An algorithmic approach to solving the Collatz/Syracuse problem

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Abstract

After introducing definitions related to the Collatz problem (Part 1), the concept of verified integers and several organizational rules around this concept are presented (Part 2). A unique logical tool, the axis of verified integers, is highlighted (Part 3). In Part 4, it is proven that all bounded trajectories without non-trivial cycles are verified. These elements allow the development of a systematic approach to solving the Collatz problem with the help of inverse graphs (Part 5). The issue of non-trivial cycles and divergent trajectories is then explored (Parts 6 and 7).

Ultimately, we arrive at two contradictory propositions:

1. Either all integers satisfy the Collatz conjecture, or
2. An infinite number of integers do not satisfy it.

This eliminates the possibility that only a small number of integers fail to satisfy the conjecture, while the rest do. The conclusion of this study leans toward the first solution : all integers satisfy the Collatz conjecture.

Presentation of the problem

The Collatz problem (also known as the (3n + 1) problem or the Syracuse problem) was first proposed by Lothar Collatz in the 1930s. Its statement is remarkably simple : the Collatz function associates each positive integer *n* with a unique successor *n′* such that:

- n′ = n / 2 if *n* is even
- n′ = (3n+1) / 2 if *n* is odd

The successive results of iterating the Collatz function from any given integer *n* are referred to as the « Syracuse trajectory of *n* ».

The Collatz problem is to determine whether all Syracuse trajectories, starting from any positive integer, eventually reach the value 1 in a finite number of iterations. The Collatz conjecture is the hypothesis that the answer is positive, meaning that all Syracuse trajectories tend toward 1.

Despite its simple formulation, this problem has remained unsolved for over 80 years. The primary difficulty lies in the fact that the operations of multiplication by (3n+1) / 2 or division by 2, depending on the parity of the numbers obtained during the successive iterations, are difficult to predict based on the initial value. These operations produce unpredictable results, resembling random phenomena.

The approach proposed in this text is algorithmic in nature. Using various logical techniques and an inverse graph-based framework, it aims to provide a positive solution to the Collatz conjecture. The significance of this approach is that other existing proofs obtained through different methods can complement this algorithmic technique, reinforcing the conclusions presented here.

PART 1

Definitions related to the Collatz problem

1) The Collatz or Syracuse function

Let's define the function C, known as the Collatz or Syracuse function, which associates with any integer n € N* a unique integer C(n) € N* such that :

if *n* is even, C(n) = n / 2

if *n* is odd, C(n) = (3n + 1) / 2

2) The trajectories T_n

For any n € N* and i € N, we will define a sequence T_n (Syracuse trajectory with the initial value *n*) composed of integers, which we will represent in the form S_i (n) € N* with :

S_0 (n) = n

$S_{i+1}(n) = C(S_i(n))$ where C is the function defined previously.

Thus, we can express T_n as follows : for any *n* and *i* belonging to N*,

$T_n = \{ n ; S_1(n) ; ... ; S_i(n) ; S_{i+1}(n) ; ...\}$

For example, we have :

$T_1 = \{ 1; 2 ; 1 ; 2 ; 1 ; ...\}$

$T_3 = \{ 3 ; 5 ; 8 ; 4 ; 2 ; 1 ; 2 ; ... \}$

3) The set of verified integers V

We will define *n* as a « verified integer » if there exists an integer *k* such that $S_k(n) = 1$

We will call V the set of verified integers.

Proving the Collatz conjecture involves proving that V = N*.

4) Verification of a trajectory

The verification of a trajectory T_n consists of determining whether this trajectory converges to 1, meaning if there exists an integer *k* such that $S_k(n) = 1$.

5) Even and Odd iterations

In the following text, the general terms « odd iteration » or « even iteration » will refer to an odd or even value, respectively, of any S_k (n) belonging to any T_n (thus leading to an odd iteration of the form $C(S_k (n)) = (3S_k (n) + 1)/2$ or an even iteration of the form $C(S_k (n)) = (S_k (n))/2$, depending on the parity of S_k (n)). For example, saying « there are three consecutive even iterations in this trajectory » means that three even values occured in succession, leading to three even iterations according to the Syracuse function.

PART 2

Organizational rules for the set of verified integers

We can initialize the set of verified integers V by successively verifying the trajectories for the first integers starting from 1 :

T_1 = { 1 ; 2 ; 1 ; 2 ... }
T_2 = { 2 ; 1 ; 2 ; 1 ... }
T_3 = { 3 ; 5 ; 8 ; 4 ; 2 ; 1 ; 2 ... }
T_5 = { 5 ; 8 ; 4 ; 2 ; 1 ; 2 ... }
T_7 = { 7 ; 11 ; 17 ; 26 ; 13 ; 20 ; 10 ; 5 ; 8 ; 4 ... }
T_9 = { 9 ; 14 ; 7 ; 11 ; 17 ... }
T_{11} = { 11 ; 17 ; 26 ; 13 ... }

For the first 11 trajectories, we observe that the set of verified integers is equal to : (1 ; 2 ; 3 ; 4 ; 5; 6 ; 7 ; 8 ; 9 ; 10 ; 11 ; 13 ; 14 ; 17 ; 20 ; 26). It is noted that the trajectories for 4, 6, 8 and 10 do not need to be studied, as their initial values are even, and they are divided by 2 in the first iteration, joining the trajectory of a previously verified value. Moreover, T_5 also did not need to be calculated because the value 5 appeared in T_3 . Therefore, we already know how the trajectory from the value 5 evolves, we just need to refer to T_3 .

We will formalize these observations. Indeed, the set of verified integers has internal rules.

1) Every value in a verified trajectory is automatically considered verified

We have : $T_1 = \{ 1 ; 2 ; 1 ; 2 ; 1 ; ...\}$

T_1 is called the trivial cycle because as soon as a trajectory reaches the value 1, it enters this trivial cycle, alterning between 1 and 2.

Suppose n € V. This means that there exists an integer *k* such that $S_k (n) = 1$.

Thus, we have $T_n = \{ n ; S_1 (n) ; \dots ; S_{k-1} (n) \} + T_1$

All $S_p (n)$ with p € [1;k-1] belong to V.

Indeed, since all $S_p (n)$ are unique integers, clearly determined and ordered by the Syracuse algorithm, we have :

$$T_{Sp(n)} = \{ S_p (n) ; S_{p+1} (n) ; \dots ; S_{k-1} (n) \} + T_1$$

As a result, we obtain the following theorem :

THEOREM 1

If n € V this implies that there exists an integer k such that S_k (n) = 1. Every integer S_p (n), for any p € [1 ; k-1], belongs to V.

Note : to take an example,
T_3 = { 3 ; 5 ; 8 ; 4 ; 2 ; 1 ; 2 ; ... }.

The odd integer 5 appears in T_3, which reaches 1 in a finite number of iterations. Therefore, 5 is a verified integer ; we know that T_5 reaches 1 in a finite number of iterations, so there is no need to calculate the successive terms of T_5 to be certain that T_5 is verified.

2) When a non-verified trajectory equals a verified integer, it is automatically verified

If one of the S_j (n) is equal to a verified integer *n'*, then *n* is verified.

For any n € N* and j € N, if S_j (n) = n' with n' € V, then there exists a finite integer *m* such that :

$T_{n'}$ = { n' ; S_1 (n') ; ... ; S_{m-1} (n') } + T_1

Hence,

T_n = { n ; S_1 (n) ; ... ; S_{j-1} (n) } + { n' ; S_1 (n') ; ... ; S_{m-1} (n') } + T_1

j and *m* are integers, so T_n reaches the integer 1 in a finite number of iterations. Therefore, *n* is a verified integer.

As a result, we obtain the following theorem :

THEOREM 2

For any n € N* and j € N*, if S_j (n) = n' with n' € V, then n € V.

Note : to take an example T_9 = { 9 ; 14 ; 7 ; 11 ; 17 ... }. The integer 7 is verified because T_7 is verified. T_9 reaches a verified integer in a finite number of iterations. Therefore, 9 is a verified integer : we do not need to continue the calculations until reaching 1 to be certain of this.

3) Every verified integer multiplied by any power of 2 is a verified integer

Suppose n € V. This implies that there exists an integer *k* such that S_k (n) = 1.

For any *d* belonging to N*, we have :

$T_{n\cdot 2^d}$ = { $n\cdot 2^d$; $n\cdot 2^{d-1}$; ... ; 2n } + { n ; S_1 (n) ; … ; S_{k-1} (n) } + T_1

k and *d* are integers, so $T_{n\cdot 2^d}$ reaches the integer 1 in a finite number of iterations.

As a result, we obtain the following theorem :

THEOREM 3

For any *n* and d € N*, if n € V, then $n \cdot 2^d$ € V.

4) Decomposition of the set of verified integers

We will decompose the set of verified integers V into three subsets :

a) a finite subset A(n) consisting of an initial block [1 ; n] of verified integers (computer calculations have shown, for example, that all integers less than $n = 10^{20}$ verify the Syracuse conjecture).

b) a finite subset B(n) of verified odd integers belonging to [n+1 ; +∞[. (B(n) consists of odd integers greater than *n* that appeared during the successive verifications of the trajectories of A(n), see **theorem 1**).

c) an infinite subset C(n) of verified even integers belonging to [n+1 ; +∞[. (C(n) consists of the set of verified integers belonging to A(n) and B(n) multiplied by all powers of 2, see **theorem 3**).

We can note that V is an infinite set. Moreover, each new verified trajectory generates an infinite number of new verified integers.

PART 3

The axis of verified integers

Let's consider the set N*, as a vertical axis, starting from 1 and extending to infinity, consisting of a pile of equal squares, each square or cell being characterized by its numerical value, indicating its position in N*.

We will refer to such an axis as the « axis of verified integers », defined as follows :

a) The cell for a given numerical value is white if the value in question does not belong to V. The cell in question is said to be unverified.

b) The cell for a given numerical value is black if the value in question belongs to V. The cell in question is said to be verified.

c) An unverifed cell can become verified.

d) Once a cell is verified, it remains so permanently.

If a trajectory during its iterations equals a verified cell, then all cells of the trajectory are verified and become black (**theorem 1 and 2**). Every verified cell multiplied by all powers of 2 is verified (**theorem 3**).

Depending on the parity of the results of the iterations, the trajectories go up (thanks to one or more successive odd iterations) or go down (thanks to one or more successive even iterations) along the axis of verified integers.

As the successive verifications of Syracuse trajectories occur, the number of verified integers along the axis increases (**theorem 1**).

Let's consider this axis as having a starting set A(n). Since this axis has a starting set A(n), it also has a finite set B(n) and an infinite set C(n) (see **Part 2 section 4)**).

PART 4

The bounded trajectories with no non-trivial cycles are verified

Let us first assume that the Syracuse trajectories we analyse are all bounded and do not contain non-trivial cycles. The problem of the axis of verified integers then transforms into a sieve problem, of an algorithmic nature.

The case of non-trivial cycles will be addressed later (**Part 6**)

The case of unbounded trajectories will also be analysed later (**Part 7**) : we will assume the existence of a divergent, hence unbounded, trajectory, and we will study the probabilities of the existence of such a trajectory.

1) The upper bound M

Let us assume that the axis of verified integers is equipped with an initial set of verified integers A(n). Consider the trajectory starting at n', a non-verified integer. We assume that the Syracuse trajectory of n'

is bounded by M.

2) The interval [n ; M] contains verified integers

On the interval [n ; 2n], all even integers are verified, since all integers in A(n) are verified.

On the interval [2n ; 4n], half of the even integers are verified.

On the interval $[n\cdot2^{i-1} ; n\cdot2^{i}]$, a proportion of $1/2^{i-1}$ of all even integers are verified.

Thus, the interval [n ; M] contains at least these verified integers, at least for those below M.

3) Use of the verified integers sieve

We assume that there is no non-trivial cycles in the considered trajectory. Thus the trajectory of n' evolves between 3 and M without entering a cycle.

By the definition of the Collatz function, C(a) ≠ a for every integer a € N*. Since the trajectory is bounded by M and we have assumed that it has no non-trivial cycle, the trajectory will take different values each time within the interval [3 ; M].

Since there are some verified integers in the interval [n ; M], the Syracuse trajectory of n' has only three possible outcomes :

a) It becomes equal to a verified integer in [n ; M] and is therefore verified.

b) It decreases below n, becomes equal to a value in A(n), and is therefore verified.

c) It continues iterating until exhausting the possibilities of non-verified integers within [n ; M]. Indeed, the trajectory cannot remain constant since C(a) ≠ a for every integer a, and it cannot take the same value twice, otherwise we would have a non-trivial cycle. This is because each trajectory follows a clearly defined path, and if it revisits a previously taken value, it will inevitably enter a non-trivial-cycle.

Once all possible non-verified integers within [n ; M] are exhausted, the next step for the trajectory is to reach a verified integer, and thus, it becomes verified.

We thus arrive at the following result :

Theorem 4

Consider an axis of verified integers with an initial set A(n). If the Syracuse trajectory of n', an arbitrary non-verified integer, is bounded and has no non-trivial cycle, then it is verified.

From this theorem, we can deduce that as long as there is no unbounded (i.e., divergent) trajectory and no non-trivial cycle, the trajectories with starting values greater than A(n) are automatically verified. The only obstacles to this verification process are therefore the possibility of divergent trajectories and trajectories with non-trivial cycles.

We will examine the probabilities of existence of such trajectories in **Parts 6 and 7**. To do so, we will first address the question of inverse graphs in **Part 5**.

PART 5

The inverse graphs

1) Presentation of the problem

We can observe that the Collatz function is not injective because, to give just an example, C(3) = C(10) = 5. This means that some values can have multiple predecessors.

Definition : if we call *z* the predecessor of *p* under the Collatz function, we have C(z) = p.

Thus, 3 and 10 are predecessors of 5. Every integer *p* has at least one predecessor, coming from an even iteration, which is simply twice the integer in question. Indeed, C(2p) = p. However, under certain conditions that we will analyze, the integer *p* may have a predecessor smaller than *p*, coming from an odd iteration.

Let's take the example of the verified integer 5 :

Starting from 5, by multiplying by powers of 2, we can see that 10, 20, 40 and 80 are, for instance, automatically verified. And some of these values have odd predecessors : for example C(3) = 5, C(13) = 20 and C(53) = 80. Since the integers 3, 13 and 53 lead to a verified integer under the Collatz function, they are considered verified (**Theorem 2**). They themselves will have predecessors, and so on. The numbers 10 and 40 do not have odd predecessors, only even ones.

Therefore, we see that starting from the integer 5, certain predecessors are known : 3, 10, 20, 40, 80. These predecessors themselves have other predecessors : 13, 53... Hence, starting from any integer *p*, we can create an infinite number of predecessors, as we will have at least the infinity of $p \cdot 2^d$ as predecessors.

2) The construction of inverse graphs

We will now present in more detail what we can call the inverse Collatz graph :

Suppose we start with an integer *p*, and we seek to determine if *p* has an odd predecessor m = 2k + 1 such that (3m+1) / 2 = p

We easily obtain m = (2p-1) / 3

Thus, 2k + 1 = (2p – 1) / 3 and p = 3k + 2.

Therefore, if p = 3k + 2, *p* admits an odd predecessor of the form m = 2k + 1

If p = 3k, then 2p - 1 = 6k – 1, which is never divisible by 3.

If p = 3k + 1, then 2p – 1 = 6k + 1, which is never divisible by 3.

Thus, if p = 3k or p = 3k + 1, *p* does not admit odd predecessor.

We can then define the inverse Collatz relation of *p*, which gives the predecessor of *p*, and which we will call I :

Definition : we can define the inverse Collatz relation I, which associates to each *p* the integers I(p), the predecessors of *p*, defined as follows:

I(p) = {2p} if p = 3k or p = 3k + 1
I(p) = {2p ; (2p – 1) / 3} if p = 3k +2

Thus, we can see that I(p) gives us the predecessors of *p* according to the Collatz application. Starting from the relation I, we can iteratively construct the inverse Collatz graph of any integer *p*, which we will call G(p).

Definition : starting from any integer *p*, we look for I(p), i.e., the predecessors of *p*. In the first step, the elements of the inverse graph G_1(p) are equal to I(p)

In the second step, we look for the predecessors of the predecessors found in the first step. In the second step, G_2(p) is equal to G_1(p), plus the new predecessors we have just found.

In step (k+1), we look for the predecessors of the predecessors found in step k. G_{k+1}(p) is equal to G_k(p), plus the predecessors just found in step (k+1).

By repeating this process indefinitely, we obtain G(p), the inverse graph of *p*.

This means that any Collatz trajectory, starting from any integer belonging to G(p), the inverse graph of *p*, will necessarily lead to *p*.

We can detail this process as follows :

Suppose $I(p) = \{p_1 ; p_2\}$

Then we have $G_1(p) = \{p_1 ; p_2\}$

Suppose $I(p_1) = \{p_3\}$ et $I(p_2) = \{p_4 ; p_5\}$

Then we have $G_2(p) = G_1(p) \cup \{ p_3 ; p_4 ; p_5\}$

Suppose $I(p_3) = \{p_6 ; p_7\}$, $I(p_4) = \{p_8\}$ et $I(p_5) = \{p_9 ; p_{10}\}$

Then we have $G_3(p) = G_2(p) \cup \{p_6 ; p_7 ; p_8 ; p_9 ; p_{10}\}$

And so on.

Example with p = 17 :

17 is of the form 3k + 2. Therefore, I(17) = {34 ; 11}.

We then analyze I(34) = {68}   I(11) = {22 ; 7}

Next I(68) = {136 ; 45}   I(22) = {44}   I(7) = {14}

Thus, we see that the inverse graph G(17) evolves as follows :

$G_1(17) = \{11 ; 34\}$

$G_2(17) = \{7 ; 11 ; 22 ; 34 ; 68\}$

$G_3(17) = \{7 ; 11 ; 14 ; 22 ; 34 ; 44 ; 45 ; 68 ; 136\}$

All values in $G_3(17)$ have trajectories leading to 17.

S_3 (136) = 17

S_3 (44) = 17

S_2 (7) = 17

3) First results

a) First of all, the inverse graph of an integer *p* includes the inverse graphs of all the elements that constitutes the inverse graph of *p*. Indeed, we progressively compute all the predecessors of *p*, then the predecessors of these predecessors, continuing this process indefinitely. The predecessors of each of these predecessors - that is, the inverse graph of each of these predecessors, will be included in the inverse graph of the initial integer *p*.

If $G(p) = \{p_1 ; p_2 ; \ldots ; p_i ; p_{i+1} ; \ldots \}$, then we have :

$G(p) = G(p_1) \cup G(p_2) \cup \ldots \cup G(p_i) \cup G(p_{i+1}) \cup \ldots$

Thus, we obtain the following result :

THEOREM 6

The inverse graph of an integer *p* includes all the inverse graphs of the elements that compose the inverse graph of *p*.

b) On the other hand, we can observe that the inverse graph of any integer *p* is an infinite set. Indeed, according to the construction of inverse graphs, each inverse graph G(p) contains at least an infinite number of values corresponding to the set of $p \cdot 2^d$ for any *d*.

Thus, we obtain the following result :

THEOREM 7

The inverse graph G(p) of any integer *p* is an infinite set.

c) Moreover, suppose we seek the inverse graph G(p) of a verified integer *p*. Since every Syracuse trajectory that starts from an element of an inverse graph G(p) leads to *p*, all these trajectories are verified (see **theorems 1 and 2**). Thus we can state that if *p* is verified, then the entire inverse graph of *p* is verified.

As a result, we obtain the following theorem :

THEOREM 8

All elements of the inverse graph of a verified integer are verified.

d) Finally, we can state that if at least one element of the inverse graph of any integer is verified, then the entire graph is verified.

Indeed, all trajectories starting from a value within the inverse graph of any integer *p* lead to that integer *p*. If one of the values in this inverse graph is equal to a verified integer, it means its trajectory leads to 1. Thus the trajectory equal to a verified integer leads not only to *p* but also to 1.

This implies that *p* converges to 1, meaning *p* is verified. Since *p* is verified, its entire inverse graph is also verified (by **theorem 8**)

Thus we arrive at the following result :

THEOREM 9

If an element of the inverse graph of any integer is verified, then the entire inverse graph is verified.

4) Application of the results to G(1)

We can observe that all inverse graphs of the verified integers are included in the inverse graph of 1, i.e., G(1), because, by definition, all the values of these inverse graphs have trajectories leading to 1 (**Theorems 6 and 8**).

The question that arises is the following : does the union of all inverse graphs of all verified integers, i.e., G(1), cover N* ? This would mean that G(1) progressively fills the axis of verified integers, leaving no unverified integer. The Collatz conjecture can, in fact, be reformulated as follows : proving that the inverse graph G(1) is equal to N*, which is equivalent to saying that all integer trajectories converge to 1.

Experimental computer verifications show that all integer trajectories up to 10^{20} converge, without exception, to 1. This means that all these integers up to 10^{20} are contained in G(1). It also implies that G(1) is at least the union of all inverse graphs of integers up to 10^{20} without exception. Since an inverse graph contains an infinite number of values, we understand that the union of all these inverse graphs will progressively fill the axis of verified integers, making it increasingly obscured in an irreversible manner (see **Part 3**).

Eric Rosendaal is one of the people who contributed to verifying the Syracuse conjecture for the first 10^{20} integers. On Eric Rosendaal's website (see *ericr.nl, "3x+1 delay records"*), it is observed that the maximum flight times for integers up to 10^{20} do not exceed 3000. This means that a maximum of 3000 iterations is required for a Syracuse trajectory to reach 1 from any starting point in the interval $[1;10^{20}]$. If we reverse the reasoning and consider the perspective of inverse graphs, this also means that a maximum of 3000 iterations of the inverse graph of 1 is sufficient to fully cover the interval $[1;10^{20}]$ in verified integers.

When we observe that 3000 iterations in the construction of G(1) are sufficient to verify all integers in the interval $[1;10^{20}]$ - that is, hundreds of billions of billions of integers - we may wonder what the upper bound of A(n) will be when billions and billions of iterations are applied to G(1). The more iterations are applied to G(1), the more the initial set A(n) will progressively cover increasingly larger intervals of N*.

We must note that the construction of inverse graphs follows an algorithmic process. This approach is therefore time-dependent. However, we can also consider the entirety of the inverse graph of 1 - or of any other integer - as already existing: the set of values leading to 1 is, at a philosophical level, already present, true, and real, even if we do not yet have the exact algorithmic and numerical confirmation of all these values at the present moment. The algorithmic approach merely reveals, step by step, the infinite and actual extent of each inverse graph, particularly that of G(1).

5) Hypothesis that G(1) does not cover all of N*

We have just seen that the union of billions upon billions of inverse graphs of already verified integers makes the axis of verified integers increasingly dark (see **Part 3**). Conversely, the set of integers that have have not yet been verified on this axis becomes increasingly discontinuous.

For a trajectory to remain unverified, it must necessarily transition only between unverified integers. However, as we verify more integers through successive iterations of G(1), the probability that any remaining Syracuse trajectories (starting above the upper bound of A(n)) will encounter a verified integer increases. This, in turn, causes these trajectories to be verified more quickly, thereby generating even more verified numbers (**Theorems 1 and 2**). Indeed, all these verified trajectories and inverse graphs evolve along the same axis - the axis of verified integers - which, with each verification step or extension of G(1), continues to be filled with verified integers.

This suggests that G(1) progressively covers all of N* as iterations accumulate. Now, let us suppose that G(1) does not entirely cover N*. This would imply the existence of "residual" integers whose trajectories do not converge to 1. Such integers, which do not belong to G(1), would necessarily be part of either a divergent trajectory or a non-trivial cycle. Indeed, these are the only remaining possibilities, since any bounded trajectory without a non-trivial cycle is automatically verified (see **Part 4**). We will analyze these two possibilities in **Part 6 and 7**, first the possibility of a non-trivial cycle and then the possibility of a divergent trajectory.

PART 6

The question of non-trivial cycles

In **Part 4,** we showed that all bounded trajectories that do not contain non-trivial cycles are verified. We will now examine the hypothesis that one of the trajectories might contain a non-trivial cycle.

If we assume the existence of a non-trivial cycle in one of the trajectories, then this trajectory must be bounded, as it cannot both enter a non-trivial cycle and diverge. These two propositions are contradictory. Indeed, once a trajectory enters a non-trivial cycle, it continues to repeat this cycle indefinitely. The trajectory is thus bounded by either the maximum value M of the non-trivial cycle or by the maximum value M′ reached by the trajectory before entering the non-trivial cycle. In both cases, a trajectory that contains a non-trivial cycle is necessarily bounded.

We can also observe that if there exists a trajectory with a non-trivial cycle, then there must be an infinite number of such trajectories. Indeed, consider the inverse graph of any value belonging to a trajectory with a non-trivial cycle. All values in this inverse graph will have trajectories leading to the trajectory that contains the non-trivial cycle. Moreover, the inverse graph of any integer is infinite (**Theorem 7**).

Thus, we obtain the following result :

THEOREM 10

If there exists a trajectory with a non-trivial cycle, then there exists an infinite number of trajectories with non-trivial cycles.

Furthermore, no element of the inverse graph of the values in a non-trivial cycle can be equal to a verified integer (**Theorem 9**). If even a single one of these values were equal to a verified integer, it would imply that at least one term of the non-trivial cycle converges to 1, which would contradict the assumption that the trajectory contains a non-trivial cycle.

Thus, we obtain the following result :

THEOREM 11

If even a single value in the union of the inverse graphs of all terms of a hypothetical non-trivial cycle is equal to a verified integer, then the cycle cannot exist, and all these values must converge to 1.

In addition to **Theorem 11**, the characteristics of a non-trivial cycle in a Syracuse trajectory impose various constraints. An analysis conducted by S. Eliahou on the structure of non-trivial cycles in Syracuse trajectories (S. Eliahou : *the 3x+1 problem: new lower bounds on non-trivial cycle lengths*) states that if the first 2^{52} integers are verified (which they are, since the first 2^{68} integers have been verified), then the lower bound for the number of iterations in a non-trivial cycle is 187,363,077. Moreover, we know that as the initial set of verified integers A(n) increases, the lower bound for the number of iterations in a non-trivial cycle also increases. If the upper bound of A(n) exceeds 2^{1000} then the lower bound for a non-trivial cycle would be approximately $3.45 \cdot 10^{500}$.

Thus, we face a double constraint : the larger the union of multiple inverse graphs of all already verified integers (as they expand through successive iterations of theses graphs), the more the axis of verified integers fills up, leaving fewer unverified integers available on this axis. This most likely leads to an ever-growing set A(n). However, the larger the initial set of verified integers A(n), the greater the lower bound for the number of iterations in a non-trivial cycle, reaching extremely large values.

A non-trivial cycle, since it cannot tend toward 1, must necessarily transition only between unverified integers. It seems difficult to conceive of non-trivial cycles with a minimum length of several hundred million unverified integers without encountering a verified integer - especially in the context of an ever-expanding set of verified integers, leading to a progressive disappearance of unverified integers.

Additionally, the verified integers forming the minimal length of a non-trivial cycle would not be randomly distributed; they would have to be perfectly ordered according to the proportions dictated by the properties of Syracuse trajectories. This further reduces the probability of such a non-trivial cycle existing. Moreover, **Theorem 10** shows that if a non-trivial cycle exists, then an infinite number of trajectories with non-trivial cycles must also exist, making this possibility even more unlikely. This would imply the existence of an infinite number of unverified integers, which contradicts various probabilistic approaches (see, for example, Terence Tao : *Almost all orbits of the Collatz map attain almost bounded values*), as well as inverse graph analyses, both of which suggest that the probability of integers failing to satisfy the conjecture is nearly zero. The increasingly discontinuous nature of the set of unverified integers makes the hypothesis of a nontrivial cycle lasting at least several hundred million iterations with unverified integers highly improbable. Additionally, **Theorem 11** requires that all values in the inverse graphs of the terms in a hypothetical non-trivial cycle must never be equal to a verified integer.

Thus, we obtain the following result :

RESULT 1

The probability of the existence of a Syracuse trajectory with a non-trivial cycle is extremely low.

PART 7

The question of divergent trajectories

In **Part 4** we showed that if all trajectories are bounded and contain no non-trivial cycles, then they are verified. We have just examined the hypothesis that trajectories might have non-trivial cycles and concluded that the probability of a trajectory containing a non-trivial cycle is very low.

We now turn to the second unresolved hypothesis from **Part 4** - namely, whether a trajectory could be divergent, meaning that it is not bounded.

First, we can observe that if a divergent trajectory exists, then an infinite number of such trajectories must exist. Consider the inverse graph of any value in the divergent trajectory. All values in this inverse graph will have trajectories leading to the divergent trajectory. Moreover, the inverse graph of any integer is infinite (**Theorem 6**).

Thus, we obtain the following result :

THEOREM 12

If a divergent trajectory exists, then an infinite number of divergent trajectories exist.

Furthermore, no element in the inverse graph of the values of a divergent trajectory can be equal to a verified integer (**Theorem 9**). If even one such value were equal to a verified integer, it would mean that at least one term in the divergent trajectory converges to 1, which would contradict the very definition of a divergent trajectory.

Thus, we obtain the following result :

THEOREM 13

If even one value in the union of the inverse graphs of all terms in a hypothetical divergent trajectory is equal to a verified integer, then the divergent trajectory cannot exist, and all such values must converge to 1.

If a divergent trajectory exists, it means that each upper bound reached by the trajectory is eventually surpassed by another upper bound, and this process continues indefinitely. This implies that the trajectory evolves within the limits of an upper bound, as we discussed in **Part 4**, without ever being equal to a verified integer. It then surpasses this bound, evolves within the limits of a new upper bound without being equal to a verified integer, surpasses it again, and so on infinitely.

The same challenge we faced in needing to find several hundred million non-verified integers to meet the minimal conditions for the existence of a non-trivial cycle applies here : we would need an infinite number of non-verified integers to support a divergent trajectory. Here again, a major difficulty arises - executing an infinite number of iterations while transitioning only between non-verified integers, never equaling a verified integer, in the context of a continuously growing set of verified integers and an almost zero probability of the existence of integers that do not satisfy the conjecture.

It is as if the increasingly discontinuous set of non-verified integers is incapable of "supporting" the infinite inverse graphs of the elements of a divergent trajectory, which would require an infinite number of non-verified integers while never equaling a verified integer.

Thus, we arrive at the following result :

RESULT 2

The probability that a divergent Syracuse trajectory exists is extremely low.

Conclusion

We showed in **Part 4** that all Syracuse trajectories that are bounded and contain no non-trivial cycles are verified without exception. In **Parts 6 and 7**, we observed that the probabilities of a Syracuse trajectory being divergent or containing a non-trivial cycle are extremely low. This is primarily due to two reasons :

1. Inverse graphs and probabilistic approaches suggest that the probability of integers failing to satisfy the Syracuse conjecture is very low.
2. A single integer failing to satisfy the conjecture would necessarily imply the existence of a divergent trajectory or a non-trivial cycle, which in turn would require an infinite number of non-verified integers. Indeed, every element of the inverse graphs of all terms within the non-trivial cycle or divergent trajectory would also need to be non-verified.

This discrepancy - between the estimated low number of integers that might not satisfy the conjecture and the necessity of having an infinite number of non-verified integers to sustain a hypothetical divergent trajectory or a non-trivial cycle with a minimum length of several hundred million terms - suggests that the probabilities of such structures existing are extremely low.

In summary, we face two contradictory hypotheses:

(a) First hypothesis : There are no divergent trajectories and no non-trivial cycles. All trajectories fall within the framework of **Part 4** and are thus verified. The Collatz conjecture holds, confirming that G(1)=N*, since all trajectories tend toward 1.

(b) Second hypothesis : At least one integer is non-verified, meaning that none of the elements in the inverse graphs of at least several hundred million non-verified integers can be equal to a verified integer.

The study we have conducted strongly suggests that the probability of G(1)=N* is extremely high, while the probability of the existence of a divergent sequence or a non-trivial cycle is extremely low.

The theoretical significance of the axis of verified integers is that multiple research avenues can build upon this concept. For example, we have seen that certain results concerning the structure of non-trivial cycles impose constraints that make the existence of such cycles highly improbable. By cross-referencing various findings - whether from inverse graph theory, probabilistic approaches, statistical verifications, or other domains - we can strengthen our confidence that the conjecture holds. The fewer theoretical cases that do not satisfy the conjecture, the less free space these increasingly rare cases will have to maneuver along the axis of verified integers. In fact, based on our current understanding, a very large number of non-verified integers would be required to even hope to establish a possibly non-verified trajectory.

The advantage of inverse graphs is that they are computationally accessible. A simple program could compute the inverse graphs of any given integer. In terms of computational efficiency, it seems far more practical to calculate the successive values of G(1) directly rather than verifying each individual trajectory within the interval $[1;10^{20}]$.

Indeed, fewer than 3,000 iterations of G(1) are sufficient to verify all integers within $[1;10^{20}]$ (see **Part 5, section 4**). By carrying out this iterative work on G(1), we could analyze the rate of evolution of this set, particularly focusing on the initial set of verified integers A(n) that constitutes it.

More and more proofs are converging toward confirming the conjecture. The goal would be to assemble all these findings into a coherent framework, leading to an almost certain confirmation that the Collatz conjecture is valid. This would serve as an approximate solution to the problem while awaiting a definitive, rigorous proof. However, it is entirely possible that the Collatz problem is undecidable (see J.H. Conway: *"On unsettleable arithmetical problems*"). This could explain why no conclusive solution has been found in over 80 years.

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