Is there an infinite number in which the sum of the divisors is the square number?

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Abstract

The infinitude of numbers whose sum of divisors is a perfect square does not appear to be directly related to prime numbers. However, upon listing such numbers, certain patterns involving primes were discovered. In this study, we show that the infinitude of numbers whose sum of divisors is a perfect square is equivalent to the infinitude of primes represented by a certain irreducible polynomial. Furthermore, we explore its connection to Bunyakovsky's conjecture. Through this analysis, we investigate a new relationship between the divisor function and the distribution of prime numbers in number theory.

$$3, 22, 66, 70, 81, 94, 214, 282, \ldots$$

Is there a rule in the sequence? At first glance, it looks like a random arrangement of numbers, but those numbers are numbers where the sum of the divisors becomes square. In other words, It is a list of numbers in which the sum of the divisors becomes square. Let's call such a sequence “DS sequence”

There is Divisor function

$$\sigma(n) = \sum_{d|n} d$$

It is a function that derives the sum of divisors. for examples

$$\sigma(3)=1+3=4$$
$$\sigma(22)=1+2+11+22=36$$
$$\sigma(66)=1+2+3+6+11+22+33+66=144$$
$$\sigma(70)=1+2+5+7+10+14+35+70=144$$
$$\sigma(81)=1+3+9+27+81=121$$
$$\sigma(94)=1+2+47+94=144$$
$$\sigma(214)=1+2+107+214=324$$
$$\vdots$$

There are squares all and we can find out the rules

$$22=11\times 2\,,66=11\times 6\,,94=2\times 47\,,214=2\times 107$$

Not all numbers are like that, but we can see a case where a double (primes) Is included in DS sequence.

And in the case of 66, six times the number (primes) is included in DS sequence.

Now we can guess Right, six times, six times of six times, six times and 107.

66 is included in A and Based on the association of 22,94,214.

$$47\times 6=282\,,107\times 6=642$$
$$\sigma(282)=1+2+3+6+47+94+141+282=576=24^2$$
$$\sigma(642)=1+2+3+6+107+214+321+642=1296=36^2$$

Surprisingly, it was established as a speculation.Then, now we have to think about 11 And 107. First, 11 and 107 are primes

$$11,47,107,\ldots$$

$$47-11=36\ ,\ 107-47=60\ ,\quad 36=6\times(6+0)\,,60=6\times(6+4)$$

So you can think of it as a sequence that grows by $6\times(6+4n)$, according to Speculation $107+6\times(6+8)=191\,,2\times 191=382\,,6\times 191=1146$

382,1146 shall be included in A.

First, 191 is a prime number. Let's see if 382 and 1146 are included in A.

$$\sigma(382)=1+2+191+382=576=24^2$$
$$\sigma(1146)=1+2+3+6+191+382+573+1146=2304=48^2$$

Satisfied , then let’s go on.

$$11,47,107,191,\ldots\ \ 191+6\times(6+12)=191+108=299$$

First, 299 is not a prime number. It is decomposed by the product of 23 and 13.

Let's see if 598 is included in A, though.

$$\sigma(598)=1+2+13+23+26+46+299+598=1008=2^4\times 3^2\times 7$$

The drug sequence does not include 598, which is twice the number of 299. Still, there is hope, Because it may be included in 11,47,107,191,299,... and only double or six times the prime number may be included in DS sequence. So let's try to develop the sequence again.

299+132=431, 431 is a prime number. Two times 431 is 862, and six times is 2586.

$$\sigma(862) = 1+2+431+862 = 1296 = 36^2$$
$$\sigma(2586) = 1+2+3+6+431+862+1293+2586 = 5184 = 72^2$$

Surprisingly satisfied again. So, 11,47,107,191,299,431,... It can be assumed that two or six times the number of prime numbers included in this sequence is included in A.

Now, let's find the general term of that sequence. There is recurrence relationsequence

$$a_n = a_{n-1} + 6\times(6+4(n-2))\ (a_1 = 11\,, n \geq 2)$$

of $\{a_n\}$ $\quad a_n = a_{n-1} + 24n - 12$

Let's $A_n = a_{n+1} - a_n$ then $\{A_n\}$is arithmetical sequence with common difference 24

Therefore, we can assume that the general term of $\{a_n\}$ is quadratic.

The grounds are as follows

<The general term of $\{b_n\}$ is a quadratic equationif $\{b_n\}$ is a monotonic increasing sequence and $\{b_{n+1} - b_n\}$ is a rithmetical sequence for all n.>

<proof>

$$b_{n+1} - b_n = c_1 + (n-1)d = c_n\ (d \neq 0)$$

$$b_n = b_1 + \sum_{k=1}^{n-1} c_k = b_1 + \sum_{k=1}^{n-1} c_1 + (k-1)d = b_1 + c_1\sum_{k=1}^{n-1} 1 + d\sum_{k=1}^{n-1}(k-1)$$

$\displaystyle\sum_{k=1}^{n-1} 1 = n-1$ And $\displaystyle\sum_{k=1}^{n-1}(k-1) = \frac{(n-1)(n-2)}{2}$

$\displaystyle\therefore\ b_n = b_1 + c_1(n-1) + \frac{d(n-1)(n-2)}{2}$, quadratic equation for n. ■

Therefore, $a_n = An^2 + Bn + C\ (A \neq 0)$

$$a_1 = A + B + C = 11\ ,\ a_2 = 4A + 2B + c = 47\ ,\ a_3 = 9A + B + C = 107$$
$$47 - 11 = 3A + B = 36\ ,\ 107 - 47 = 5A + B = 60$$
$$60 - 36 = 2A = 24\ \therefore A = 12\ ,\quad B = 0\quad ,\quad C = -1$$
$$\therefore a_n = 12n^2 - 1$$

Thus, double and six times the primes of the $12n^2 - 1$ - form are included in B.

Thus, if there are infinitely many primes in the form $12n^2-1$, DS sequence is an infinite sequence. In other words, there are infinitely many numbers in which the sum of the divisors becomes the square number.

Now, let's prove that for every prime number in the $12n^2-1$ form, double and six times are included in DS sequence.

<proof>

$\sqrt{\sigma(2p)}$, $\sqrt{\sigma(6p)} \in N$ about $\forall\ p = 12n^2-1$

$$\sigma(2p) = 1+2+p+2p = 3p+3$$
$$\sigma(6p) = 1+2+3+6+p+2p+3p+6p = 12p+12 = 2^2 \times \sigma(2p)$$
$$\therefore \sqrt{\sigma(2p)} \in N \Leftrightarrow \sqrt{\sigma(6p)} \in N$$
$$p = 12n^2-1\ ,\ \sigma(2p) = 3(12n^2-1)+3 = 36n^2 = (6n)^2$$

■

Here we can extend for prime p as follows.

$$p = n^k\sigma(q)^{m-1} - 1 \Rightarrow \sqrt{\sigma(qp)} \in N\ (k,m \in 2N\ ,\ \ q,n \in N)$$

<proof>

<Lemma1> $\gcd(a,b) = 1 \Rightarrow \sigma(ab) = \sigma(a)\sigma(b)\ (a,b \in N)$

<proof>

$\forall\, d_1, d_2\ s.t.\ d_1|a\, , d_2|b \qquad d_1d_2|ab$

$$\sigma(a) = \sum_{d_1|a} d_1\ ,\ \sigma(b) = \sum_{d_2|\,b} d_2\ ,\ \sigma(ab) = \sum_{d_1|a}\sum_{d_2|b} d_1d_2\ (\because a \perp b \Leftrightarrow \forall\, d_1 \neq d_2)$$

$$\therefore \sum_{d_1|a}\sum_{d_2|b} d_1d_2 = (\sum_{d_1|a} d_1)(\sum_{d_2|b} d_2) = \sigma(a)\sigma(b)$$

Except for 1.

■

<Lemma2> For prime p with $p = n^k\sigma(q)^{m-1} - 1$, p and q are always sub to each other, except for p=q. (n,q is a natural number, k,m is an even number)

<proof>

You can use <Lemma1> to express the sum of divisors of all natural numbers

$$\forall n \in N\ ,\ n = \prod_{k=0}^{m} p_k^{a_k}\ ,\ (m = \prod_{k=0}^{k} (a_k + 1))$$

by Fundamental theorem of arithmetic

And for any prime power $\sigma(p^r) = 1 + p + p^2 + \cdots + p^r$

This is sum of geometric sequence. thus $\sigma(p^r) = \dfrac{1 - p^{r+1}}{1 - p}$ s

$$\sigma(n) = \sigma(\prod_{k=0}^{m} p_k^{a_k}) = \prod_{k=0}^{m} \sigma(p_k^{a_k}) = \prod_{k=0}^{m} (\frac{1 - p_k^{a_k+1}}{1 - p_k})$$

$\because p_j^{a_j} \perp p_i^{a_i}\quad (i,j \in \{k\}, i \neq j)$ $\quad \therefore \sigma(q) = \prod_{k=0}^{m} (\dfrac{1 - p_k^{a_k+1}}{1 - p_k})$ thus

$$\exists\ 1 \neq r \wedge r | q \quad \therefore 1 \equiv 1 \pmod r,\ r^k \equiv 0 \pmod r$$

$$\therefore \sigma(q) \equiv 1 \pmod r\ ,\ \{\sigma(q)\}^{m-1} \equiv 1 \pmod r$$

$$\therefore p \equiv n^k - 1 \pmod r$$

therefore We can think of the necessary and sufficient conditions for p and q to coprime each other.

$$p - n^k + 1 \equiv 0 \pmod r \Leftrightarrow n^k \equiv 0 \pmod r \Leftrightarrow \gcd(p,q) \geq 2$$

In other words, when there is a common divisor r of p and q and we divide p, p and q are not coprime to each other.

And since p is a prime number and r is not 1, if p is a divisor of q, then r=p.

Therefore, if p is a divisor of q and not q, then p and q are not cows to each other.

Suppose that p is a divisor of q and not q.

Then p is the divisor of q, so p is equal to or less than q. But since it is not the same, p is less than q.

Now , let's find a contradiction.

q is a natural number, so it is 1 or greater than 1. When q is greater than 1

$$q > 1 \Rightarrow \sigma(q) \geq 1 + q$$

$p \geq n^k \sigma(q)^{m-1} - 1 \geq n^k (1+q)^{m-1} - 1 \geq q$ Because the minimums of n, k, and m are 1, 2, and 2, respectively. Thus, it was concluded from the original proposition that p was greater than or equal to q. And because p and q are not the same, p is greater than q. Thus, p is less than q, Since "p is a contradiction, it is a contradiction of the same divisor, it cannot be a weak divisor. Thus, p and q are always coprime to each other, except when p and q are equal. ■

Now let's prove this proposition using lemma 1, 2.

$p = n^k \sigma(q)^{m-1} - 1 \Rightarrow \sqrt{\sigma(qp)} \in N \ (k,m \in 2N, \ q,n \in N)$

P and q are always coprime by lemma 2

$\sigma(qp) = \sigma(q)\sigma(p)$ by lamme 1

P is prime $\sigma(p) = 1 + p$ thus

$\sigma(qp) = \sigma(q)(n^k \sigma(q)^{m-1} - 1) + \sigma(q) = n^k \sigma(q)^m$

$k,m \in 2N \Rightarrow \frac{k}{2}, \frac{m}{2} \in N \therefore \sqrt{n^k \sigma(q)^m} = n^{\frac{k}{2}} \sigma(q)^{\frac{m}{2}} \in N$ ■

Thus $\sigma(qp)$ is included in DS sequence about primes $p = n^k \sigma(q)^{m-1} - 1$

And we can simplify it and think of the case where k,m is 2. Then

$p = \sigma(q) n^2 - 1$ And this can be thought of as a quadratic equation for n.

$\sigma : N \rightarrow N$ is neither surjective nor injective but $\sigma(N) \subset N$ thus It is an integer coefficient quadratic equation.

$\sigma(14) = 24$, $\sigma(15) = 24$ thus Not injective

There is no α that satisfies $\sigma(\alpha) = 2$, so it does not surjective

$\sigma(1) = 1, \sigma(p) = 1 + p, \min(p) = 2$, $\sigma(2) = 3 \therefore \sim \exists \ \alpha \ s.t. \ \sigma(\alpha) = 2$

However $Card(N) = \aleph^0$, $\sigma(N) \subset N$, thus $Card(\sigma(N)) = \aleph^0$

Therefore If there are infinitely many primes in the form $p = \sigma(q) n^2 - 1$, then there are infinitely many DS sequence.

However, if q is the number of B, the only p that is satisfied is 3.

<proof>

If q is a number of DS sequence, then the sum of the divisors of q is a square number. Then it is decomposed in the natural number.

$p = (\sqrt{\sigma(q)}\, n + 1)(\sqrt{\sigma(q)}\, n - 1)$ In order to be a primes here, one of the two must be 1. thus $\sqrt{\sigma(q)}\, n = 2$, $n = \frac{2}{\sqrt{\sigma(q)}}$ Thus, when $\sigma(q)$ is a square number, the prime number of the $p = \sigma(q) n^2 - 1$-form is only when

$n = \frac{2}{\sqrt{\sigma(q)}}$, and then p is 3.

■

Thus, if there are infinite prime numbers in form $p = \sigma(q)n^2 - 1$ when q is not included in DS sequence The sum of the divisors is also numerous.

Now let's look at the case where p=q.

Case of $p = q$, $\sigma(qp) = \sigma(p^2) = 1 + p + p^2$ don't become square

<proof>

Suppose that there is a square form $1 + p + p^2$.

$1 + p + p^2 = n^2 \quad (n \in N)$

$1 + p + p^2 - n^2 = 0$ It is a quadratic equation for p, and the discriminant must be a square number to be established.

Because it's an integer.

$D = 1^2 - 4(1 - n^2) = 4n^2 - 3 = k^2 \quad (k \in N)$

$(2n + k)(2n - k) = 3$ And both terms are natural numbers, and we break down 3 in to two natural numbers

The only way is to multiply 1 and 3 $2n + k > 2n - k$ thus

Must $2n + k = 3$, $2n - k = 1$

thus $n = k = 1$

$D = 1$ and $p = \frac{1 \pm \sqrt{D}}{2} = \frac{1 \pm 1}{2} = 0 \vee 1$

The roots 0 and 1 are not both prime numbers. This equation has only 0, 1, but since all of them are not prime numbers, there is no equation that is prime number and square number. Therefore, there is no prime number p that $1 + p + p^2$ becomes a square number. ■

Therefore, when p=q, the number included in DS sequence cannot be expressed as p.

For primes p and natural numbers q,n, if there are infinite prime numbers in the form

$p=\sigma(q)n^2-1$, there are infinite numbers in which the sum of the divisors becomes square. And the converse holds true.
This is because all terms of DS sequence are in the form of the product of p and q With respect to p and q that satisfy the above equation.

We can think of $\sigma(q)$ as a constant and think of it as a $f(n)=\sigma(q)n^2-1$-form a two-variable polynomial It can be interpreted as containing a myriad of primes with $f(N)$ for $f: N \rightarrow N$
It is equivalent to the above two propositions.

And this is one particular form of Bunyakovsky's conjecture.
Bunyakovsky's conjecture is for an integer coefficient of higher order than 1, for an irreducible polynomial $p(x)$ Considering the range $p(N)$ of the natural set N, $p(N)$ Is a conjecture that one of the following two must be true.

1. greatest common divisor of $p(N)$ isn't 1
2. $p(N)$ contains an infinite number of prime numbers

Bunyakovsky conjecture that $p(N)$ is not satisfied with both at the same time.
In other words, the range of the integer coefficient-based polynomial includes an infinitely large number of prime numbers.
But Bunyakovsky conjecture is still an open conundrum of number theory.

If Bunyakovsky's conjecture is true, there are infinite numbers in which the sum of the divisors is squared.
When $\sigma(q)$ isn't square , $f(n)=\sigma(q)n^2-1$ is integer coefficient irreducible Polynomial , If Bunyakovsky's conjecture is true, there are infinite prime numbers in the form $\sigma(q)n^2-1$.
Therefore, there is an infinite number in which the sum of the divisors, which is the equivalence proposition, is a square number.
Converse doesn't hold up

Conclusion.
If Bunyakovsky's conjecture is true, there are infinite numbers in which the sum of the divisors is squared.