On Duality between Lines and Points

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Abstract

There are several notions of duality between lines and points. This note shows that all these can be studied in a unified way. Most interesting properties are independent of specific choices.

It is also shown that dual mapping can be its own inverse or preserve relative order (but not both).

1 Dual Mapping

A non-vertical line is determined by two parameters— e.g., m and c in $y = mx + c$ or a and b in $\frac{x}{a} + \frac{y}{b} = 1$. And a point in 2-d also requires two parameters (coordinates, e.g., x and y in Cartesian and r and θ in polar).

Various authors suggest different ways to map points to lines and vice-versa

Ja'Ja'[4] Point (r, s) is mapped to line $y = rx + s$ and line $y = mx + c$ is mapped to point $(-m, c)$

O'Rourke[1] O.Rourke [1] suggests two mappings: the first maps line $y = mx + c$ to point (m, c) and conversely.

The other mapping, the one he actually uses, maps point (r, s) to line $y = 2rx - c$ and conversely.

Berg et.al.[3] point (r, s) is mapped to line $y = rx - s$, and the dual of line $y = mx + c$ is the point $(m, -c)$.

Let us look at general dual mapping in which point (r, s) is mapped to the line $y = \alpha rx + \beta s$. And a line $y = mx + c$ gets mapped to the point $(\mu m, \lambda c)$.

Observation 1 *[1] There is a one-to-one correspondence between all non-vertical lines and all points in the plane.*

We want the following property to be true for the mapping:

If a point p lies on a line L, then dual of line L, say $d(L)$ (is a point) $d(L)$, which should lie on

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dual of point p, say $d(p)$ (which is a line $d(p)$).

A general line through point (r, s) will be $y - s = m(x - r)$. This line gets mapped to a point $(\mu m, \lambda(s - mr))$. As this point must lie on line $y = \alpha rx + \beta s$, we get

$$\begin{aligned} \lambda(s - mr) &= \alpha(\mu m) + \beta s \text{ or} \\ s(\lambda - \beta) &= m(\alpha\mu + r\lambda \text{ or} \\ \lambda &= \beta \text{ and} \\ \alpha\mu &= -\lambda \text{ or} \\ \alpha\mu &= -\beta \end{aligned}$$

Thus, the **permissible transforms** will map
point (r, s) to line $y = \alpha(rx - \mu s)$ and will map
line $y = mx + c$ to point $(\mu m, -\alpha\mu c)$.

Lemma 1 *If $d(\ )$ is a permissible dual transform, then a point p lies on a line L, then the dual of line L, $d(L)$ lies on $d(p)$, the dual of point p.*

Corollary 1 *[1] Two lines L_1 and L_2 intersect in a point P, iff, dual $d(P)$ passes through $d(L_1)$ and $d(L_2)$.*

As we have two free parameters, we can impose additional "desirable" conditions. Two popular desirable conditions are

1. If the dual of point p is line L, then the dual of line L should be point p[3, 1, 5].
2. If point p lies above line L, then the dual of p should lie above the dual of L[4].

Let us first look at the first condition. Dual of line $y = (\alpha r)x + (-s\mu\alpha)$ will be point $(\mu(\alpha r), (-\mu\alpha)(-s\mu\alpha))$. For this point to be (r, s) we need $\mu\alpha = 1$.

Lemma 2 *If $\alpha\mu = 1$, then dual d is its own inverse.*

Remark These transforms have only one free parameter. Point (r, s) gets mapped to line $y = \alpha rx - s)$, and line $y = mx + c$ gets mapped to the point $(\frac{1}{\alpha}m, -c)$.

Let us now look at the second condition. A point (r, s) is above the line $y = mx + c$ if $s - rm > c$ or $0 > c + rm - s$ or $c + rm - s < 0$.

The dual of point (r, s) is the line $y = \alpha(rx - \mu s)$, and the dual of line $y = mx + c$ is the point $(\mu m, -\alpha\mu c)$. Line $y = \alpha(rx - \mu s)$ is above the point $(\mu m, -\alpha\mu c)$, if

$$(-\alpha\mu c) - \alpha r(\mu m) < -\alpha\mu s$$

or

$$-(\alpha\mu)(c + rm - s) < 0$$

As $c + rm - s < 0$, $-(c + rm - s) > 0$ or $\alpha\mu < 0$.

Lemma 3 *If $\alpha\mu < 0$, then if a point p lies above line L, then $d(p)$ lies above $d(L)$. And if*

If $\alpha\mu > 0$, then if a point p lies above line L, $d(p)$ lies below $d(L)$ (i.e. order gets reversed).

Observation 2 *Both Conditions can not be simultaneously satisfied.*

If we define the *vertical distance*[5] between a point (r, s) and line $y = mx + c$ to be $|mr + c - s|$, then, from the proof of lemma, in the dual space the vertical distance gets scaled by $|\alpha\mu|$.

Parallel lines $y = mx + c$ and $y = mx + b$ will be mapped to points $(\mu m, -\alpha\mu c)$ and $(\mu m, -\alpha\mu b)$. Or (vertical) distance between these two points is $|\alpha\mu(b - c)|$.

Thus, to preserve vertical distance $\alpha\mu = \pm 1$.

Har-Peled [5, Exercise 31.2] observes that no duality can preserve exact orthogonal distances between points and lines.

2 Applications

Assume S is a set of points in the plane. A point p is on the (upper) convex hull, if and only if there is a non-vertical line L through p s.t., all points of S are below the line L.

There is a line in dual space for each point of S (including p). The dual of L is a point which lies on line $d(P)$. All dual lines $d(S)$ should be on one side of $d(L)$. Or $d(L)$ lies on (upper or lower, depending on the sign of $\alpha\mu$) envelope of lines $d(S)$.

Let us consider the intersection of half-planes. Each half-plane is either $y \leq m_i x + c_i$, which will be called **top constraint** or $y \geq m_i x + c_i$, which will be called **bottom constraint**.

The "feasible region" for top (respectively, bottom) constraints will be a convex region. Assume point (x_0, y_0) is on the boundary of this convex region. As it satisfies all top constraints, it is below lines $y = m_i x + c_i$ (if i is a top constraint). And as it is on the boundary, it is on at least one such line. In dual space, point (x_0, y_0) gets mapped to a line (say L_0), and each line $y = m_i x + c_i$ will get mapped to a point (say p_i). Moreover, all points p_i in dual space will be on one side of the line L_0, and at least one point will be on this line. Thus, line L_0 will be tangent to the convex hull.

An edge e in the convex hull (in dual space) is a line between two dual points, say $d(L_1)$ and $d(L_2)$. This edge e will correspond to a point (say X) in the untransformed domain. As (dual) line $d(X)$ (which contains the segment e) passes through points $d(L_1)$ and $d(L_2)$, point X will lie on lines L_p and L_q, i.e., point X will be the point of intersection of these two lines.

Thus, line segments of the convex hull define the extreme points of feasible solution space, and both sets occur in the same relative order. Let us assume we have determined the convex hull. We can, similarly, deal with bottom constraints B. As both these hulls are sorted (say on x coordinate), these can be merged in linear time. The merged sets define a vertical slab, in which there are at most two boundary segments, one from lower and one from upper. Hence, we can determine the boundary in linear time.

In linear programming, assume we wish to maximise $cx + dy$. As each corner point (p_i, q_i) of solution space, we find $V_i = cp_i + dq_i$ and find the largest V_i. This again takes linear time if the solution space is known.

The kernel of a polygon is the region from which the entire polygon is visible. We interpret each polygon edge as a half-plane (the side containing the interior). The intersection of these regions will give the kernel.

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