}, \tag{40}$$

$$\Gamma^r_{\theta\theta} = -r^*, \quad \Gamma^r_{\phi\phi} = -r^*\sin^2\theta, \tag{41}$$

$$\Gamma^\theta_{r\theta} = \frac{c^2}{2GM}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \tag{42}$$

$$\Gamma^\phi_{r\phi} = \frac{c^2}{2GM}, \quad \Gamma^\phi_{\theta\phi} = \cot\theta. \tag{43}$$

Since we consider a purely radial motion, we have $u^\theta = u^\phi = 0$. Furthermore, all Christoffel symbols that include the indices θ or ϕ appear in terms that necessarily include at least one factor of u^θ or u^ϕ in the second term on the right-hand side of Eq. (34). This leads to:

$$f^\theta = f^\phi = 0. \tag{44}$$

Thus, only f^t and f^r need to be computed.

At late times, the components of the four-velocity are given by:

$$u^r = \frac{dr}{d\tau} = \frac{-c\beta}{\sqrt{1-\beta^2}}\sqrt{1 - \frac{2GM}{c^2 r}} = \frac{-c\beta}{\sqrt{1-\beta^2}}\sqrt{\frac{r^*}{r^* + \frac{2GM}{c^2}}} \approx \frac{-c\beta}{\sqrt{1-\beta^2}}\sqrt{\frac{c^2 r^*}{2GM}}, \tag{45}$$

$$u^t = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\beta^2}}\frac{1}{\sqrt{1 - \frac{2GM}{c^2 r}}} = \frac{1}{\sqrt{1-\beta^2}}\sqrt{\frac{r^* + \frac{2GM}{c^2}}{r^*}} \approx \frac{1}{\sqrt{1-\beta^2}}\sqrt{\frac{2GM}{c^2 r^*}}. \tag{46}$$

To compute the components of the four-force, we first determine $\frac{du^r}{d\tau}$ and $\frac{du^t}{d\tau}$. As $\tau = \tau(r^*)$, using the chain rule:

$$\frac{du^r}{d\tau} = \frac{du^r}{dr^*}\frac{dr^*}{d\tau} \approx \frac{c^2\beta^2}{1-\beta^2}\frac{c^2}{4GM}, \tag{47}$$

$$\frac{du^t}{d\tau} = \frac{du^t}{dr^*}\frac{dr^*}{d\tau} \approx \frac{c\beta}{1-\beta^2}\frac{1}{2r^*}. \tag{48}$$

2.4.1 Calculation of f^r

The radial component of the four-force is given by:

$$\frac{f^r}{m} = \frac{du^r}{d\tau} + \Gamma^r_{tt}(u^t)^2 + 2\Gamma^r_{tr}u^t u^r + \Gamma^r_{rr}(u^r)^2. \tag{49}$$

Substituting the known values:

$$\frac{f^r}{m} = \frac{1}{1-\beta^2}\frac{c^4}{4GM}. \tag{50}$$

2.4.2 Calculation of f^t

The temporal component of the four-force is given by:

$$\frac{f^t}{m} = \frac{du^t}{d\tau} + \Gamma^t_{tt}(u^t)^2 + 2\Gamma^t_{tr}u^t u^r + \Gamma^t_{rr}(u^r)^2. \tag{51}$$

Since $\Gamma^t_{tt} = 0$ and $\Gamma^t_{rr} = 0$, we get:

$$\frac{f^t}{m} = \frac{-c\beta}{1-\beta^2}\frac{1}{2r^*}. \tag{52}$$

It can be observed that the "power" required to slow down the body diverges as it approaches the Schwarzschild radius. However, it remains finite at any finite distance from the horizon.

3 Conclusion

The results indicate that the finiteness of proper time is not a sufficient criterion for determining whether an object crosses the event horizon. The ability to send and recover signals at arbitrarily late times demonstrates that the body never crosses the Schwarzschild radius. This suggests that the clocks of its system may slow down to such an extent that the finite proper time interval never elapses. By analogy, the usual interpretation of gravitational collapse should thus be reconsidered.

References

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