The mean and sum of primes in a given interval and their applications in the proof of the Goldbach conjecture

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7/2/2025

Contents

Abstract

In this research a formulation of the approximate sum of primes is presented. With it also is presented the mean of primes. The formulations are used to confirm the validity of the Binary Goldbach conjecture through establishing the interval containing Goldbach partition primes of a set of even numbers. A general Goldbach partition theorem is established by which Goldbach conjecture is proved. Finally the paper seeks to get a better prime counting function than li (x)

Keywords Sum of primes; mean of primes; Interval of Goldbach partition; proof of the conjecture; Variable Euler number; Goldbach partition theorems

Introduction

In the paper reference [1] a formulation was established for determining the interval that contains n primes.
In this paper a formulation will be presented for determining the sum and mean of a set of cosecutive primes.
The conept of mean of primes is important in the Goldbach partition of composite even numbers. Goldbach conjecture implies that every integer greater than 1 is a mean of at least one pair of primes. Consider primes in the interval $(1, x]$. How do we approximate their mean?

An approximation for sum of primes

Sum of integers from 1 to x

$$s_x = \frac{x(1+x)}{2} = (\frac{x+0.5}{\sqrt{2}})^2 - (\frac{1}{2\sqrt{2}})^2 \tag{1}$$

Now the sum of primes up to x is given by:

$$s_{p\leq x} \approx \frac{(\frac{x+0.5}{\sqrt{2}})^2}{\ln(\frac{x+0.5}{\sqrt{2}})} \tag{2}$$

Therefore

$$x \approx \sqrt{2\ln(\frac{x+0.5}{\sqrt{2}})s_{p\leq x}} \tag{3}$$

By (2)

$$\ln(x+0.5) \approx (\frac{x+0.5}{\sqrt{2s_{p\leq x}}})^2 + \ln\sqrt{2} \tag{4}$$

Therefore

$$\ln x \approx (\frac{x}{\sqrt{2s_{p\leq x}}})^2 + \ln\sqrt{2} \tag{5}$$

Therefore

$$x \approx \sqrt{2(\ln x - \ln\sqrt{2})s_{p\leq x}} \tag{6}$$

Therefore:

$$s_{p\leq x} \approx \frac{0.5x^2}{\ln x - \ln\sqrt{2}} \tag{7}$$

Using the prime number theorem the mean of primes is given by

$$m_{p\leq x} \approx \frac{0.5x\ln x}{\ln x - \ln\sqrt{2}} \tag{8}$$

The equation (8) can be interpreted to mean that in the interval

$$(1 < m = \frac{0.5x\ln x}{\ln x - \ln\sqrt{2}} < 2m) \tag{9}$$

there exist primes for the Goldbach partition of $2m$. We will illustrate ir by an example

Example 1 Solve $4 = \frac{0.5x\ln x}{\ln x - \ln\sqrt{2}}$ and hence find the primes for the Goldbach partition of 8.

solution Using the appropriate calculator $(x_1, x_2) = (1.4087859, 7.8238029)$.

This result means there exists a pair of Goldbach partition primes for 8 in the interval $(1.4087859, 4)$ and $(4, 7.8238029)$. The prime pair is $(3, 5)$.

Note that the primes in this interval are 2, 3, 5 and 7. Their actual mean is $\frac{17}{4} = 4.25$. The mean of 3, 5 and 7, however is 5.

Example 2 Use equation to estimate the sum of the primes in the interval $(1, 11)$. Approximate their mean.

Solution and matters arising $s_{p\leq 11} \approx \frac{0.5\times 11^2}{\ln 11 - \ln\sqrt{2}} = 29.4931801850427$.

The actual sum is 28.

now by prime number theorem $\pi(1, 11) \approx 4.58735630566671$. The approximate mean of the primes is $\frac{29.4931801850427}{4.58735630566671} = 6.43765256776959$. The actual mean is 5.6. The actual mean of the odd primes is 6. 5 This would represent half the mean of even numbers 12 and 14.
Primes in the interval (1, 14] with a mean of 6.5 are $(3, 11, 5, 7)$.

We can make some additional notes. The primes in the interval (1, 14) with a mean of 4 are $(3, 5)$ those with a mean of 5 are (3, 5 , 7). Those with a mean of 6 are (5, 7). Those with a mean of 7 are (3, 7, 11). Those with a mean of 8 are (3, 5, 11, 13). Those with a mean of 9 are (5, 7, 11, 13). The primes with a mean of 2 are (2,2) those with a mean of 3 are (3,3). In general primes in the interval (1, 2n) are consist of primes with mean ranging from 2 to $n + 1$.

Example 3 Find the interval containing primes having a sum of 200.

solution Solve the equation $\frac{0.5x^2}{\ln x} = 200$ and you obtain the results x_1 = 1.4213745 x_2 = 38.168042 The primes in the interval $(1, 38)$ add up to 197.

Goldbach partition theorems

Generalized Goldbach partition theorem 1

The generalized Goldbach partition theorem 1 was proved in the paper reference [3]. It states as follows: The square of every positive integer greater than 1 is equal to the sum of the square of an integer greater or equal to zero and and a Goldbach partiton partition semiprime. The Goldbach partition theorem (1) implies that:

$$m = \sqrt{n^2 + p_1 p_2} || m > 1 \tag{10}$$

Here m represents the mean of the Goldbach partition primes. It also implies that

$$p_2 - p_1 = 2\sqrt{m^2 - p_1 p_2} || m \geq \sqrt{p_1 p_2} \tag{11}$$

The generalized Goldbach partition theorem (1) establishes that the interval containing semiprimes for the Goldbach partition of the composite even number $2m$ is

$$3(2m - 3) \leq p_1 p_2 \leq m^2$$

also:

$$p_1 p_2 + (m - p_1)^2 = m^2$$

meaning also that

$$m = \sqrt{p_1 p_2 + (m - p_1)^2} \tag{12}$$

$$p_2 - m = m - p_1 = \sqrt{m^2 - p_1 p_2} \tag{13}$$

Generalized Goldbach partition theorem 2

Goldbach partition theorem 2

The Goldbach partition theorem 2 is about the number of Goldbach partition primes in the interval $(m, 2m)$. In paper reference [3] it was established that if the composite even number $2m$ is nonsemiprime then the number of Goldbach partition semiprimes is given by:

$$\pi(m, 2m) \geq R(2m) \geq 1$$

Every semiprime even number has automatically at least 1 Goldbach partition pair. Theorem 2 follows from the inequality

$$m = \sqrt{(m-p_1)^2 + p_1p_2} < p_2 < 2m = 2\sqrt{(m-p_1)^2 + p_1p_2} | n > 0$$

Goldbach partition theorem 3

Given two prime numbers p_1 and p_2 with a mean of N the composite even numbers numbers in the closed interval $[2N-2, 2N+2]$ have primes in the open interval $(1, 2N)$ for their complete Goldbach partition (N is a positive integer greater than 1).

Proof The Largest possible prime for the Goldbach partition of the even number $2N+2$ is $2N-1$. The Largest prime for the Goldbach partition of composite even number $2N-2$ is $2N-5$. Therefore the interval the open interval $(1, 2N)$ contains primes for the Goldbach partition of composite even numbers 2N -2, 2N and 2N + 2. Q.E.D.

Implications of theorem 3 The theorem means that all the primes for the complete Goldbach partition of 6, 8 and 10 are in the interval (1, 8). The primes for the the complete Goldbach partitions of 12, 14, 16 are in the interval $(1, 14)$. The primes for the complete Goldbach partitions of 18, 20, 22 are in the interval $(1, 20)$ and so on. This means effectively that every composite even number has at least one Goldbach partition.

Generalization of Goldbach partition theorem 3 In its generalization, the Goldbach partition theorem implies that primes less than $2x$ are used for Goldbach parttition of all composite even numbers in the interval $[4, 2x+2]$. Thus the Binary Goldbach conjecture is true.

The concept of variable Euler number and it's applications to the prime counting function

In this section we shall look at different ways the Euler's number can be derived. The Euler number is central to the prime number theorem. The number theorem For the purpose of this paper we will define the variable Euler number as

$$e_x = (1 + \frac{1}{x})^x \tag{14}$$

also

$$e_{\frac{1}{x}} = (1 + x)^{\frac{1}{x}} \tag{15}$$

Now

$$\frac{e_x}{e_{\frac{-1}{x}}} = \frac{(1+\frac{1}{x})^x}{(x+1)^{\frac{-1}{x}}} \tag{16}$$

Now:

$$\lim_{x\to\infty} \frac{(1+\frac{1}{x})^x}{(x+1)^{\frac{-1}{x}}} = e \tag{17}$$

So that

$$\pi(x) \sim \frac{x}{\ln_{\frac{(1+\frac{1}{x})^x}{(x+1)^{\frac{-1}{x}}}} x} \tag{18}$$

Also

$$e_{\sqrt[n]{x}} = (1+\frac{1}{\sqrt[n]{x}})^{\sqrt[n]{x}} \tag{19}$$

and

$$e_{\frac{1}{\sqrt[n]{x}}} = (1+\sqrt[n]{x})^{\frac{1}{\sqrt[n]{x}}} \tag{20}$$

$$\frac{e_{\sqrt[n]{x}}}{e_{\frac{-1}{\sqrt[n]{x}}}} = \frac{(1+\frac{1}{\sqrt[n]{x}})^{\sqrt[n]{x}}}{(1+\sqrt[n]{x})^{\frac{-1}{\sqrt[n]{x}}}} \tag{21}$$

$$\lim_{\sqrt[n]{x}\to\infty} = \frac{(1+\frac{1}{\sqrt[n]{x}})^{\sqrt[n]{x}}}{(1+\sqrt[n]{x})^{\frac{-1}{\sqrt[n]{x}}}} = e \tag{22}$$

A closer look at the prime number theorem

Consider the identity function

$$\pi(x) = \frac{x}{\ln_{e_x} e_x^{\frac{x}{\pi(x)}}} \tag{23}$$

If

$$x = e_x^{\frac{x}{\pi(x)}} \tag{24}$$

then

$$e_x = x^{\frac{\pi(x)}{x}} = \sqrt[\frac{x}{\pi(x)}]{x} \tag{25}$$

Thus $e_2 = \sqrt{2} = 1.4142135623731$; $e_3 = \sqrt[3/2]{3} = 2.0800838230519$; $e_5 = \sqrt[5/3]{5} = 2.62652780440377$ $e_{97} = \sqrt[97/26]{97} = 3.40829167303047$ and so on. By the prime number theorem

$$\lim_{x\to\infty} \sqrt[\frac{x}{\pi(x)}]{x} = e \tag{26}$$

since

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln x}} = 1 \tag{27}$$

From the foregoing analysis there is a justification for constructing a prime counting function with a variable logarithmic base. The equation (21) will be a basis for establishing a viable and variable logarithmic base given by:

$$e_x = \frac{e_{\sqrt{x}}}{e_{\frac{-1}{\sqrt{x}}}} = \frac{(1+\frac{1}{\sqrt{x}})^{\sqrt{x}}}{(1+\sqrt{x})^{\frac{-1}{\sqrt{x}}}} \approx \sqrt[\frac{x}{\pi(x)}]{x} \tag{28}$$

This would also mean that:

$$x \approx (\frac{(1+\frac{1}{\sqrt{x}})^{\sqrt{x}}}{(1+\sqrt{x})^{\frac{-1}{\sqrt{x}}}})^{\frac{x}{\pi(x)}} \tag{29}$$

This would also mean that

$$\pi(x) \approx \frac{x \ln \frac{(1+\frac{1}{\sqrt{x}})^{\sqrt{x}}}{(1+\sqrt{x})^{\frac{-1}{\sqrt{x}}}}}{\ln x} \tag{30}$$

By the above formula the number of primes upto 10^{23} is given by: $\pi(10^{23} \approx (10^{23} ln(((1+(1/root(10^{23})))^root(10^{23}))/((1+root(10^{23}))^(-1)/root(10^{23}))))))/ln(10^{23})) = 1.88823688605574 * 10^{21}$. The actual number is 1,925,320,391,606,803,968,923. The worked number has 98 percent accuracy.

Pages 12 to 15 demonstrate the success of the logarithmic base in achieving with an accurate prime counting function.

summary and conclusion

A reliable formulation was achieved for establishing the sum and mean of a set of consecutive primes. It has also been established that given a pair of primes of mean n, then the composite even numbers [4 , 2n + 2] can be partitioned by primes in the interval $(1, 2n)$. This itself confirms that the Goldbach conjecture is true. A new an highly accurate prime counting function has been achieved.

References

[1]Samuel Bonaya Buya, The interval containing n primes. http://vixra.org/abs/2502.0005

[2]Lauren^iu Panaitopol, Intervals containing prime numbers. NNTDM 7 (2001), 4,111-114

[3] Samuel Bonaya Buya and John Bezaleel Nchima (2024). A Necessary and Sufficient Condition for Proof of the Binary Goldbach Conjecture. Proofs of Binary Goldbach, Andrica and Legendre Conjectures. Notes on the Riemann Hypothesis. International Journal of Pure and Applied Mathematics Research, 4(1), 12-27. doi: 10.51483/IJPAMR.4.1.2024.12-27.

[4]Samuel Bonaya Buya (2024l, Confirming Buya's and Bezaleel's proof of the Binary Goldbach conjecture using Bertrand's postulate https://papers.ssrn.com/sol3/Papers.cfm?abstract_

Images for demonstration

Figure 1: Primes for Goldbach partition of even numbers in the interval $[16, 20]$

Figure 2: Primes for complete Goldbach partition of even numbers in the closed interval $[6, 52]$

Now we know that $\pi(100) = floor\frac{100}{\ln_{\sqrt{11}}100} = 26$.

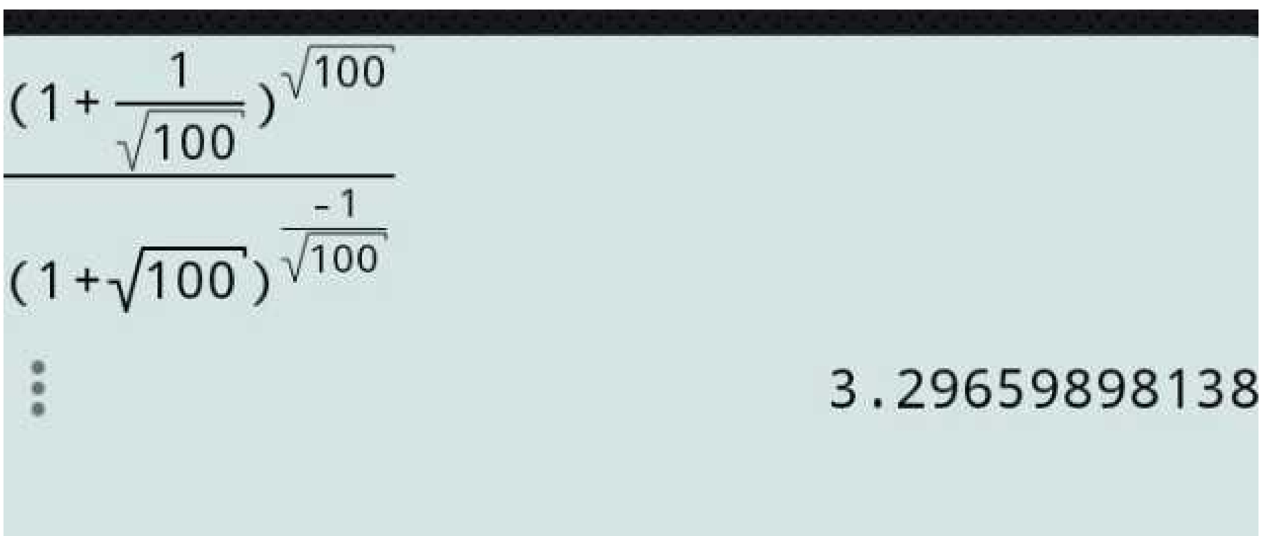
$\pi(200) = floor\frac{200}{\ln_{\sqrt{12}}200} = 46$.

Figure 3: Primes for complete Goldbach partition of composite even numbers in the closed interval $[6, 102]$

Conducting tests for the most appropriate variable Euler number for the prime counting function

Example Testing equation 19 We will use the equation below as a prime counting function.

$$\pi(x) \sim \frac{x}{\ln_{\frac{(1+\frac{1}{\sqrt[n]{x}})^{\sqrt[n]{x}}}{(1+\sqrt[n]{x})^{\frac{-1}{\sqrt[n]{x}}}}} x} \tag{31}$$



$$\pi(100) = \frac{100}{\ln_{3.29659898138} 100} = 25.90330600492$$

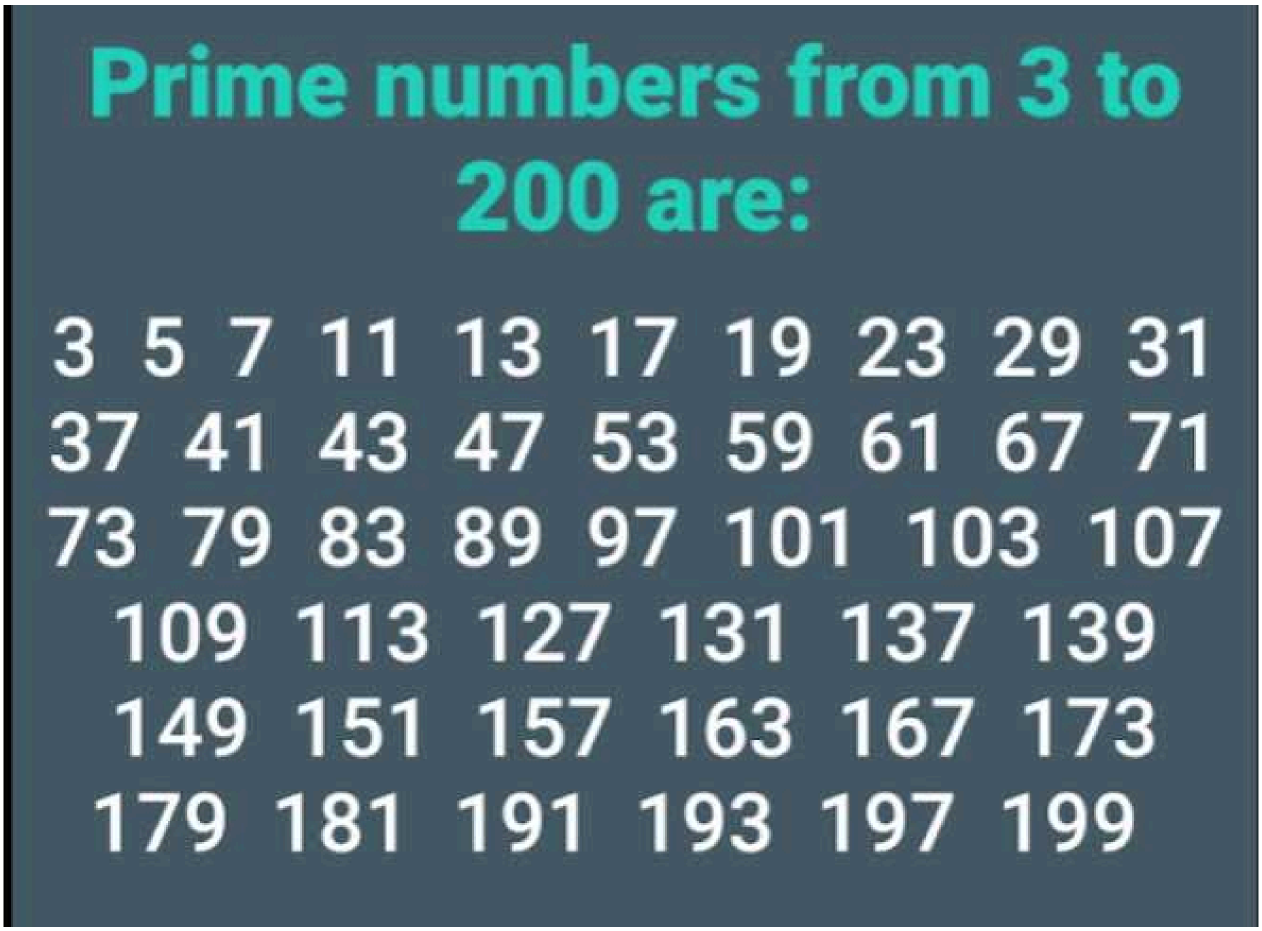


Figure 4: Primes for complete Goldbach partition of even numbers in the in the closed interval [6, 202]

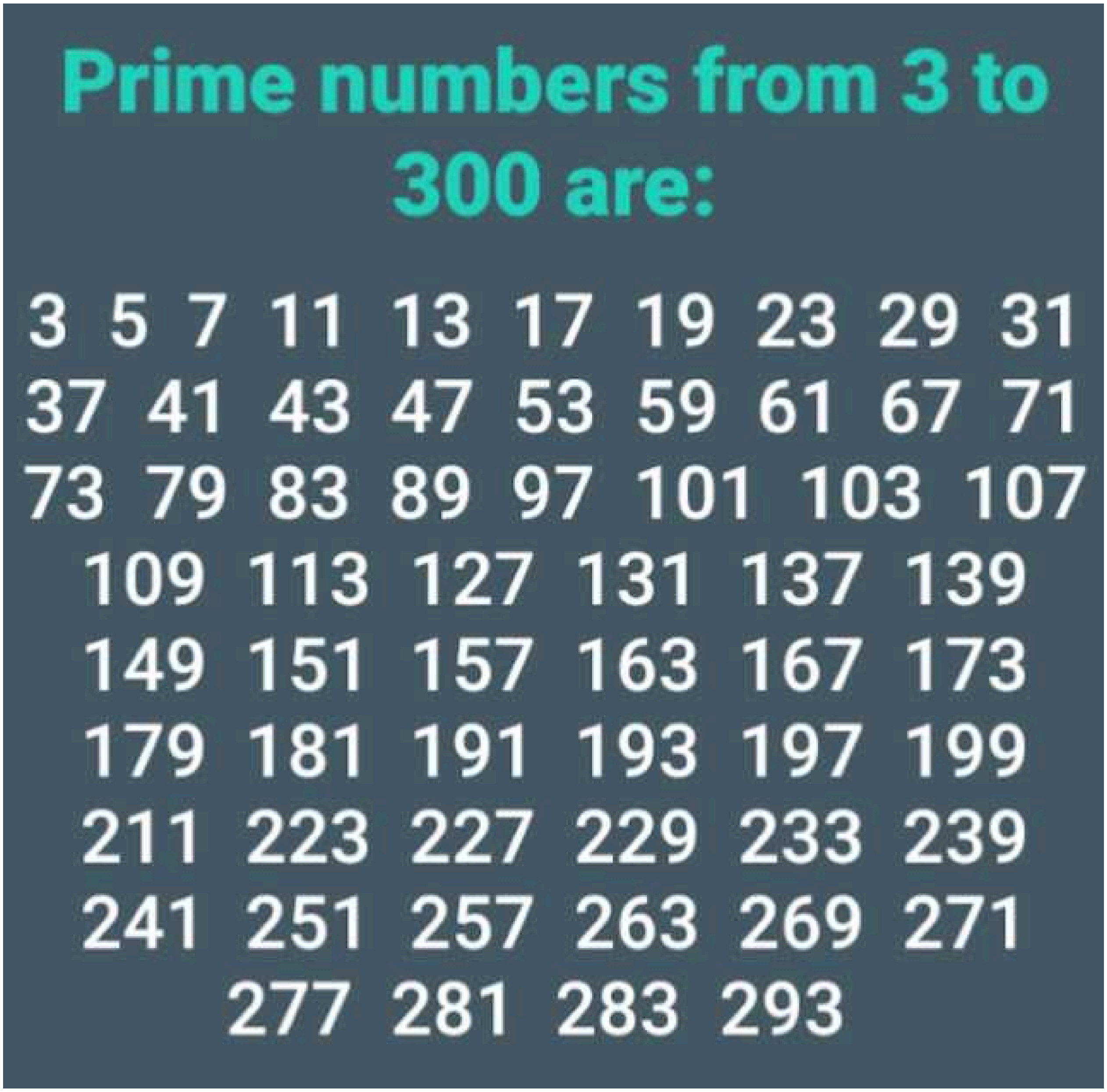


Figure 5: Primes for complete Goldbach partition of even numbers in the closed interval [6, 302]

Figure 6: $\pi(20000) = 2,262$

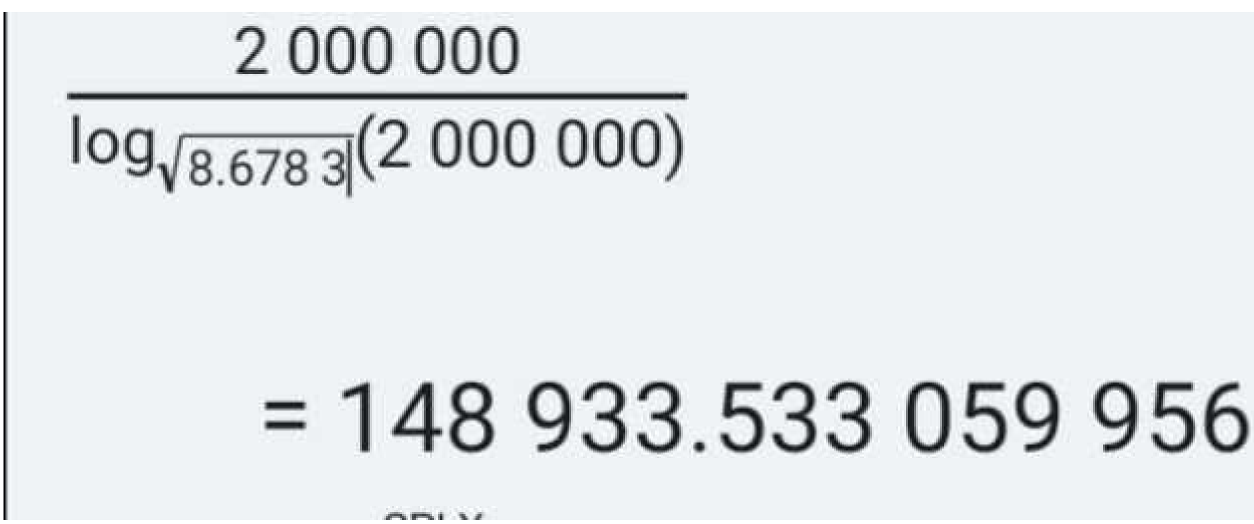


Figure 7: $\pi(2000000) = 148933$

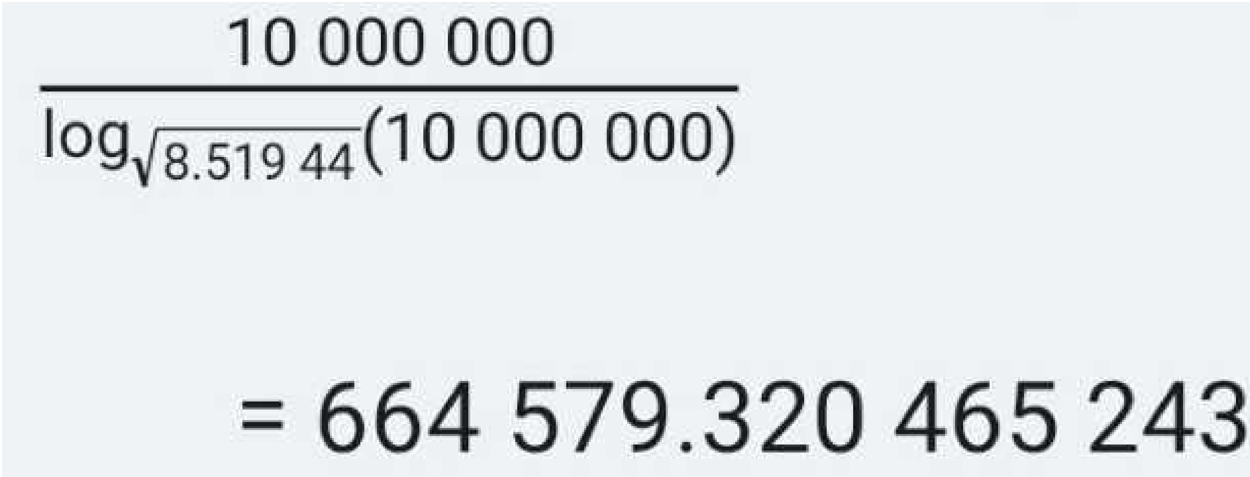


Figure 8: $\pi(10000000) = 664,579$

note that: $\ln_{3.29659898138} 100 = 3.86051108615$, a very accurate estimate of the mean gap of primes up to 100.

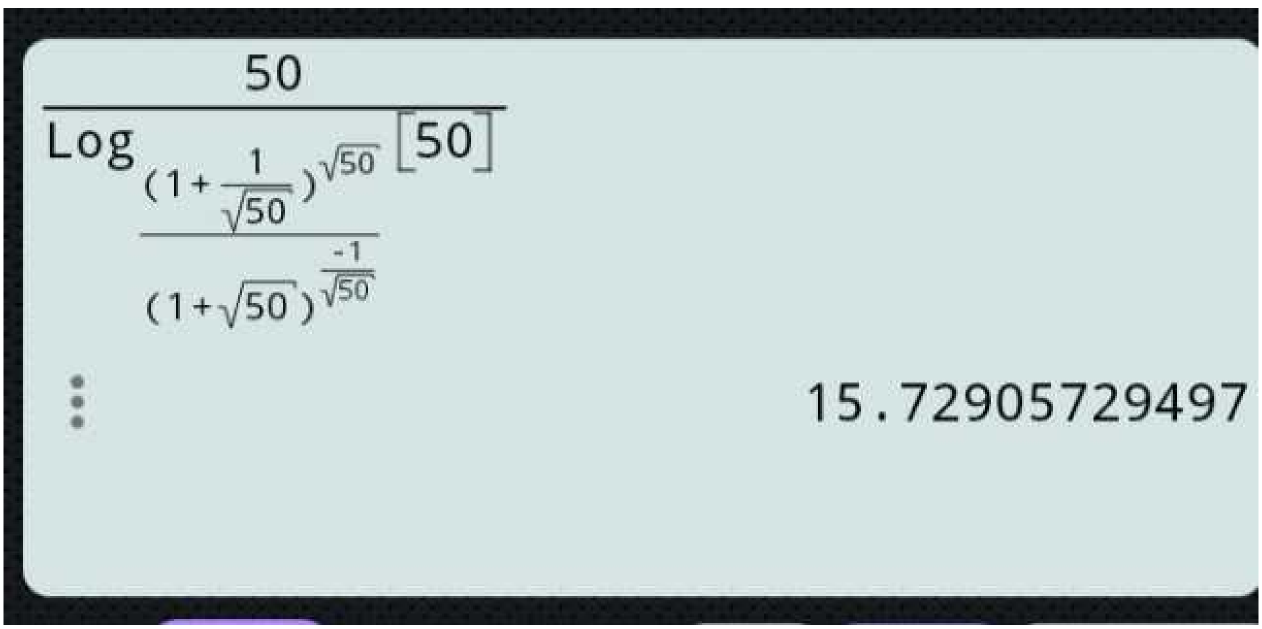


Figure 9: Determining the number of primes to less than 50. The actual number is 15

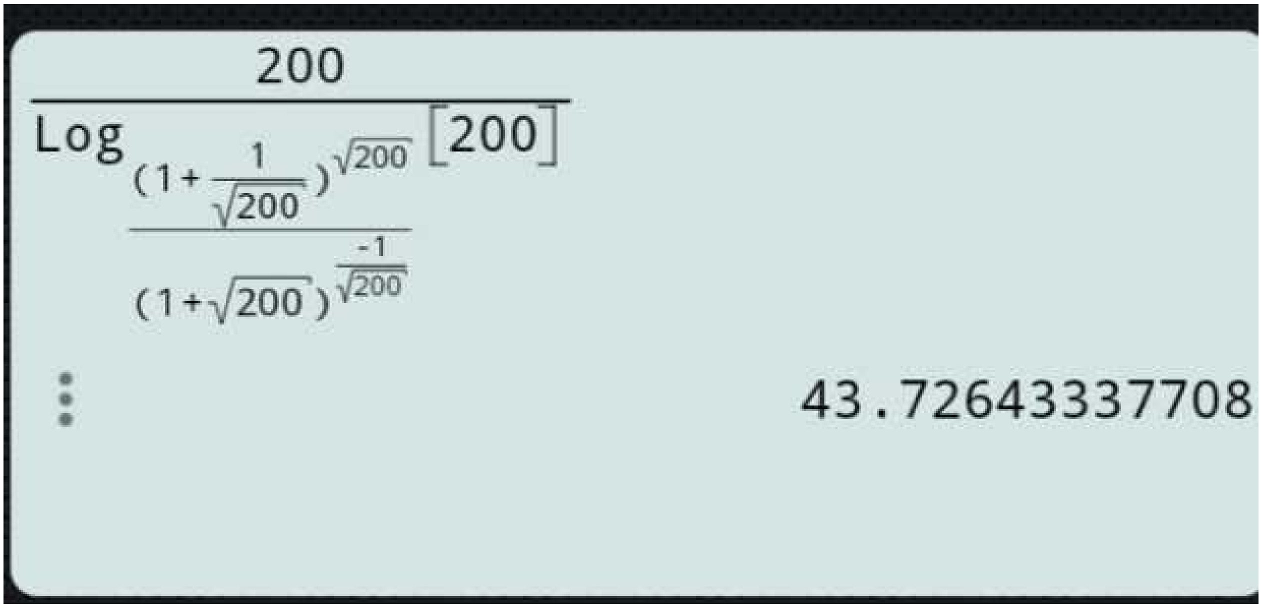


Figure 10: Determining the number of primes to less than 200. The actual number is 46

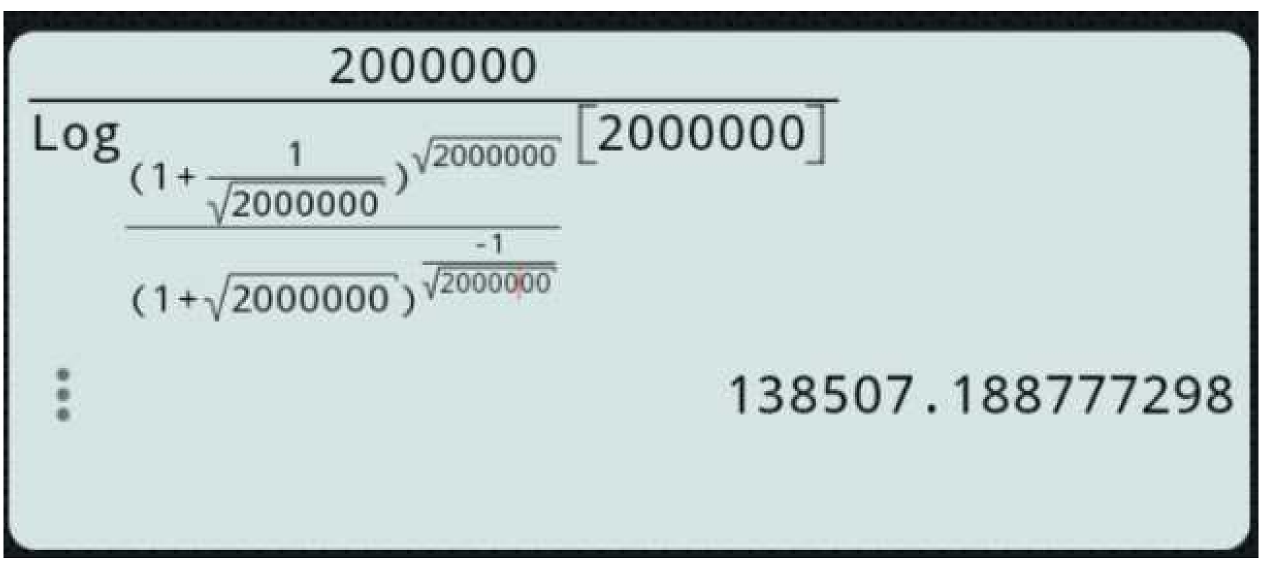


Figure 11: Determining the number of prime up to 2 000 000. The actual number is 148 933