Reducing Uncertainty through the Application of Empirical Symbolism: Delos and Adelos

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Abstract.

Throughout history, humanity has sought to limit error in order to gain more precise insights and optimize its tools, be they physical or conceptual.

This work explores the intricate relationship between geometry, mathematics, and the propagation of uncertainty, with particular attention to methods that either mitigate or avoid the amplification of uncertainty in mathematical, geometric, and applied contexts.

The study begins by examining the foundational concepts underlying uncertainty in mathematical models, exploring how various geometric and topological structures can be leveraged to better understand and control the flow of uncertainty across different domains. A new number convention, the so-called empirical number, is introduced, enabling a more accessible assessment of uncertainty propagation. Particular focus is placed on those constructions that actively work to reduce uncertainty, offering insights into techniques that prevent the cascading effect of errors, a challenge often encountered in both theoretical and applied mathematics.

Through the use of geometric principles, this work provides novel approaches to managing the inherent uncertainties in complex systems, ranging from simple algebraic problems to intricate applications. It highlights methods such as error propagation reduction, geometrically optimized models, and innovative adaptations to traditional methods that reduce computational or conceptual uncertainty. These techniques are of significant theoretical importance and are also crucial in practical applications, where precision and reliability are paramount, particularly in applied mathematics.

By addressing both the philosophical and practical dimensions of uncertainty, this work paves the way for a refined understanding of the interaction between mathematical theory and real-world applications, offering tools to navigate complex, uncertain environments with greater confidence and precision. The exploration of these ideas provides new insights into the role of uncertainty in mathematical systems, particularly in those constructions that prioritize stability and error mitigation over mere approximation.

The exposition begins with the definition of the empirical number, followed by the wide application of the Monte Carlo method to assess the functionality presented. It also recalls the theory of error propagation, addresses basic segment operations, and reexamines the Pythagorean theorem, which plays a crucial role in limiting error propagation. Further, the work discusses the applications of these ideas in calculus, specifically in the propagation of uncertainty through differentiation and integration.

1. Definition of Empirical Numbers.

Introduction.

In the field of probability and applied sciences, random variables represent a crucial tool for modelling uncertainty. A random variable assigns a numerical value to every possible outcome of a random event, allowing for a mathematical description of complex phenomena such as experimental results, financial market fluctuations, or the lifespan of a product.

Random variables are widely used in statistics, physics, finance, engineering, and many other disciplines. They can model discrete events, such as the roll of a die, or continuous events, such as measuring a physical quantity subject to fluctuations. In this context, tools like the probability distribution function, the probability density function, and the cumulative distribution function are essential for describing the distribution of probabilities across the various possible outcomes.
Mathematical operations on random variables, such as addition, multiplication, and differentiation, require advanced techniques such as convolutions, integral transforms, and stochastic calculus. The latter, in particular, is used to analyse processes that evolve randomly over time, such as Brownian motion in physics or fluctuations in financial markets. However, these techniques can be complex.

Despite their power, the use of random variables presents some practical difficulties. Operations like the sum or product of random variables often require complex convolutions, which can be difficult to interpret, shifting the focus from the conceptual aspect to the technical one. Additionally, in the case of continuous variables, differential calculus requires complex tools like the Ito derivative, further complicating the analysis.
In many real-world applications, such as experimental physics or engineering, one often deals with data that has intrinsic uncertainty. However, managing this uncertainty through the formalism of random variables can be overly complex, especially when the uncertainty is treated qualitatively rather than quantitatively. For example, one might wonder how uncertainty propagates in operations such as the sum of two segments without delving into detailed calculations related to the specific distribution.

To overcome these limitations, I propose in this document the use of empirical numbers, a new approach that integrates uncertainty directly into the structure of the number itself. An empirical number consists of two parts: Delos, which represents the exact or nominal value of a quantity, and Adelos, which reflects the uncertainty or potential variability associated with it.
This dual structure allows for a clear representation of both the determined value of a quantity and its uncertainty, simplifying many mathematical operations and enhancing intuitive understanding.

Let's now look at the advantages of using empirical numbers.

> Clear Separation of Components: By explicitly separating the nominal value (Delos) from the uncertainty (Adelos), empirical numbers offer a transparent framework for managing quantities in the real world. This clarity simplifies operations like addition, multiplication, and differentiation, directly accounting for uncertainty without the need for complex probabilistic methods, allowing us to maintain focus on the entity in its original entirety. Adelos is not just an uncertainty to be reduced, but also a potential to be enhanced.

Enhanced Ontological Capacity: Beyond their mathematical utility, empirical numbers offer a richer ontological perspective. The distinction between Delos and Adelos reflects a more nuanced understanding of reality, where measurable quantities are not simple fixed points, but entities with intrinsic variability. This duality captures both the tangible and potential aspects of a magnitude, making empirical numbers a more complete tool for representing the world in a symbolic way.

Simplified Calculations: Integrating uncertainty into the definition of empirical numbers allows for simplified operations compared to traditional methods. For example, in the addition of two empirical numbers, uncertainty propagation follows predefined rules, eliminating the need for convoluted transformations. This approach not only streamlines calculations but also makes them more intuitive, especially in fields like experimental physics and engineering.

Adaptability to Quantum Computing: With the advent of quantum computing, empirical numbers could adapt particularly well to these new computational paradigms. Quantum computers, which inherently operate with superpositions and probabilistic states, could manage empirical numbers naturally, further simplifying complex calculations and improving computational efficiency.

In conclusion, the introduction of empirical numbers, with their clear distinction between Delos and Adelos, represents an important step forward in the way we manage and understand uncertain quantities. This approach is in no way intended to replace or negate the use of random variables, which remain fundamental tools for dealing with probabilities. However, on an ontological level, I believe that empirical numbers and their subsequent treatment offer a more complete and powerful framework for representing the complexity of the real world. By embracing empirical numbers, we can achieve greater precision and understanding in both theoretical and applied contexts, paving the way for new advancements in mathematics and science.

The empirical number

In the following discussion, we will introduce a new class of numbers, called Empirical Numbers, which represent a formal extension of real numbers. We will denote this class with the symbol $\mathbb{E}$. These numbers are introduced to represent entities composed of an exact part and an uncertain part. The exact part could, for example, represent the average of an entity, while the uncertain part could express its variability.

An empirical number is an ordered pair of real numbers (d, a), where ‘d’ and ‘a’ are real numbers. An empirical number can be written in the form $\boldsymbol{d}_{(\boldsymbol{a})}$, where $\boldsymbol{d}$ represents the nominal value and **a** identifies the uncertainty.

The part **d** is called Delos (Δήλος), a Greek term meaning "visible" or "clear." The name reflects the concept of its mathematical and symbolic properties of exactness, evoking the "realm of hyperuranic ideas."
The part **a** will be called Adelos (ἀδήλος), a Greek adjective meaning "undefined," "obscure," or "uncertain." This term describes something unclear or ill-defined, emphasizing its properties of indeterminacy.
The part **a** will also be referred to as "potentiality" for reasons that will be clarified later.

An empirical number can be graphically interpreted as a segment whose endpoints are blurred:

An empirical number, when representing a physical variable, has the two quantities **a** and **d** expressed in the same unit of measurement.

Examples of quantities expressed with empirical numbers are:

- $25_{(3)}°C$: the conditioned temperature of a room is 25°C with a variability of ±3°C.
- $180_{(2)}cm$: a person's height varies depending on posture by ±2 cm.
- $1.5_{(0.2)}hours$: the duration of a trip, indicating that the average trip time is 1.5 hours (equivalent to 1 hour and 30 minutes) with a variability of ±0.2 hours. In other words, the trip time may vary between 1.3 hours and 1.7 hours.
- $50_{(5)}$€: the price of a service, which can range from 45 to 50 euros.

A measurable quantity includes both an expected part, such as the average height of a population, and a part indicating the variability around this expected value, such as the range of height variation within the group. Variability can be described using different distribution characteristics and confidence intervals. In this discussion, we will consider among others a Gaussian distribution (normal curve) as a model for variability, without losing generality, as various distributions (e.g., uniform or Weibull) behave analogously in the study of variability propagation. As can be easily observed, distributions different from the normal, such as the uniform or Weibull distributions, lead to different values for the standard deviation but exhibit identical behaviour under the operations we will describe, allowing for increased generality in our considerations.

A key aspect is that a certain entity, such as a person in a population, has an expected height value and an actual measure that rarely coincides exactly with the expected value, but falls within the predicted variability range. Therefore, the term Adelos represents the potential of height, expressed through a specific measure.
The empirical number does not express a defined and exact quantity, but represents a quantity (Delos) and its corresponding potential for expression within a defined range (Adelos).

Random variables represent how probabilities are distributed in a sample space, while variables expressed by empirical numbers are more representative of the uncertainty around an expected value; the perspective is quite different. The empirical number is focused on the value, the *Delos*, and the uncertainty of its localization in the sample space, while the random variable provides a global view of probability density, especially in the case of continuous random variables.

In the traditional conception of random variables, uncertainty is treated as an external element to the entity being studied. The underlying idea is that of a betting game: uncertainty is seen as a contingent factor, like a roulette in which we bet on an outcome, attempting to predict future results through probability. While this view is useful in many statistical contexts, it is reductive as it reduces the concept of chance to an external dynamic that does not capture the essence of the entity itself.
The traditional probabilistic representation, embodied by random variables, sees chance as a set of possible outcomes of a phenomenon, where each outcome is associated with a probability. This approach, however, risks flattening the concept of chance into a logic of prediction and betting. It reduces the phenomenon to a mere issue of percentages and probabilities, a forecasting game where chance is seen as a disturbance that alters an ideal behaviour, rather than as an intrinsic property of the world.

In the view of empirical numbers, chance is not an external entity to the object in question, but rather an intrinsic quality of the entity represented by the *Delos*. The *Delos*, representing the expected value of a magnitude, cannot be separated from its *Adelos*, the uncertainty that surrounds it and represents its variability. This view is much more significant, as it reflects an idea of chance as a necessity rather than as a contingency. Uncertainty is not something we add ex post to model the random behaviour of a variable, but an intrinsic characteristic of reality.

From this ontological conception arises a different approach to the propagation of uncertainty. While random variables focus on probability distributions and density functions, empirical numbers shift the focus to how uncertainty propagates through operations on the variables themselves through a construction process.

The measure.

The measurement process is the operation through which an empirical number, initially composed of an exact part (*Delos*) and an uncertain part (*Adelos*), is reduced to a unique and determined value.

During measurement, the *Adelos* component, representing the uncertainty or potential variability of the empirical number, is reduced to zero, leaving only the *Delos*. In other words, measurement eliminates the potential variability associated with the quantity, producing a pure mathematical and unique value, which can be treated with traditional mathematical techniques.

For example, a temperature measurement initially expressed as $25_{(2)}°C$, meaning with an uncertainty of ±2°C, is reduced to $23{,}983_{(0)}°C$ after measurement, or alternatively to $26{,}151_{(0)}°C$, indicating that the value determined by the measurement no longer has uncertainties associated with it.

This does not imply that measurement is intrinsically free of uncertainty, but rather that the act of measurement provides a single value that, by its nature, reduces *Adelos*, the potential uncertainty.
Empirical symbolism enables effective treatment of quantities, synthesizing within the symbol both *Delos* and *Adelos*, which together represent the entire range of variability of the variable. This variability is articulated in possible values through its potential, expressed by *Adelos*.

Measurement is an event within a process that determines the manifestation of potentiality in a specific, determined value, fundamentally random, within the variability range defined by *Adelos*. The measurement process ensures that, starting from the potentialities expressed by the empirical number, a specific value emerges within the range defined by *Adelos*, with the latter being completely reduced.

$$\boldsymbol{d}_{(\boldsymbol{a})} \xrightarrow[\boldsymbol{Measure}]{} \boldsymbol{d}_{(\boldsymbol{0})} \tag{1}$$

This does not imply that the measurement event is devoid of absolute uncertainty, but rather that from this event arises a specific numerical value, one that can be processed using classical tools and subsequently moved to the Platonic 'Hyperuranion'
We can imagine the measurement process as a mechanism that extracts a specific real number from the range of variability defined by the *Adelos* of an empirical number. Measurement thus becomes the event that transforms an indeterminate potential into a determined real value.

The collection of different measurement events allows us to obtain distinct values $\boldsymbol{d}_{(\boldsymbol{0})}$ which, when considered together, can reconstruct the original space of the empirical number $\boldsymbol{d}_{(\boldsymbol{a})}$. The reverse process, from the manifestation of individual measurements to the reconstruction of the original empirical number, is conceptually possible. Through the collection of a series of measurements, we can estimate both the *Delos* (the nominal value) and the *Adelos* (the uncertainty). This process requires statistical techniques such as calculating the mean and standard deviation, or the range, to reconstruct an empirical number that accurately represents the measured quantity, even accounting for its intrinsic uncertainty.

To formalize a probabilistic theory based on empirical numbers, two phases are followed: the first concerns the definition of the domain of empirical numbers and the morphisms that operate on it, while the second focuses on the transition towards a codomain of traditional numbers through probabilistic measures.

Let an empirical number d_a be defined as an ordered pair composed of an exact part (Delos) and an uncertain part (Adelos):

$$d_a = (x_d, x_a) \; with \; x_d \in \mathbb{R}, x_\mathrm{a} \in \mathbb{R}^+$$

- Let x_d represent the exact value (deterministic part, Delos).
- Let x_a represent the uncertainty or variability associated (stochastic part, Adelos).

Therefore, the domain D of empirical numbers is given by the set::

$$D = \{(x_d, x_a) \mid x_d \in \mathbb{R}, x_a \geq 0$$

Let us define morphisms $\phi: D \to D$ that transform empirical numbers into other empirical numbers. These morphisms preserve the structure of empirical numbers, meaning they act on both the Delos part and the Adelos part.

For example, if we consider an addition operation between two empirical numbers

$$\phi_+(x_a, y_a) = \left(x + y, \sqrt{1^2 + 1^2}\right) = \left(x + y, \sqrt{2}\right)$$

The Delos part follows the usual rules of addition.
The Adelos part follows the quadratic sum, similar to error propagation, where the total uncertainty increases as the square root of the sum of the squares of the individual uncertainties (these calculations will be addressed in the next chapters).

Other morphisms can include operations such as multiplication, subtraction, division, and so on, with similar rules for the propagation of the Adelos part.

Generalization: for a generic operation $f(x_a, y_a)$ between two empirical numbers, the transformation is expressed as:

$$\phi_f(x_a, y_a) = \left(f_d(x, y), g_a(f_d, x, y)\right)$$

where $g_a(f_d, x, y)$ represents a function that describes how the uncertainty propagates as a function of the operation f. These operations will be addressed in the following chapters.f.

Let's introduce a measurement theory that allows the transition from a domain consisting of empirical numbers to a codomain of traditional (measurable in a probabilistic sense) real numbers. We define a measurement morphism $\mu: D \to \mathbb{R}$ that "collapses" the empirical number into a real value. This morphism can be based on a probability distribution associated with the Adelos part:

$$\mu(d_a) = \mathbb{E}(\mathrm{d} + \xi) \; with \; \xi \sim P(0, ka)$$

Where ξ is a random variable distributed according to a probability distribution P with mean 0 and standard deviation equal to ka , which represents the uncertainty.

In this way, the measurement of an empirical number becomes a real value that incorporates both the Delos value and the Adelos uncertainty, expressed in the form of a probability distribution. The probabilistic theory applied to the Adelos part can follow the rules of classical probability, with distributions associated with the variability.

The codomain obtained through the measure morphisms μ will be a set of real numbers ℝ, which can be studied according to the rules of probability. Each empirical number is mapped to a real number according to its probability distribution, allowing the use of standard techniques in probabilistic analysis.

In general, the measure morphism can be defined as:

$$\mu : (x_d, x_a) \mapsto \mathbb{E}(X)\ con\ X = x_d + \xi,\quad \xi \sim D(x_a)$$

where $D(x_a)$ represents a distribution related to the uncertainty x_a, which may vary depending on the situation (normal, uniform, etc.).

Once the codomain $\mu(D) \subseteq \mathbb{R}$ is obtained, we can apply classical probabilistic theory to study the distribution of the values $\mu(d_a)$. This include:

- Analysis of the resulting distributions (mean, variance).
- Operations on distributions. Statistical and probabilistic inference on the initial empirical system.

This formalization allows us to study empirical numbers both in terms of their deterministic value (Delos) and their uncertainty (Adelos), through morphisms that preserve the empirical structure. Moreover, the measurement morphisms enable the transfer of empirical numbers to traditional numbers, linking them to a classical probabilistic theory based on measurement rules and distributions.
We now define the inverse measurement morphism, which, starting from a traditional real number, returns to the domain of empirical numbers as an estimate. We can introduce a map μ^{-1} that reconstructs the empirical number d_a from a real number r ∈ ℝ.

The inverse morphism $\mu^{-1}: \mathbb{R} \to D$ must take a real value r (which can represent an observed estimate, an average value, etc.) and return an empirical number (x_d, x_a), i.e., d_a, where:

- x_d is the estimated exact value (Delos).
- x_a is the estimate of the associated uncertainty (Adelos).

The inverse morphism can be defined as follows:

$$\mu^{-1}(r) = (r, \hat{x}_a)$$

where r ∈ ℝ is the real value, and $\hat{x}_a$ represents an estimate of the Adelos uncertainty, derived based on various considerations related to the context of the problem (such as the variance of measurements, standard error, or an estimated probability distribution).
To estimate $\hat{x}_a$, we can rely on various sources:

- Observational data: If we have access to a sample of measurements, x_a can be estimated by calculating the standard deviation of the measurements. In this case, $\hat{x}_a$ could be defined as the standard deviation σ of the measurements:

$$\hat{x}_a = \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(r_i - r)^2}$$

where r_i are the observations, and r is the mean.

In sampling situations, $\hat{x}_a$ could be based on the standard error of the mean, calculated as:

$$\hat{x}_a = \frac{\sigma}{\sqrt{n}}$$

where σ is the sample standard deviation, and n is th sample size.

If empirical data is unavailable, we can assume a probability distribution for the uncertainty, such as a normal distribution $N(0, \hat{x}_a)$, and estimate $\hat{x}_a$ based on an initial hypothesis or a known distribution.

The inverse morphism is not necessarily a perfect inversion (i.e., a one-to-one correspondence), as empirical numbers include an uncertainty component that cannot always be fully determined from a single real value. The map μ^{-1}may return a range or a set of possible empirical numbers, with varying levels of Adelos, based on the additional information available.
In practice, the estimation of uncertainty depends on the data and context, but the general structure of the μ^{-1} map follows the same concept: a real value is reconstructed into an empirical number consisting of a Delos estimate and an Adelos estimate.

2. **Monte Carlo Method.**[I]

Throughout the discussion, frequent reference will be made to Monte Carlo simulations to more explicitly illustrate certain characteristics under consideration.
The Monte Carlo method is a mathematical simulation technique used to solve problems that may be theoretically deterministic but are complex or impossible to resolve analytically due to their stochastic nature or the large number of variables involved. This method is named after the famous Monte Carlo casino, as it relies on probabilistic principles and the use of empirical numbers.

In our simulations, we will consider, for simplicity, normal distributions, also known as Gaussian distributions. It is a continuous probability distribution that can be defined as follows:

(2)

$$d_{(a)} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where:

- μ is the mean of the distribution (Delos of the empirical number),
- σ is the standard deviation (Adelos of the empirical number),
- σ^2 is the variance.

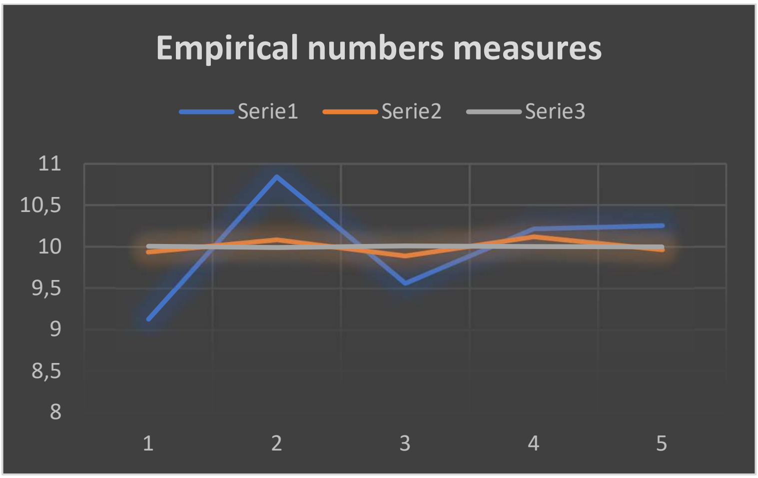
We now simulate, using the Monte Carlo method, five measurements of the empirical number. $10_{(1)}$:

9.127 10.843 9.562 10.215 10.253

We now evaluate the measurements of an empirical number $10_{(0.1)}$, which is much less variable:
9.937 10.086 9.891 10.121 9.965

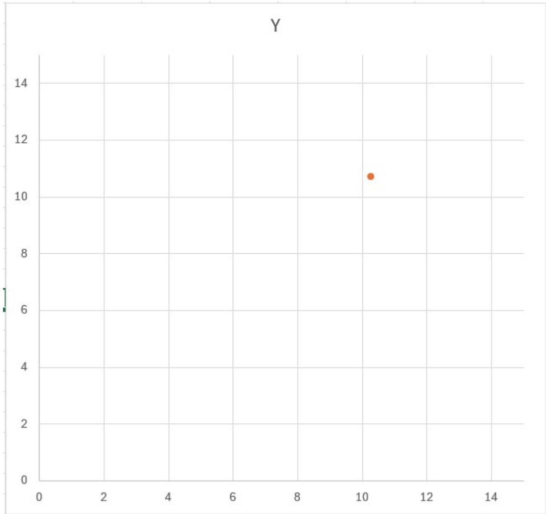
And now with the empirical number $10_{(0.01)}$:
10.005 9.990 10.011 9.998 9.996

Below, the three series are compared:

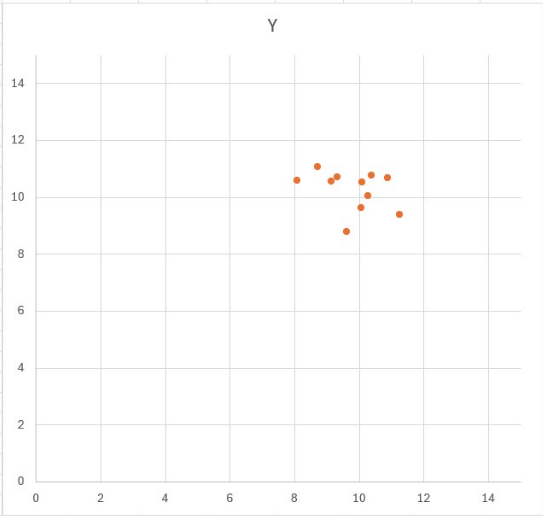


As seen from the graph, the three series present measured values around the Delos, but Series 1 exhibits a much more pronounced Adelos compared to the other two, while Series 3 shows a much lower Adelos than the others.

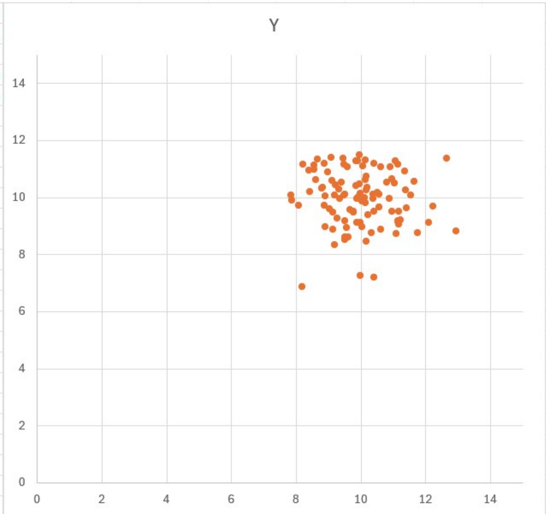
Let us now show a pair of empirical numbers $10_{(1)}$,$10_{(1)}$ in the X,Y plane with different measurements, and starting from these measurements, the estimation of the pair of empirical numbers.



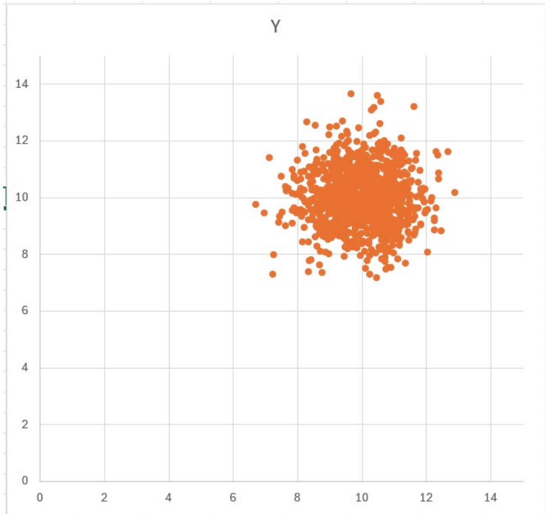
1 measure; $10.267_{(-)}$ $10.698_{(-)}$



10 measures; $9.803_{(0.703)}$ $10.244_{(0.941)}$



100 measures; $9.986_{(1.043)}$ $9.958_{(0.972)}$



1000 measures; $10.013_{(0.983)}$ $9.993_{(1.001)}$

In this context, the Monte Carlo method can be seen as a concrete exemplification of the concept of Delos and Adelos. Each Monte Carlo simulation operates as a mechanism that extracts numerical values from a set of potentials, represented by all possible data configurations within the defined range. In other words, each individual simulation is a measurement event that materializes one of the many potentials in the domain of the real.

In this case, Delos represents the set of possible manifestations of empirical values, while Adelos defines the limits within which these manifestations can occur, as in the case of the standard deviation in a normal distribution. In the Monte Carlo method, the individual values generated, while appearing real and unique, are in fact part of a larger whole, consisting of the entire space of possibilities described by probabilities. Each value emerges from this space, shaped by the play of probabilities and confined by the limits imposed by Adelos.

Furthermore, in Monte Carlo simulations, thousands of values are technically generated through the measurement process. The totality of these values expresses the empirical number in an extremely synthetic way, providing a statistical representation of the entire range of potentialities. The empirical reality, however, often manifests through only a few measurement events, akin to individual fragments of a larger picture, which by themselves fail to provide a comprehensive view. It is like judging the arrival of spring by observing a single swallow: a single event cannot capture the complexity of the whole.

The empirical number, on the other hand, manages to provide this broader view, aggregating the multiple potentialities and allowing us to grasp the overall essence of the phenomenon. Without this broader perspective, the overall design of reality would be lost, hidden behind the apparent randomness of individual measurement events. Only through the accumulation and synthesis of data can we discern the order hidden behind the chaos of individual measurements.

3. **Propagation of uncertainty and the Gradient Norm**

Mathematical operations on empirical numbers follow different rules compared to classical algebraic ones due to their nature. While the Delos part adheres to the usual rules of real numbers, the Adelos part follows a characteristic mechanism.

The technique developed here for the calculations of Adelos derives from the study of error propagation, based on the following concepts.[II]

Consider a function of two or more variables: $f(x_1, x_2, .., x_n)$ where variables $x_1, x_2, .., x_n$ are not correlated, and each of them is associated with its own uncertainty. $\Delta x_1, \Delta x_2, .., \Delta x_n$.
The extended uncertainty Δf of the function f can be calculated from the uncertainties of each variable x_i by computing the partial derivatives of f with respect to them:

(3)

$$\Delta f = \Delta f(x_1, x_2, .., x_n, \Delta x_1, \Delta x_2, .., \Delta x_n) = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \Delta x_i\right)^2}$$

The relation (3) holds in most cases where the function involved is sufficiently regular and is adequate for describing the effects of small variations in influencing factors and accidental errors. However, in some cases, there may be strong interactions between factors, which could require the inclusion of higher-order terms in the Taylor series expansion, including mixed terms.
In order to simplify the study of the propagation of Adelos in arithmetic operations, we will consider the isovariable case where $\Delta x_1 = \Delta x_2, = \cdots = \Delta x_n = a$ denoting 'a' as the generic uncertainty treated as the Adelos of the terms of the function, and Af as the extended uncertainty, interpreted as the Adelos of the function. In this case, relation (3) becomes:

(4)

$$Af = a \cdot \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2}$$

The function under the square root is nothing other than the Norm of the Gradient of the function under consideration:

(5)

$$\| \nabla f(x) \| = \sqrt{((\partial f/\partial x_1)^2 + (\partial f/\partial x_2)^2 + \cdots + (\partial f/\partial x_n)^2)}$$

In which the terms $\left(\frac{\partial f}{\partial x_i}\right)^2$ can be considered as characteristic functions of the sensitivity of the propagation of the Adelos a.

In fact, the arguments will mainly focus on the study of these functions and their influence in various computational contexts.

For brevity, we will denote these terms as D_i and thus relation (4) becomes:

(6)

$$Af = a \cdot \sqrt{\sum_{i=1}^{n} D_i}$$

Relation (6) can be expressed by relation (7) in the non-isovariable case:

(7)

$$Af = \sqrt{\sum_{i=1}^{n} D_i \cdot a_i^2}$$

From the notation introduced for the empirical number, it follows that any physical quantity will be expressed using the notation $d_{(a)}$, where a identifies the Adelos of its value d, representing the Delos.

We consider ow the propagation of errors with Covariance = 1

Let x_a and y_a be two empirical numbers, the covariance between x_a and y_a, denoted as $cov(x_a, y_a)$, measures the linear relationship between the two empirical variables. If $cov(x_a, y_a) = 1$ the adelos 'a' are perfectly correlated as the first variable is a copy of the second, like in a measurement process:

$$\boldsymbol{x}_{(\boldsymbol{a})} \xrightarrow[\boldsymbol{Measure}]{} \boldsymbol{x}_{(\mathbf{0})} = \boldsymbol{y}_{(\mathbf{0})}$$

Propagation of Errors Formula in additions operations:
The propagation of uncertainty for two quantities $x_a$and $y_a$that are combined, for example by addition or subtraction, can be expressed as:

(8)

$$a_z = \sqrt{a_x^2 + a_y^2 + 2\,cov(x_a, y_a)}$$

Where a_z represents the adelos of the result. If the covariance is 1, the formula becomes:

(9)

$$a_z = \sqrt{a_x^2 + a_y^2 + 2}$$

Example for a Hypotenuse Calculation:
Consider two segments x_a and y_a with covariance = 1. We are interested in calculating the hypotenuse $h_{ah}$of a right triangle where $x_a$and $y_a$represent the two catheti.
The length of the hypotenuse h is given by: $h = \sqrt{x^2 + y^2}$

To calculate the uncertainty in h, we apply the propagation of errors for a function of two variables, considering their covariance:

(10)

$$a_h = \sqrt{(\frac{\partial h}{\partial x})^2 a_x^2 + (\frac{\partial h}{\partial y})^2 a_y^2 + 2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}cov(x_a, y_a)}$$

First, compute the partial derivatives:

$$\frac{\partial h}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \frac{\partial h}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

Substituting these into the error propagation formula, we get:

(11)

$$a_h = \sqrt{(\frac{x}{\sqrt{x^2 + y^2}})^2 a_x^2 + (\frac{y}{\sqrt{x^2 + y^2}})^2 a_y^2 + 2\frac{x}{\sqrt{x^2 + y^2}}\frac{y}{\sqrt{x^2 + y^2}}} = \sqrt{a^2 + \frac{2xy}{x^2 + y^2}}$$

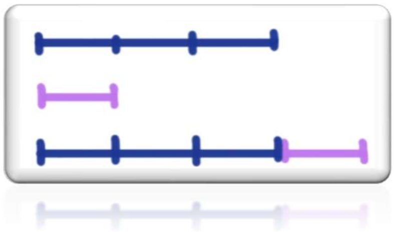
Considering $a = a_x = a_y$
This formula takes into account the uncertainties in both $a_x$and a_y as well as their covariance of 1. The resulting a_h gives the uncertainty in the hypotenuse h.

Previous examples demonstrates that including covariance considerably complicates the calculations. In most of our discussions on uncertainty propagation, we will assume the independence of empirical variables. By treating the elements of the calculation as independent random variables, we can simplify the analysis and focus on the fundamental aspects of uncertainty propagation.

We will evaluate the Adelos in some computational operations in the following chapters.

4. Mathematical Formalization and Operations

Sum of two segments



This is the case of two aligned segments.
Let $x1_{(a)}$ and $x2_{(a)}$ be two segments; suppose we join them in such a way as to obtain the sum segment s_{na} where the initial Adelos a will become Af times the initial one:

(12)

$$D_1 = \left(\frac{\partial(x1 + x2)}{\partial x1}\right)^2 = 1$$

(13)

$$D_2 = \left(\frac{\partial(x1 + x2)}{\partial x2}\right)^2 = 1$$

(14)

$$Af = \sqrt{2}$$

And therefore, the sum segment is given by:

(15)

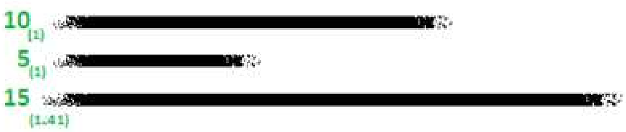
$$x1_{(a)} + x2_{(a)} = (x1 + x2)_{(\sqrt{2}a)}$$

where we note that the uncertainty of the sum segment increases relative to the uncertainties of the two initial segments by a factor of $\sqrt{2}$.

By numerically simulating two segments with values of $10_{(1)}$ and $5_{(1)}$, respectively, we obtain with 1000 samples:

$$9.985_{(1.025)} \hat{\mp} 4.974_{(1.006)} = 14.959_{(1,433)}$$

Graphically, the sum of two segments can be represented as follows:



The sum of empirical numbers, subtractions, multiplications, and divisions.

Generalizing, when the summation operation is repeated over m segments, the overall adelos increases according to the square root of m:

(16)

$$x1_{(a)} \hat{\mp} x2_{(a)} \hat{\mp} \ldots \hat{\mp}\, xm_{(a)} = \sum_{m}^{1} xm_{(\sqrt{m}a)}$$

The operator " $\hat{\mp}$ " indicates a sum between empirical numbers with covariance 0.

By numerically simulating four segments with values of $10_{(1)}$, $5_{(1)}$, $4_{(1)}$, $2_{(1)}$, respectively, we obtain, with 1000 samples, a doubling of the adelos as expected:

$$9.985_{(1.025)} \hat{\mp} 4.974_{(1.006)} \hat{\mp} 4.033_{(0.966)} \hat{\mp} 2.003_{(0.980)} = 20.996_{(1,941)}$$

We will use the operator "+" to indicate a symbolic sum (hyperuranic) even for empirical numbers. A symbolic operation refers to an operation that does not manifest in the external world but remains confined to thought (the covariance is 1).
In this case, let us consider that the expression is realized by the measurement for the first segment, while the subsequent applications are purely symbolic relative to the measured value:

(17)

$$\boldsymbol{d_{(a)} \xrightarrow[Measure]{} d'_{(0)}\ \ d'_{(0)} \underset{\Sigma}{\rightarrow} md'_{(ma)}}.$$

The segment to be added is expressed mentally and therefore does not change, and the sum of m identical symbolic segments will be:

(18)

$$x1_{(a)} + x1_{(a)} + \cdots + x1_{(a)} = mx1_{(ma)}$$

Simulating the sum of four symbolic segments with the characteristic $10_{(1)}$ using 1000 samples:

$$9.985_{(1.025)} + 9.985_{(1.025)} + 9.985_{(1.025)} + 9.985_{(1.025)} = 39.939_{(4,101)}$$

The results remain the same when considering distributions other than the normal one, such as the uniform or Weibull distribution.
In a sum, the adelos always increases proportionally to the number of sums "m" according to the relationship:

(19)

$$\widehat{a_{\Sigma}} = \sqrt{m}$$

If one wishes to construct a segment of any length starting from a small segment of arbitrary length, the adelos grows sub-linearly towards infinity according to $\sqrt{m}$.
The linear sum or subtraction of segments is an operation that considerably increases the adelos and progressively absorbs the delos, making it indistinguishable.
Simulating the sum of 1000 segments of length $1_{(1)}$ each, we obtain the following results from 5 consecutive simulations: 1044, 1001, 1024, 924, 954.

As will be seen in the next section, it is possible to construct segments without progressively increasing the adelos increment.

The operation of summing m equal segments is not generally the same as the operation of multiplying a segment by a factor of m.

(20)

$$\sum_{i=1}^{\hat{m}} x_{i(a)} = m \,\hat{\cdot}\, x_{(\sqrt{m}a)} \neq m \cdot x_{(a)} = mx_{(ma)}$$

The operation of subtracting m equal segments $x_{(a)}$ from a certain quantity $d_{(a)} = mx_{(a)}$ is not generally the same as the operation of dividing a segment by m.

(21)

$$d_{(a)} - \sum_{i=1}^{\hat{m}} x_{(a)} = x_{(\sqrt{m+1}a)} \neq \frac{d_{(a)}}{m} = x_{(a/m)}$$

Simulating with $d_{(a)} = 40_{(1)}$ $10_{(1)}$ and $x1_{(a)}, x2_{(a)}, x3_{(a)} = 10_{(1)}$ and m=3, we get $x_{(a)} = 9{,}987_{(1{,}944)}$

It should be noted that this also holds in the case of subtracting two segments, and in particular, we observe that if the two segments are equal, $x1_{(a)} = x2_{(a)}$ the difference:

(22)

$$x1_{(a)} \hat{-} x2_{(a)} = 0_{(\sqrt{2}a)}$$

Graphically, it is as if after the operation, only the adelos part remains as a residue:

In general, the difference between two equal empirical numbers does not result in a zero outcome. Simulating respectively, $x1_{(a)} = 10_{(1)}$ and $x2_{(a)} = 10_{(1)}$ with 1000 samples, the result obtained is $-0.013_{(1.413)}$.

Another consideration that can be made is that adding and subtracting the same quantity from a certain value does not yield the same initial value. Indeed:

(23)

$$x1_{(a)} \hat{+} x2_{(a)} \hat{-} x2_{(a)} = x1_{(\sqrt{3}a)} \neq x1_{(a)}$$

In the simulation, we obtain the following results for the same values as before: $10.100_{(1.744)}$.

While performing symbolically with the subtraction operation:

(24)

$$x1_{(a)} \hat{+} x2_{(a)} - x2_{(a)} = x1_{(a)}$$

In the simulation, we obtain for the same values as before $9.999_{(0.992)}$.

The non-isovariable case can be calculated with equation (7), therefore:

(25)

$$Af = \sqrt{\sum_{i=1}^{n} a_i^2}$$

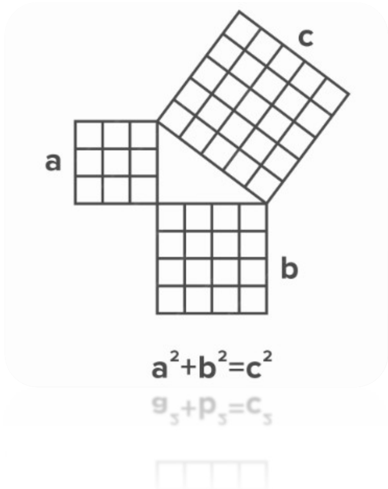
Simulating the sum of two numbers, $10_{(2)}$ and $5_{(1)}$, the sum becomes $15_{(\sqrt{5})}$ we obtain with 1000 samples:

$$9.969_{(2.051)} \widehat{\mp} 4.974_{(1.006)} = 14.948_{(2,28)}$$

5. The Pythagorean Theorem and Empirical Numbers

In this paragraph, we will encounter two significant properties of the Pythagorean Theorem. The first challenges the validity of the theorem when empirical numbers are employed. The second, on the other hand, unveils an exclusive property that allows for the construction of all natural numbers starting from a unitary element, with adelos remaining unchanged throughout the entire construction. This fact is particularly noteworthy.

Pythagorean Theorem

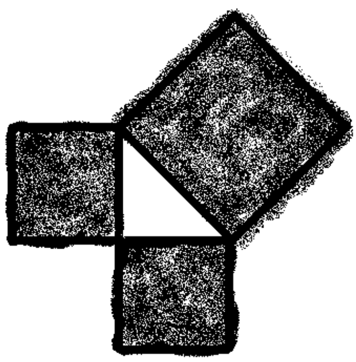


The Pythagorean Theorem states that the sum of the areas of the squares constructed on the two legs of a right triangle is equal to the area of the square constructed on the hypotenuse. However, this relationship does not hold when dealing with quantities expressed using empirical numbers. From result (11), we observe that:

$$Square\ on\ leg\ 1_{(a)} + Square\ on\ leg\ 2_{(a)} = \ Square\ on\ Hypotenuse_{(\sqrt{2}a)}$$

The absolute adelos of the area increases by a factor of $\sqrt{2}$, while the relative adelos stay the same.

It is as if the square constructed on the hypotenuse fades with each successive construction by an amount equal to $\sqrt{2}$



The $\sqrt{2}$, an irrational number that cannot be expressed as a finite fraction, embodies the infinity of uncertainty. The Pythagoreans, who initially sought to understand the world through their rigorous numerical framework, were confronted with a profound crisis when they encountered the irrationality of $\sqrt{2}$. This number challenged their belief in the finiteness and order of mathematical constructs, leading them into an existential and philosophical crisis about the nature of the infinite and its implications. On this rock, they were struck, finding their project stranded. They were so close to reaching the shore and finding stable ground, if only they had fully embraced the irrational hypotenuses to squared integers. These constructions will be developed further later in this chapter.

We can also consider constructing the squares starting from the legs according to the relationship $leg1_{(a)}{}^2 + leg2_{(a)}{}^2$

(26)

$$D_1 = \left(\frac{\partial(leg1^2 + leg2^2)}{leg1}\right)^2 = 4\, leg1^2$$

(27)

$$D_2 = \left(\frac{\partial(leg1^2 + leg2^2)}{leg2}\right)^2 = 4\, leg2^2$$

(28)

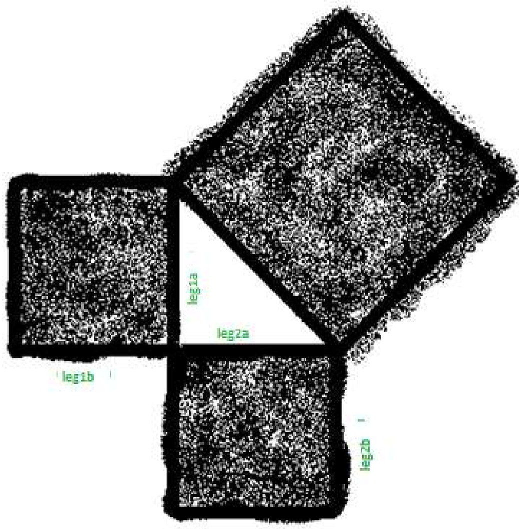
$$Af = 2\sqrt{leg1^2 + leg2^2} = twice\ the\ hypotenuse$$

In this case, the adelos increases by a factor double that of the hypotenuse. By numerically simulating two legs with values of $10_{(1)}$ and $10_{(1)}$, and a second with legs of $20_{(1)}$ and $20_{(1)}$, we obtain with 1000 samples:

$9.989_{(0.998)}$, $10.034_{(1.018)}$ → $202.487_{(28,469)}$
$19.983_{(1.045)}$, $20.080_{(1.002)}$ → $804.615_{(57,8)}$

In the case where we consider the construction legs as four empirical numbers, the adelos is calculated as the square constructed on the hypotenuse:

$leg1a_{(a)} \times leg1b_{(a)} + leg2a_{(a)} \times leg2b_{(a)}$



(29)

$$Af = \sqrt{leg1a^2 \times leg1b^2 + leg2a^2 \times leg2b^2}$$

The adelos grows less compared to the previous calculation. It is observed that, depending on the setup of the calculation, even though the delos is equivalent, this does not hold true for the adelos.

By numerically simulating the 4 legs with values of $10_{(1)}$, and a second simulation with legs of $20_{(1)}$, we obtain with 1000 samples:

$10.057_{(0.937)}$, $10.024_{(0.988)}$, $10.019_{(0.994)}$, $9.910_{(1.018)}$ → $200.048_{(19,702)}$
$19.981_{(0.987)}$, $20.033_{(0.993)}$, $20.030_{(1.031)}$, $20.093_{(0.984)}$ → $802.737_{(40,385)}$

Interesting constructions from the theorem

A key aspect of uncertainty propagation study in this paper is that, under certain conditions, the overall uncertainty does not necessarily increase and in some cases, it may even decrease. The hypotenuse calculation serves as a crucial example where the interplay between uncertainty components, can lead to unexpected results. Analysing this case in detail will provide insights into how empirical numbers behave in structured calculations constructions.

We now aim to determine the hypotenuse given the two independent legs, which is also referred to as the calculation of the Euclidean norm.

Let $x_{(a)}$ and $y_{(a)}$ be the legs of a triangle; the value of the hypotenuse $d_{(a)}$ will be:

(30)

$$D_1 = \left(\frac{\partial(\sqrt{x^2+y^2})}{\partial x}\right)^2 = \frac{x^2}{x^2+y^2}$$

(31)

$$D_2 = \left(\frac{\partial(\sqrt{x^2+y^2})}{\partial y}\right)^2 = \frac{y^2}{x^2+y^2}$$

$$Af = 1$$

Let us denote by $\overset{\perp}{\rightarrow}$ the operation of calculating the Euclidean norm, that is, the calculation of the hypotenuse length:

(32)

$$x_{(a)}, y_{(a)} \overset{\perp}{\rightarrow} \sqrt{x^2+y^2}_{(a)}$$

<u>this result tells us that this operation leaves the adelos unchanged.</u>

By numerically simulating two segments with values of $10_{(1)}$ and $10_{(1)}$, we obtain with 1000 samples:

$$10.578_{(0.997)}\ ,\ 10.041_{(0.987)} \xrightarrow{\perp} 14.968_{(0,983)}$$

The property of adelos conservation in the construction of the hypotenuse can be extended to the calculation of the Euclidean norm in any dimension:

(33)

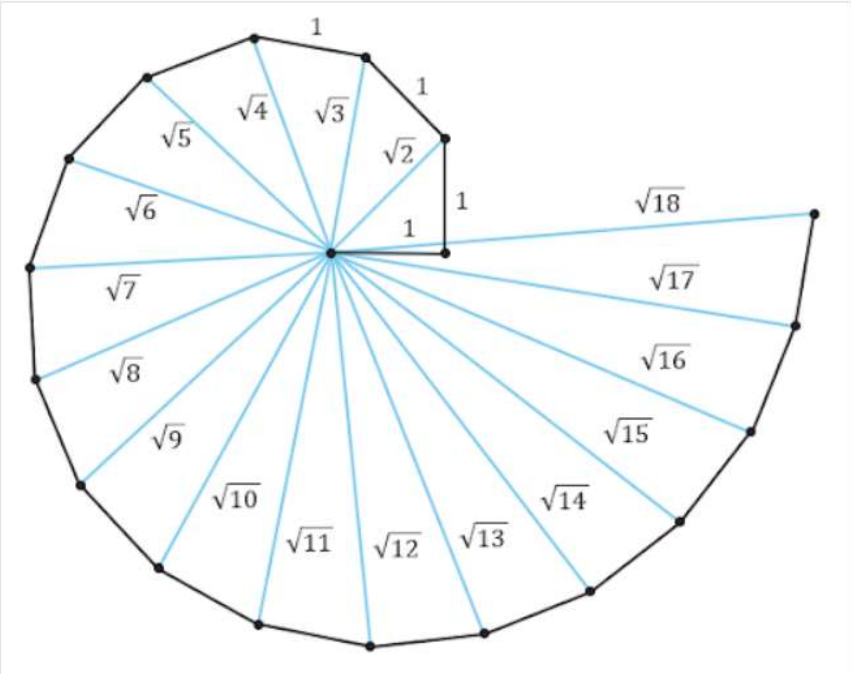
$$\boldsymbol{Af} = ||\,\boldsymbol{\nabla}||\boldsymbol{x}_{(\boldsymbol{a})}||\,|| = \mathbf{1}$$

We have just seen that the calculation of the Euclidean norm is an isovariable operation. Starting from a unit length segment $x_{(a)}$ we can construct a segment of any length and unit uncertainty **a** as a subsequent construction of hypotenuse segments and their squares starting from segments:

(34)

$$\boldsymbol{x}_{(\boldsymbol{a})} \overset{\boldsymbol{n}\perp}{\rightarrow} \boldsymbol{n}\boldsymbol{x}_{(\boldsymbol{a})}\ \forall\, \boldsymbol{n}^{\mathbf{2}} \in \mathbb{Z}$$

Geometrically, we can construct a segment corresponding to any Natural number through the construction of the Pythagorean spiral of Theodorus of Cyrene, as shown in the following diagram:



With this method, N^2 constructions are required starting from the unit length to obtain the length N, which can be as large as desired while keeping the adelos unchanged and equal to its initial value.

Now, let us examine a method that requires a smaller number of constructions to obtain a Natural number N:

2 is obtained as follows: $\mathbf{1}_{(a)}, \mathbf{1}_{(a)} \xrightarrow{1\perp} \sqrt{2}_{(a)}$ $\quad \sqrt{2}_{(a)}, \sqrt{2}_{(a)} \xrightarrow{2\perp} \mathbf{2}_{(a)}$

3 is obtained as follows: $1_{(a)}, 2_{(a)} \xrightarrow{1\perp} \sqrt{5}_{(a)}$ $\quad \sqrt{5}_{(a)}, 2_{(a)} \xrightarrow{2\perp} \mathbf{3}_{(a)}$

4 as: $\sqrt{2}_{(a)}, 1_{(a)} \xrightarrow{1\perp} \sqrt{3}_{(a)}$ $\quad \sqrt{3}_{(a)}, 2_{(a)} \xrightarrow{2\perp} \sqrt{7}_{(a)}$ $\quad \sqrt{7}_{(a)}, 3_{(a)} \xrightarrow{3\perp} \mathbf{4}_{(a)}$

5 (Pythagorean number) as: $3_{(a)}, 4_{(a)} \xrightarrow{1\perp} \mathbf{5}_{(a)}$

6 as: $\sqrt{2}_{(a)}, 3_{(a)} \xrightarrow{1\perp} \sqrt{11}_{(a)}$ $\quad \sqrt{11}_{(a)}, 5_{(a)} \xrightarrow{2\perp} \mathbf{6}_{(a)}$

7 as: $2_{(a)}, 3_{(a)} \xrightarrow{1\perp} \sqrt{13}_{(a)}$ $\quad \sqrt{13}_{(a)}, 6_{(a)} \xrightarrow{2\perp} \mathbf{7}_{(a)}$

8 as: $\sqrt{11}_{(a)}, 2_{(a)} \xrightarrow{1\perp} \sqrt{15}_{(a)}$ $\quad \sqrt{15}_{(a)}, 7_{(a)} \xrightarrow{2\perp} \mathbf{8}_{(a)}$

9 as: $1_{(a)}, 4_{(a)} \xrightarrow{1\perp} \sqrt{17}_{(a)}$ $\quad \sqrt{17}_{(a)}, 8_{(a)} \xrightarrow{2\perp} \mathbf{9}_{(a)}$

10 as: $\sqrt{3}_{(a)}, 4_{(a)} \xrightarrow{1\perp} \sqrt{19}_{(a)}$ $\quad \sqrt{19}_{(a)}, 9_{(a)} \xrightarrow{2\perp} \mathbf{10}_{(a)}$

And so on.

The rule for the construction of all natural numbers N is given as:

(35)

$$N_{(a)} = (N-1)_{(a)} \perp \sqrt{2N-1}_{(a)}$$

And where the irrational hypotenuse can be constructed as:

(36)

$$\sqrt{2N-1}_{(a)} = \sqrt{\left\lfloor \frac{2N-1}{2} \right\rfloor} \perp \sqrt{2N-1-\left\lfloor \frac{2N-1}{2} \right\rfloor}_{(a)}$$

Where the symbol ⌊ ⌋ denotes the greatest integer less than or equal to the expression within the parentheses.

It is easy to see that any quantity can be constructed by recursively using a previously constructed quantity with an integer or a smaller irrational hypotenuse.

The required number of constructions is:

(37)

$$(N-1)+\left\lfloor\frac{2N-1}{2}\right\rfloor-\left\lfloor\frac{\sqrt{2N-1}+1}{2}\right\rfloor+1+\left\lfloor\frac{N}{2}\right\rfloor-\left\lfloor\frac{\sqrt{N}}{2}\right\rfloor$$

Where different colours indicate integer terms, odd squares, and even squares, according to the following distinction. For N=10, this method requires 21 constructions compared to the 99 required by the spiral of Theodorus.

Integers: 9,8,7,6,5,4,3,2,1

Odd squares $\sqrt{19}$, $\sqrt{17}$, $\sqrt{15}$, $\sqrt{13}$, $\sqrt{11}$, ~~$\sqrt{9}$,~~ $\sqrt{7}$, $\sqrt{5}$, $\sqrt{3}$

Even squares $\sqrt{10}$, $\sqrt{8}$, $\sqrt{6}$, ~~$\sqrt{4}$,~~ $\sqrt{2}$

We will call this construction **"Τριμερὴς Γένεσις"** *(Trimerȅs Génesis)* — Tripartite Genesis.

The Tripartite Genesis harmoniously integrates integers, odd squares, and even squares, avoiding the direct construction of irrational hypotenuses as done, for example, in the Spiral of Theodorus. This makes it more compatible with an "orthodox Pythagorean" view of mathematics.

The tripartite genesis of Natural numbers exhibits gaps, much like prime numbers. However, in the latter case, the gaps do not follow a regular pattern as they do for perfect squares. For prime numbers, the absence of numbers grows in an unpredictable manner, with spacings increasing progressively and irregularly. In contrast, perfect squares create gaps according to a well-defined progression, dictated by the quadratic growth of their sequence.

This analogy suggests that both the tripartite genesis of natural numbers and the distribution of prime numbers share a structure with voids, yet with a fundamental difference: while perfect squares impose a geometric regularity in their distribution, prime numbers appear in a more chaotic manner, albeit following certain global statistical laws, such as the Prime Number Theorem.

One could thus hypothesize a generalization of the concept of "structured gap" in natural numbers, distinguishing between Delian gaps (perfect squares) and Adelian gaps (prime numbers).

In a geometric construction, we can imagine an iterative process in which, starting from a base (e.g., a unit segment x_a), all natural numbers can be generated through the construction of right-angled triangles. The hypotenuse can be interpreted as an "operator" that generates natural numbers from a simple element.

The construction of an empirical number is an intrinsically sequential and hierarchical process that must pass through all the previous stages, with no possibility of "mental shortcuts." A deep understanding of such a number requires awareness of all these intermediate steps. While it is possible to name or use an empirical number without mentally retracing the entire conceptual scale, its true "mathematical construction" necessarily implies this complete path, reflecting the complexity and layering of the very concept of number.

Functions equivalent in the Delos but not in the empirical field

The fundamental property of calculating the hypotenuse and its Adelos holds when the operation of constructing the square is performed empirically. This property also holds for other constructions but is not universally valid in general.

For example, the same result can be obtained starting from an analogous relation:

(38)

$$d = \frac{y}{sin\left(arctg\frac{y}{x}\right)}$$

(39)

$$D_1 = \left(\frac{\partial(\frac{y}{sin\left(arctg\frac{y}{x}\right)})}{\partial x}\right)^2 = \frac{1}{\frac{y^2}{x^2}+1}$$

$$D_2 = \left(\frac{\partial(\frac{y}{sin\left(arctg\frac{y}{x}\right)})}{\partial y}\right)^2 = \frac{y^2}{x^2+y^2}$$

$$Af = 1$$

By numerically simulating two segments with values of $10_{(1)}$ and $10_{(1)}$, respectively, we obtain, using 1000 samples, the same result as equation (24):

$x=10.005_{(1.005)}$, $y=9.987_{(1.020)}$ $d=14.174_{(1,007)}$

However, it does not hold, for instance, with a different construction:

(40)

$$d = \frac{y}{sin(\alpha)}$$

It produces the same Delos but does not leave the Adelos unchanged:

(41)

$$D_1 = \left(\frac{\partial(\frac{y}{sin(\alpha)})}{\partial\alpha}\right)^2 = y^2\, cotg^2(\alpha)\, cosec^2(\alpha)$$

(42)

$$D_2 = \left(\frac{\partial(\frac{y}{sin(\alpha)})}{\partial y}\right)^2 = cosec^2(\alpha)$$

(43)

$$Af = cosec(\alpha)\sqrt{(y^2\, cotg^2(\alpha) + 1)}$$

That for $\alpha \to 0$; $U \to \infty$

By numerically simulating two segments with values of $0.005_{(0.001)}$ and $10_{(0.001)}$, respectively, we observe, using 1000 samples, an increase in the Adelos by three orders of magnitude:

$\alpha = 0.005_{(0.001)}$, $y=10_{(0.001)}$ $d=10.304_{(2,22)}$

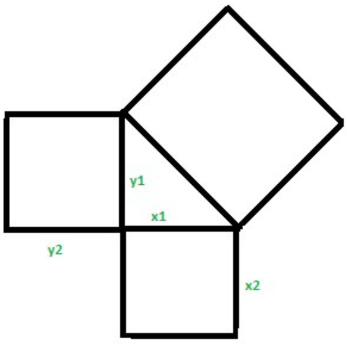
The conclusion is that, as previously observed for the addition and subtraction of the same quantity, certain invariance properties of operations valid in symbolic calculations no longer hold when performed with empirical numbers. Even the geometric construction of the operation itself may not always be equivalent, as seen in the earlier constructions of the hypotenuse segment.

Decreasing the adelos

We now aim to construct the hypotenuse by considering 4 segments:

(44)

$$x1_{(a)}, x2_{(a)}, y1_{(a)} y2_{(a)} \xrightarrow{\perp} \sqrt{x1\, x2 + y1\, y2}$$



The Adelos of this function is calculated as:

$$Af \ = \frac{1}{2}\sqrt{\frac{x1^2 + x2^2 + y1^2 + y2^2}{x1\, x2 + y1\, y2}}$$

Noting that the legs x1, x2 and y1, y2 are two measurements of the same empirical variable, the expression simplifies as:

(45)

$$Af \ = \frac{1}{2}\sqrt{\frac{2x^2 + 2y^2}{x^2 + \ y^2}} = \frac{\sqrt{2}}{2}$$

The relationship holds for the normal distribution but can be computed for any other distribution.

What is particularly interesting is that, unlike the case of constructing squares, in this case, the Adelos not only does not depend on the measurements of the legs and remains constant, but it even decreases. One might think that the repeated application of this operation could reduce the Adelos to an arbitrarily small value, but never to zero.

To reduce the Adelos to an arbitrarily small value, ε, n constructions must be performed according to the following relationship:

(46)

$$n = \frac{ln(\varepsilon)}{ln(\frac{\sqrt{2}}{2})}$$

Considering the sequence of constructed legs, one would arrive at the quantity:

$1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, 3 \ldots \sqrt{n}$

If, for example, one started with an initial Adelos of 1 mm and aimed to reach an Adelos equivalent to the Planck length, approximately 1,616 X 10^{-32}mm it would require n=212 constructions. Starting from a leg 10 mm long, one would arrive at a triangle with leg $10_{(a)}$ mm and $10 \times \sqrt{212} \approx 145{,}6_{(\varepsilon)}$ mm

In terms of the length of the initial unit segment, it must therefore have a length equal to:

(47)

$$u = \frac{L}{\sqrt{n}}$$

Where L is the length of the segment one wishes to obtain with an Adelos smaller than ε.

This implies that the first n constructions will have an Adelos greater than ε.

To reduce the Adelos of the initial constructions, one could consider starting from the constructed hypotenuses and reconstructing the legs. This operation generally increases the Adelos, but for specific constructions, it may reduce it.

Starting from the hypotenuse and working backward to one of the legs, the uncertainty is not invariant and depends on the initial values themselves:

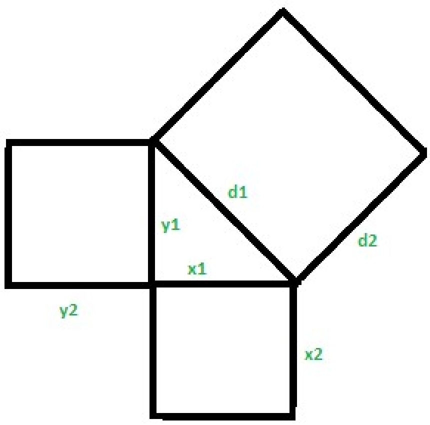
(48)

$$x = \sqrt{d^2 - y^2}_{(2\sqrt{d^2+y^2})}$$

If we perform the inverse construction of (38):

(49)

$$x1_{(a)}, x2_{(a)}, d1_{(a)} d2_{(a)} \xrightarrow{\perp'} \sqrt{d1\, d2 - x1\, x2}$$



The Adelos of this function is calculated as:

$$Af = \frac{1}{2}\sqrt{\frac{d1^2 + d2^2 + x1^2 + x2^2}{d1\,d2 - x1\,x2}}$$

Noting that the legs x1, x2 and the hypotenuses d1, d2 are similar, the expression simplifies as:

(50)

$$Af = \frac{\sqrt{2}}{2}\sqrt{\frac{d^2 + x^2}{d^2 - x^2}}$$

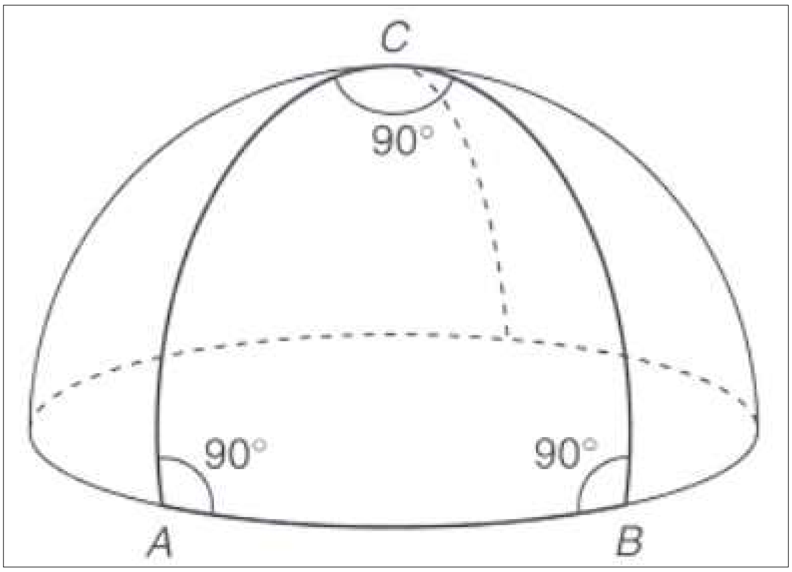
With appropriate choices of the hypotenuse and the leg, it is possible to obtain an Adelos of the calculated leg smaller than 1, leading to a reduction of the Adelos.

It can be easily calculated that with a ratio of d and x less than $\sqrt{3}$, the Adelos is reduced.

For example, starting with d=2$_{(a)}$ and x=1$_{(a)}$ the second leg is obtained as $\sqrt{3}_{(\sqrt{5/6}a)}$

Alternatively, starting with d=41$_{(a)}$ and x=9$_{(a)}$ the second leg is obtained as $40_{(\sqrt{881/1600}a)}$

We now ask whether what was found for the right triangle in Euclidean space also holds for a spherical surface, and therefore, we will proceed with measuring a geodesic on a spherical surface.



Consider the Pythagorean theorem for spherical triangles, that is, the cosine of the hypotenuse is equal to the product of the cosines of the two legs.

Let $\boldsymbol{r}$ be the radius of the sphere, $\boldsymbol{a}$ the horizontal side, and $\boldsymbol{b}$ the vertical side. The distance between two points is given by the relation:

(51)

$$d = r\, arcos\,(cos\frac{a_{(u)}}{r}\;\; cos\frac{b_{(u)}}{r})$$

(52)

$$Af = \sqrt{\frac{(sen^2\frac{a}{r}\; cos^2\frac{b}{r} + sen^2\frac{b}{r}cos^2\frac{a}{r})}{1 - cos^2\frac{a}{r}\;\; cos^2\frac{b}{r}}}$$

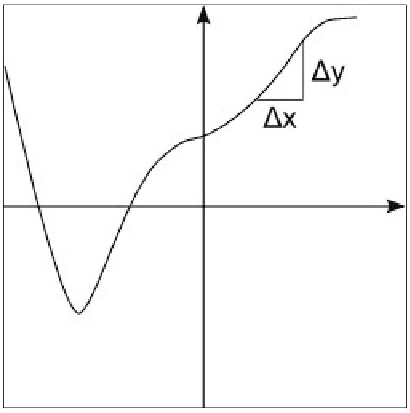
The uncertainty in the measurement of the geodesic is not constant and varies as a function of the distance between the points considered, oscillating between 0 and 1.

In principle, this behaviour allows for the identification of the space in which the measurements are made, and through the analysis of the Adelos, it is possible to determine whether we are in a Euclidean or non-Euclidean space.

6. Application in Calculus

The Derivation

Let $y = f(x)$ be a function defined on an interval [a,b], and let $\frac{\Delta y}{\Delta x}$ be the incremental ratio of the function in the vicinity of a point x_0 that lies within the interval.



From Calculus, we know that if, as the increment Δx of the variable tends to zero, the limit of the incremental ratio of a function in the vicinity of one of its points exists and is finite, then this limit is the derivative of the function at that point:

$$f'(x0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Geometrically, the meaning of the derivative of a function at a given point is that it represents the slope of the tangent line to the curve at that point.

When working with empirical data, the derivative can be approximated using finite differences. Suppose we have two measurements of the same variable x_a, the approximated derivative can then be expressed as:

(53)

$$f'\left(x_{(a)}\right) = \frac{f\left(x_{(a)}\right) \hat{-} f\left(x_{(a)}\right)}{x_{(a)} \hat{-} x_{(a)}} = d_{(a')}$$

The measurements are distinct and not identical, of the same variable. This methodology, based on finite differences, allows us to approximate the derivative without resorting to infinitesimal calculus, that is:

(54)

$$f'\left(x_{(a)}\right)_{(a')} = \frac{f\left(\overrightarrow{\overset{1}{x_{(a)}}}\right) - f\left(\overrightarrow{\overset{2}{x_{(a)}}}\right)}{\overrightarrow{\overset{1}{x_{(a)}}} - \overrightarrow{\overset{2}{x_{(a)}}}} = d_{(a')}$$

The notation $\overrightarrow{\overset{1}{x_{0_{(a)}}}}$ refers to the first measurement, and $\overrightarrow{\overset{2}{x_{0_{(a)}}}}$ refers to the second measurement, where the values of the measurements cannot be identical. Moreover, a’ represents the Adelos of the derivative f’.

Let's see in the following table some formulas for the fundamental derivatives:

Functions	Delos of y' (d)	Adelos of y' (a')
Constant Function $y = k$	0	0
Power Function $y = x^n, n \in \mathbb{R}$	nx^{n-1}	$\frac{\lvert n(n-1)\rvert}{\sqrt{2}} x^{n-2}$
$y = kx$	k	0
$y = \frac{1}{x}$	$-\frac{1}{x^2}$	$\sqrt{2}x^{-3}$
$y = \sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\frac{1}{4\sqrt{2}}x^{-\frac{3}{2}}$
$y = \sqrt[3]{x}$	$\frac{1}{3x^{\frac{2}{3}}}$	$\frac{2}{9\sqrt{2}}x^{-\frac{5}{3}}$
$y = \frac{1}{x^2}$	$-\frac{2}{x^3}$	$\frac{6}{\sqrt{2}}x^{-4}(*)$
$y = \frac{1}{x^3}$	$-\frac{3}{x^4}$	$\frac{12}{\sqrt{2}}x^{-5}(*)$
Absolute value Function $y = x$	$\frac{\lvert x\rvert}{x}$	0
Logarithmic Function $y = \ln(x)$	$\frac{1}{x}$	$\frac{1}{\sqrt{2}x^2}(*)$
Exponential Function $y = e^x$	e^x	$\frac{e^x}{2\sqrt{2}}(*)$
Trigonometric Function $y = \sin x$	$\cos x$	$\frac{\sin x}{\sqrt{2}}(*)$
$y = \cos x$	$-\sin x$	$\frac{\cos x}{\sqrt{2}}(*)$
$y = \tan x$	$\frac{1}{\cos^2 x}$	$\propto x$ (**)

(*) Empirically found form

(**) Form not yet determined

Integration

Let $y = f(x)$ be a function defined and bounded on a closed interval $[\alpha;\beta]$. Geometrically, the definite integral represents the area under the curve of the function and the x-axis between $x = \alpha$ and $x = \beta$.

Let $\{x_0, x_1, \dots, x_n\}$ be a partition of the interval $[a;b]$ such that $\alpha = x_0 < x_1 < \cdots < x_n = \beta$, and let $\Delta x_i = x_i - x_{i-1}$ be the widths of the subintervals. Consider a point $x_{i_{(a)}}$ within each subinterval $[x_{i-1} - x_i]$. To measure the definite integral, we calculate the area as:

(55)

$$\int_{\alpha}^{\beta} f(x_{(a)}) = \sum_{i=0}^{n} f\left(x_{i_{(a)}}\right) \Delta x_i \;=\; d_{(a')}$$

Here, the goal is not to study the expression of the Adelos for every algebraic function, but to mention that there are functions where the Adelos increases and others where it decreases, as we develop the integral calculus.

The calculation can be carried out by using the Fundamental Theorem of Calculus with the employment of empirical variables:

(56)

$$\int_{\alpha}^{\beta} f(x_{(a)}) = F(\beta) - F(\alpha) \;=\; d_{(a')}$$

where $F'(x_{(a)}) = f(x_{(a)})$.

In this case, the calculation of the Adelos is directly applied by using equation (7).

7. **Conclusions.**

This work presents an innovative formalization of uncertainty propagation through the introduction of empirical numbers, offering a new perspective on the nature of uncertainty in mathematics. The most significant aspect lies not in specific numerical results, but in the conceptual framework that enables the systematic study of propagation and reduction of the Adelos.

Three key contributions are of particular relevance:

1. The formalization of empirical numbers provides a rigorous mathematical tool for tracing the propagation of uncertainty through various mathematical operations. This allows for understanding how the Adelos propagates differently depending on the mathematical constructions used, as demonstrated in the emblematic case of the Pythagorean theorem.
2. The discovery that certain mathematical constructions can maintain or even reduce the Adelos, as shown in geometric constructions based on the Pythagorean theorem. This property opens new perspectives for designing algorithms and mathematical constructions that inherently minimize the propagation of uncertainty.
3. The extension of this framework to differential and integral calculus, which reveals how uncertainty propagates through fundamental operations of mathematics, providing new tools for analyzing the robustness of mathematical models.

These results have profound implications, both theoretical and practical:

- In pure mathematics, they offer a new way of conceiving mathematical constructions, where the management of Adelos becomes an explicit design criterion.
- In applied sciences, they provide tools to design algorithms and computational methods that actively control the propagation of uncertainty.
- In scientific methodology, they enable the quantitative evaluation of the robustness of mathematical constructions with respect to initial uncertainty.

This new perspective transforms the study of uncertainty from a problem of post hoc management to an intrinsic element of mathematical construction, opening new research directions in number theory, geometry, and mathematical analysis. In particular, the possibility of constructing sequences that reduce the Adelos suggests the existence of fundamental mathematical structures yet to be explored in the context of quantified uncertainty.

References and Notes.

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