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ABOUT THE RIEMANN HYPOTHESIS

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1. Introduction

Bernhard Riemann made the hypothesis, that is here proposed to confirm, that the complex xi (ξ) function zeros are real [2] (p.139). The eta (η) Dirichlet function will also be proposed to be used for a proof that the Riemann zeta function nontrivial zeros, which are linked to the xi zeros, have real part equal to $\frac{1}{2}$.

2. The zeros of ξ

Theorem 2.1. *There exists a real sequence $(a_n)_{n\in\mathbb{N}}$ such that the Riemann ξ function can be written such as, for $t \in \mathbb{C} : \xi(t) = \sum_{n=0}^{\infty}(-1)^n|a_n|t^{2n}$.*

Proof. According to Riemann [2] (p.138), for $t \in \mathbb{C}$:

$$\xi(t) = 4\int_1^{\infty} \frac{d(x^{\frac{3}{2}}\psi'(x))}{dx} x^{-\frac{1}{4}} \cos(\frac{1}{2}t\log(x))\,dx.$$

So, as Riemann typed: "Diese Function... ...lässt sich nach Potenzen von tt in eine... ...convergirende Reihe entwickeln.", which can be translated as "This function... ...allows itself to be developed in powers of tt... ...as a converging series.", the Riemann ξ function can be such as, for $t \in \mathbb{C}$: $\xi(t) = \sum_{n=0}^{\infty} a_n(t^2)^n$, where for $n \in \mathbb{N}$:

$$a_n = 4\,\frac{(-1)^n}{2^{2n}(2n)!}\int_1^{\infty} \frac{d(x^{\frac{3}{2}}\psi'(x))}{dx} x^{-\frac{1}{4}} (\log(x))^{2n}\,dx.$$

The function $\frac{d(x^{\frac{3}{2}}\psi'(x))}{dx} = \pi x^{\frac{1}{2}} \sum_{n=1}^{\infty}\left(n^2\pi x - \frac{3}{2}\right) n^2 e^{-n^2\pi x}$ being positive on $(1;+\infty)$, and having a finite limit at 1^+, for all $n \in \mathbb{N}$, $(-1)^n a_n > 0$, which gives the theorem 2.1. □

Noting for $z \in \mathbb{C}$, $\mathrm{Arg}(z)$ as the principal argument of z being in $(-\pi;\pi]$, $(u_n(t))_{n\in\mathbb{N}} = (a_n t^{2n})_{n\in\mathbb{N}}$, follows this theorem:

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Theorem 2.2. *For any t in $\mathbb{C}$ such that $\Re(t) \neq 0$, $\Im(t) \in (-\frac{1}{2}, \frac{1}{2})$ and $\xi(t) = 0$, t is real.*

Proof. Let be $t = a + ib \in \mathbb{C}$ such that $a \neq 0, b \in (-\frac{1}{2}; \frac{1}{2})$, and $\xi(t) = 0$. Thus,

$$\sum_{n=0}^{\infty} u_{n+1} = -a_0. \tag{1}$$

Let us name $z = \sum_{n=0}^{\infty} |u_{n+1}(t)| e^{i \operatorname{Arg}(u_n(t))}$ (convergence confirmed below). We have:

$$\begin{aligned}
\Im(z) &= \Im\left(\sum_{n=0}^{\infty} |u_{n+1}(t)| e^{i\frac{1}{2} \operatorname{Arg}(u_n^2(t))}\right) \\
&= \Im\left(\sum_{n=0}^{\infty} |u_{n+1}(t)| e^{i\frac{1}{2} \operatorname{Arg}\left(\left(\frac{u_n(t)}{u_{n+1}(t)}\right)^2 u_{n+1}^2(t)\right)}\right) \\
&= \Im\left(\sum_{n=0}^{\infty} |u_{n+1}(t)| e^{i\frac{1}{2} \operatorname{Arg}\left(t^{-4} u_{n+1}^2(t)\right)}\right) \\
&= \Im\left(\sum_{n=0}^{\infty} |u_{n+1}(t)| e^{i \operatorname{Arg}\left(t^{-2} u_{n+1}(t)\right)}\right) \\
&= \Im\left(\sum_{n=0}^{\infty} u_{n+1}(t) e^{i \operatorname{Arg}\left(t^{-2}\right)}\right) \\
&= \Im\left(-a_0 e^{i \operatorname{Arg}(t^{-2})}\right) \text{ from the equation (1)}
\end{aligned}$$

$$\Im(z) = -a_0 \sin(\operatorname{Arg}(t^{-2})). \tag{2}$$

And:

$$\begin{aligned}
\Im(z) &= \Im\left(\sum_{n=0}^{\infty} |u_{n+1}(t)| e^{i \operatorname{Arg}(u_n(t))}\right) \\
&= \Im\left(\sum_{n=0}^{\infty} u_{n+1}(t) e^{i \operatorname{Arg}\left(\frac{u_n(t)}{u_{n+1}(t)}\right)}\right) \\
&= \Im\left(\sum_{n=0}^{\infty} u_{n+1}(t) e^{i \operatorname{Arg}(-t^{-2})}\right) \\
&= \Im\left(-a_0 e^{i \operatorname{Arg}(-t^{-2})}\right) \text{ from the equation (1)}
\end{aligned}$$

$$\Im(z) = a_0 \sin(\operatorname{Arg}(t^{-2})). \tag{3}$$

Thus, thanks to the equations (2) and (3): $a_0 \sin(\operatorname{Arg}(t^{-2})) = 0$. Then, $t^2 = a^2 - b^2 + i2ab$ is real, $b = 0$ because $a \neq 0$, and t is real. ☐

Therefore, as Riemann typed [2] (p.138) "...so kann die Function $\xi(t)$ nur verschwinden, wenn der imaginäre Theil von t zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$ liegt.", which can be translated as "...it follows that the function $\xi(t)$ can only vanish if the imaginary part of t lies between $\frac{1}{2}i$ and $-\frac{1}{2}i$", with the lemma 2.1 is proposed that the Riemann hypothesis is confirmed.

Lemma 2.1. *For any t in $\mathbb{C}$ such that $\Re(t) = 0$ and $\Im(t) \in (-\frac{1}{2}, \frac{1}{2})$, $\xi(t)$ is not null.*

Proof. We proceed by contradiction. Let be t in $\mathbb{C}$ such that $\Re(t) = 0$, $\Im(t) \in (-\frac{1}{2}, \frac{1}{2})$ and $\xi(t) = 0$. Then, $\xi(t) = \sum_{n=0}^{\infty} |a_n|(\Im(t))^n$. Because of the symmetry of the zeros about the real axis (which can be proved by saying that $\xi(\bar{t}) = 0$ using the polar form of the xi expression of the theorem 2.1), $a_0 = 0$. But a_0 is positive. We have the lemma 2.1. $\square$

3. The nontrivial zeros of ζ

In 1859, Riemann wrote: $\Gamma(\frac{s}{2}+1)(s-1)\pi^{\frac{-s}{2}}\zeta(s) = \xi(t)$ where $s = \frac{1}{2} + it$. Let us now propose to use the Dirichlet eta (η) function, present in an analytic continuation expression of ζ, on $0 < \Re(s) < 1$, where are its nontrivial zeros.

Lemma 3.1. *For any a in $(0, \infty)$, any n in $\mathbb{N}$, any s in $\mathbb{C}$, if $\Re(s)$ is in $(0, 1)$ and $\zeta(s) = 0$, then $\int_0^{\infty} \frac{x^{s-1}}{e^{xa^{-4n}}+1}\,dx = 0$.*

Proof. Let be a a positive real number, n in $\mathbb{N}$, and s in $\mathbb{C}$, such that $\Re(s) \in (0, 1)$ and $\zeta(s) = 0$. Lang [1] (p.157) and Spiegel [3] (line 15.82) give, for $\Re(s) > 0$:

$$\zeta(s) = \frac{\eta(s)}{1-2^{1-s}} \quad \text{and} \quad \eta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x+1}\,dx.$$

Then, $\zeta(s) = 0$ implies that $\int_0^{\infty} \frac{x^{s-1}}{e^x+1}\,dx = 0$, and the variable change $x = ua^{-4n}$ gives the lemma 3.1. $\square$

Theorem 3.1. *For any s in $\mathbb{C}$, if $\Re(s)$ is in $(0, 1)$, $\Im(s) \neq 0$ and $\zeta(s) = 0$, then $\Re(s) = \frac{1}{2}$.*

Let be $s = \sigma + it$ a nontrivial zero, with σ in $(0, 1)$ and t a positive real number, and let be $a = \exp\left(\frac{\pi}{2t}\right)$. The lemma 3.1 gives that:

$$\int_0^1 \frac{x^{s-1}}{e^{xa^{-4n}}+1}\,dx = -\int_1^{\infty} \frac{x^{s-1}}{e^{xa^{-4n}}+1}\,dx$$

n going to infinity, the dominated convergence theorem makes the left term converge to $\frac{1}{2s}$. Thus,

$$\begin{aligned}\Re\left(\frac{1}{2s}\right) + \underset{n\to\infty}{o(1)} &= \Re\left(-\int_1^{+\infty} \frac{x^{s-1}}{e^{xa^{-4n}}+1}\,dx\right)\\ &= -\int_1^{\infty} \frac{x^{-\sigma}\cos\left(t\,\frac{1-\sigma}{\sigma}\ln(x)\right)}{1+\exp\left(a^{-4n}\,x^{\frac{1-\sigma}{\sigma}}\right)}\,dx, \text{with the variable change } x = u^{\frac{1-\sigma}{\sigma}}\\ &= \int_0^{1} \frac{x^{-\sigma}\cos\left(t\,\frac{1-\sigma}{\sigma}\ln(x)\right)}{1+\exp\left(a^{-4n}\,x^{\frac{1-\sigma}{\sigma}}\right)}\,dx\end{aligned}$$

with the same variable change applied to $\Re\left(\int_0^{\infty} \frac{x^{s-1}}{e^{xa^{-4n}}+1}\,dx\right) = 0$.

n going to infinity, the dominated convergence theorem makes this last term converge to

$$\frac{1}{2}\int_0^1 x^{-\sigma}\cos\left(t\,\frac{1-\sigma}{\sigma}\ln(x)\right)dx,$$

which equals $\frac{1}{2}\,\frac{(1-\sigma)^2}{\sigma}\,\frac{1}{(1-\sigma)^2+t^2}$. Then, because this term equals $\Re\left(\frac{1}{2s}\right)$:

$$\frac{(1-\sigma)^2}{\sigma}\,\frac{1}{(1-\sigma)^2+t^2} = \frac{\sigma}{\sigma^2+t^2}.$$

This equation implies that $t^2((1-\sigma)^2-\sigma^2) = 0$, and t being non null, $\sigma = \frac{1}{2}$. The zeta nontrivial zeros being symmetric about the real axis (which can be proved saying that $\zeta(\bar{s}) = 0$ using eta), with the lemma 3.2 is proposed that the Riemann hypothesis is confirmed.

Lemma 3.2. *For any s in $\mathbb{C}$ such that $\Im(s) = 0$, and $\Re(s)$ in $(0,1)$, $\zeta(s)$ is not null.*

Proof. We proceed by contradiction. Let be $s \in \mathbb{C}$ such that $\Re(s) = \sigma \in (0;1), \Im(s) = 0$, and $\zeta(s) = 0$. Then $\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^{\sigma}} = 0$ and the series $\left(\sum_{n=0}^{m} (2n+2)^{-\sigma} - (2n+1)^{-\sigma}\right)_{m\in\mathbb{N}^*}$ converges to 0. We define, for $x \in [1,\infty), f(x) = x^{-\sigma}$. With the mean value theorem, exists a sequence $(b_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}: \frac{f(2n+2)-f(2n+1)}{(2n+2)-(2n+1)} = f'(b_n) = -\sigma b_n^{-\sigma-1}$. Then, $\sum_{n=0}^{+\infty} f'(b_n) = 0 = \sum_{n=0}^{+\infty} \frac{1}{b_n^{\sigma+1}}$. But, for all n in $\mathbb{N}, b_n$ is in $(2n+1, 2n+2)$ and therefore positive, we have the lemma 3.2. □

References

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[2] B. Riemann. Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse. *Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, **7**:136–139, November 1859.

[3] M. R. Spiegel. *Mathematical Handbook of Formulas and Tables*. Schaum. Mac Graw Hill, January 1986.

Abstract. Here is proposed to confirm the Riemann hypothesis.