The mean and sum of primes in a given interval and their applications in the proof of the Goldbach conjecture

Samuel Bonaya

7/2/2025

Contents

Abstract

In this research a formulation of the approximate sum of primes is presented. With it is also presented the mean of primes. The formulations are used to confirm the validity of the Binary Goldbach conjecture through establishing the interval containing Goldbach partition primes of a set of even numbers. Finally the paper seeks to get a better prime counting function than li (x)

Keywords Sum of primes; mean of primes; Interval of Goldbach partition; proof of the conjecture; Variable Euler number

Introduction

In the paper reference [1] a formulation was established for determining the interval that contains n primes.
In this paper a formulation will be presented for determining the sum and mean of a set of cosecutive primes.
The conept of mean of primes is important in the Goldbach partition of composite even numbers. Goldbach conjecture implies that every integer greater than 1 is a mean of at least one pair of primes. Consider primes in the interval $(1, x]$. How do we approximate their mean?

An approximation for sum of prime

Sum of integers from 1 to x

$$s_x = \frac{x(1+x)}{2} = (\frac{x+0.5}{\sqrt{2}})^2 - 0.25 \tag{1}$$

Now the sum of primes up to x is given by:

$$s_{p\leq x} \approx \frac{(\frac{x+0.5}{\sqrt{2}})^2}{\ln(\frac{x+0.5}{\sqrt{2}})} \tag{2}$$

Therefore

$$x \approx \sqrt{2\ln(\frac{x+0.5}{\sqrt{2}})s_{p\leq x}} \tag{3}$$

By (2)

$$\ln(x+0.5) \approx (\frac{x+0.5}{\sqrt{2s_{p\leq x}}})^2 + \ln\sqrt{2} \tag{4}$$

Therefore

$$\ln x \approx (\frac{x}{\sqrt{2s_{p\leq x}}})^2 + \ln\sqrt{2} \tag{5}$$

Therefore

$$x \approx \sqrt{2s_{p\leq x}(\ln x - \ln\sqrt{2})} \tag{6}$$

Therefore:

$$s_{p\leq x} \approx \frac{0.5x^2}{\ln x - \ln\sqrt{2}} \tag{7}$$

Using the prime number theorem the mean of primes is given by

$$m_{p\leq x} \approx \frac{0.5x\ln x}{\ln x - \ln\sqrt{2}} \tag{8}$$

The equation (8) can be interpreted to mean that in the interval

$$(1 < m = \frac{0.5x\ln x}{\ln x - \ln\sqrt{2}} < 2m) \tag{9}$$

there exist primes for the Goldbach partition of $2m$. We will illustrate ir by an example

Example 1 Solve $4 = \frac{0.5x\ln x}{\ln x - \ln\sqrt{2}}$ and hence find the primes for the Goldbach partition of 8.

solution Using the appropriate calculator $(x_1, x_2) = (1.4087859, 7.8238029)$.

This result means there exists a pair of Goldbach partition primes for 8 in the interval $(1.4087859, 4)$ and $(4, 7.8238029)$. The prime pair is $(3, 5)$.

Note that the primes in this interval are 2, 3, 5 and 7. Their actual mean is $\frac{17}{4} = 4.25$.

Example 2 Use equation to estimate the sum of the primes in the interval $(1, 11)$. Approximate their mean.

Solution and matters arising $s_{p\leq 11} \approx \frac{0.5\times 11^2}{\ln 11 - \ln\sqrt{2}} = 29.4931801850427$.

The actual sum is 28.

now by prime number theorem$\pi(1, 11) \approx 4.58735630566671$. The approximate mean of the primes is $\frac{29.4931801850427}{4.58735630566671} = 6.43765256776959$. The actual mean is 5.6. The actual mean of the odd primes is 6. 5 This would represent mean of even numbers 12 and 14.
Primes in the interval (1, 14] with a mean of 6.5 are $(3, 11, 5, 7)$.

Identification of the interval containing primes for the Goldbach partition of a given even number

Theorem: Interval containing primes for Goldbach partition of composite even numbers and proof of the Binary Goldbach conjecture Given two prime numbers p_1 and p_2 with a mean of N the composite even numbers numbers in the closed interval $[2N-2, 2N+2]$ have primes in the open interval $(1, 2N)$ for their complete Goldbach partition (N is a positive integer greater than 1).

Proof The Largest possible prime for the Goldbach partition of the even number $2N+2$ is $2N-1$. The Largest prime for the Goldbach partition of composite even number $2N-2$ is $2N-5$. Therefore the interval the open interval $(1, 2N)$ contains primes for the Goldbach partition of composite even numbers 2N -2, 2N and 2N + 2. Q.E.D.

Implications of the theorem The theorem means that all the primes for the complete Goldbach partition of 6, 8 and 10 are in the interval (1, 8). The primes for the the complete Goldbach partitions of 12, 14, 16 are in the interval $(1, 14)$. The primes for the complete Goldbach partitions of 18, 20, 22 are in the interval $(1, 20)$ and so on. This means effectively that every composite even number has at least one Goldbach partition.
Thus the Binary Goldbach conjecture is true.

The concept of variable Euler number and it's applications to get a better prime counting function

In this section we shall look at different ways the Euler's number can be derived. The Euler number is central to the prime number theorem. The number theorem For the purpose of this paper we will define the variable Euler number as

$$e_x = (1 + \frac{1}{x})^x \tag{10}$$

also

$$e_{\frac{1}{x}} = (1 + x)^{\frac{1}{x}} \tag{11}$$

Now

$$\frac{e_x}{e_{\frac{-1}{x}}} = \frac{(1 + \frac{1}{x})^x}{(x + 1)^{\frac{-1}{x}}} \tag{12}$$

Now:

$$\lim_{x \to \infty} \frac{(1 + \frac{1}{x})^x}{(x + 1)^{\frac{-1}{x}}} = e \tag{13}$$

So that

$$\pi(x) \sim \frac{x}{\ln_{\frac{(1+\frac{1}{x})^x}{(x+1)^{\frac{-1}{x}}}} x} \tag{14}$$

Also

$$e_{\sqrt[n]{x}} = (1 + \frac{1}{\sqrt[n]{x}})^{\sqrt[n]{x}} \tag{15}$$

and

$$e_{\frac{1}{\sqrt[n]{x}}} = (1 + \sqrt[n]{x})^{\frac{1}{\sqrt[n]{x}}} \tag{16}$$

$$\frac{e_{\sqrt[n]{x}}}{e_{\frac{-1}{\sqrt[n]{x}}}} = \frac{(1+\frac{1}{\sqrt[n]{x}})^{\sqrt[n]{x}}}{(1+\sqrt[n]{x})^{\frac{-1}{\sqrt[n]{x}}}} \tag{17}$$

$$\lim_{\sqrt[n]{x}\to\infty} = \frac{(1+\frac{1}{\sqrt[n]{x}})^{\sqrt[n]{x}}}{(1+\sqrt[n]{x})^{\frac{-1}{\sqrt[n]{x}}}} = e \tag{18}$$

Conducting tests for the most appropriate variable Euler number for the prime counting function

Now we know that $\pi(100) = floor \frac{100}{\ln_{\sqrt{11}} 100} = 26.$

$\pi(200) = floor \frac{200}{\ln_{\sqrt{12}} 200} = 46.$

Figure 1: $\pi(20000) = 2,262$

Example Testing equation 18 We will use the equation below as a prime counting function.

$$\pi(x) \sim \frac{x}{\ln_{\frac{(1+\frac{1}{\sqrt[n]{x}})^{\sqrt[n]{x}}}{(1+\sqrt[n]{x})^{\frac{-1}{\sqrt[n]{x}}}}} x} \tag{19}$$

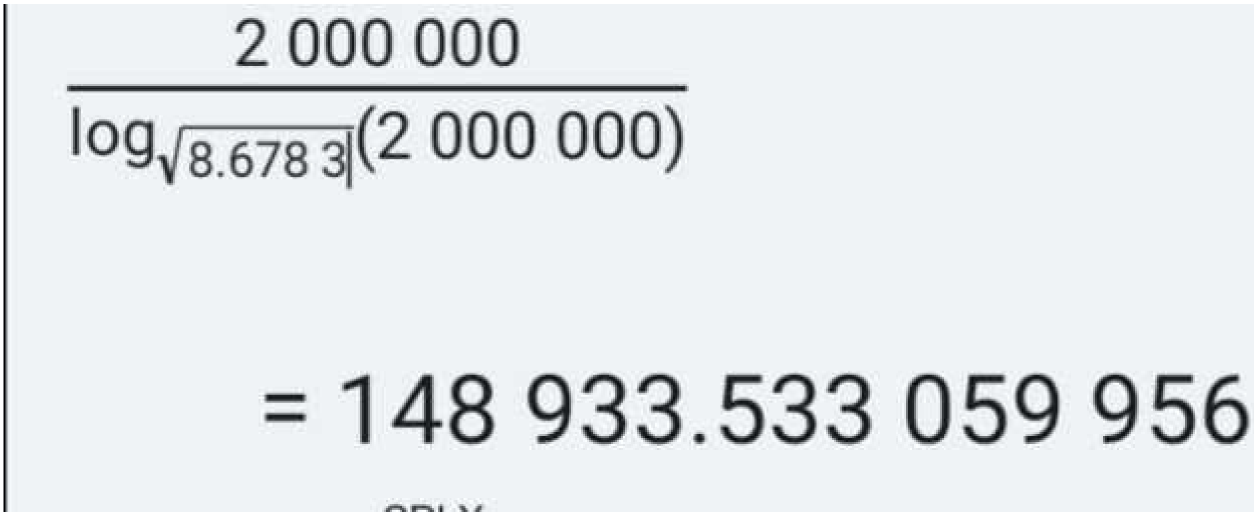

Figure 2: $\pi(2000000) = 148933$

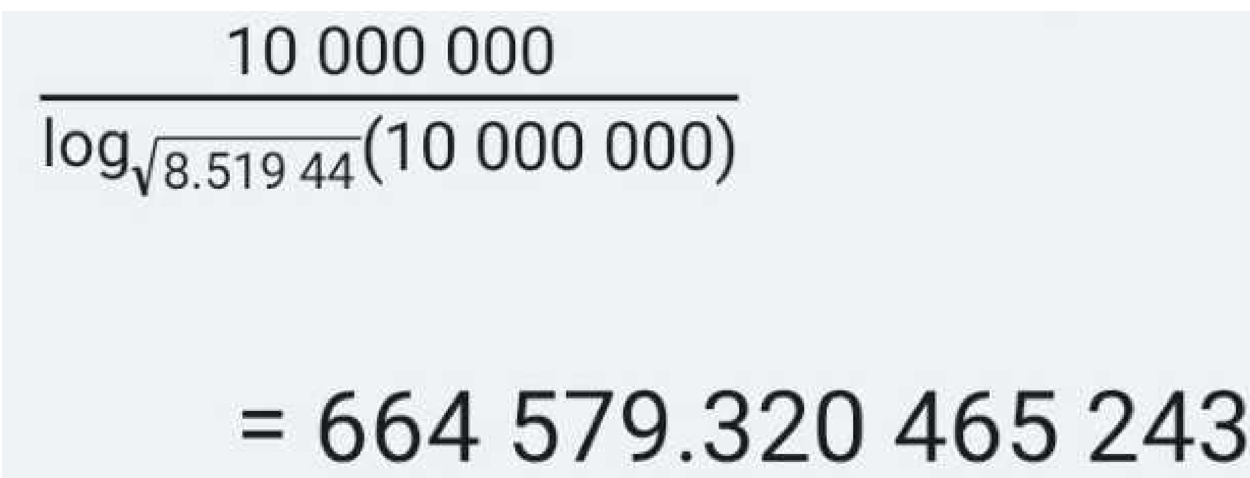

Figure 3: $\pi(10000000) = 664,579$

$$\pi(100) = \frac{100}{\ln_{3.29659898138} 100} = 25.90330600492$$

note that: $\ln_{3.29659898138} 100 = 3.86051108615$, a very accurate estimate of the mean gap of primes up to 100.

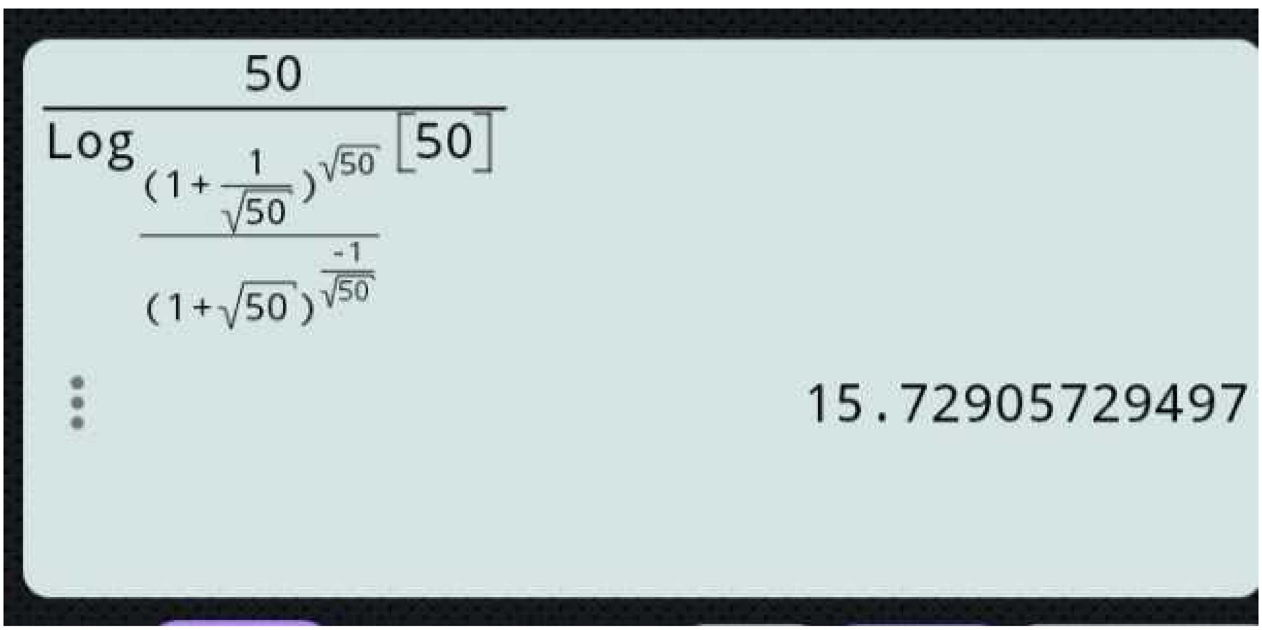

Figure 4: Determining the number of primes to less than 50. The actual number is 15

Other results

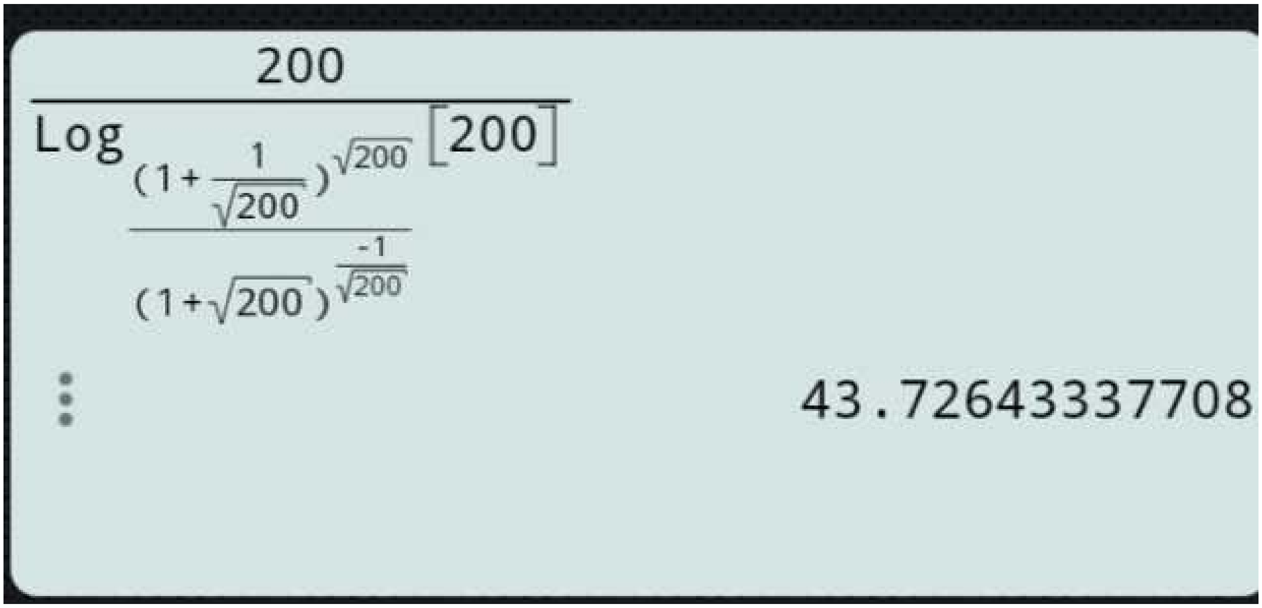

Figure 5: Determining the number of primes to less than 200. The actual number is 46

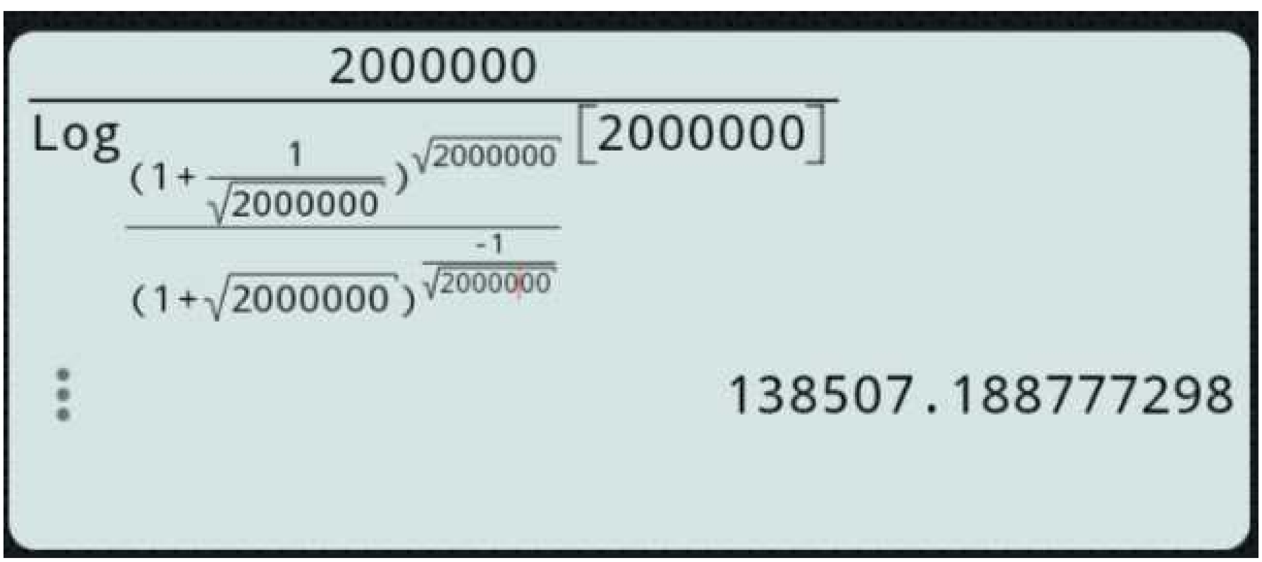

Figure 6: Determining the number of prime up to 2 000 000. The actual number is 148 933

summary and conclusion

A reliable formulation was achieved for establishing the sum and mean of a set of consecutive primes. It has also been established that given a pair of primes of mean n, then the composite even numbers [2n -2, 2n, 2n + 2] can be partitioned by primes in the interval $(1, 2n)$. This itself confirms that the Goldbach conjecture is true. A new an highly accurate prime counting function has been achieved.

References

[1]Samuel Bonaya Buya, The interval containing n primes. http://vixra.org/abs/2502.0005

[2]Lauren^iu Panaitopol, Intervals containing prime numbers. NNTDM 7 (2001), 4,111-114