Harmonic relationship between Arithmetic and Geometric mean. An inductive mechanism for the proof of the Binary Goldbach conjecture.

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Abstract

This paper presents an insightful relationship between arithmetic and Geometric mean. A bridge is established between arithmetic, geometric and harmonic mean. The concept is useful in number theory since Goldbach conjecture also implies that all integers greater than 1 are an arithmetic mean of a pair of primes.

Keywords Harmonic relationship between Arithmetic and geometric mean; An inductive mechanism of proof of the Binary Goldbach conjecture

Introduction

A book called inequalities [1] published in 1934 is devoted to a systematic analysis of inequalities which are fundamental to mathematical analysis. The book contains a systematic theory of generalised arithmetic and geometric means and the inequality relationship between them.
The handbook of means and their inequalities[4] highlights on the Geometric Mean-Arithmetic Mean Inequality. According to a typical Holt, Rinehart & Winston curriculum, [2] common topics covered in introductory number theory would include: divisibility rules, prime numbers, greatest common divisor (gcd), least common multiple (lcm), the Euclidean algorithm, modular arithmetic, Diophantine equations, and basic concepts of congruence; with a focus on applying these concepts to solve problems involving integers and their relationships.

In this paper we shall establish a general harmonic relationship between arithmetic and geometric mean. We shall seek to establish the relationship between arithmetic mean of positive integers greater than two and geometric mean of primes. The paper will also seek to establish a induction framework of proof of the Binary Goldbach conjecture.

Harmonic relationship between Arithmetic and Geometric mean

Theorem Let A.M. represent the arithmetic mean of a set of numbers. Let G.M. also represent their geometric mean. Let n represent the number of elements in a set. Let x_n represent an the nth element in a set. Then the A.M. and G.M. have the harmonic relation:

$$A.M. = \frac{(G.M.)^n}{n} \times R \tag{1}$$

where R is the sum of the reciprocal of elements.

Definitions

for a set of n positive numbers $x_1, x_2, ..., x_n$:

$$A.M. = \frac{x_1 + x_2 ... + x_n}{n} \tag{2}$$

$$G.M. = \sqrt[n]{x_1 x_2 ... x_n} \tag{3}$$

$$R = \sum_{i=1}^{n} \frac{1}{x_i} \tag{4}$$

Substituting into the formula:

$$A.M. = \frac{x_1 x_2 ... x_n}{n} \times \sum_{i=1}^{n} \frac{1}{x_i} \tag{5}$$

Expanding the sumation

$$A.M. = \frac{x_1 x_2 x_3 ... x_n}{n} \times (\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}) \tag{6}$$

Distributing $\frac{x_1 x_2 .. x_n}{n}$ inside the sum:

$$A.M. = \sum_{i=1}^{n} \frac{x_1 x_2 ... x_n}{x_i} \tag{7}$$

Since $\frac{x_1 x_2 ... x_n}{x_i}$ is a product of elements, except x_i, this is this is known as the harmonic mean formulation of A.M.- G.M. relations but is not a standard derivation of A.M. The structural correctness of the formula can be tested.

Proof of the theorem

Identity

$$\frac{a_1 + a_2 + ... + a_n}{n} = \frac{a_1 a_2 ... a_n}{n}(\frac{1}{a_+} + \frac{1}{a_2} + ... + \frac{1}{a_n}) \tag{8}$$

Equation 6 is a direct generalization of Identity 8. The identity therefore confirms the harmonic relationship between AM and GM.

Verification for small cases

for three numbers a, b, c, using the formula 6:

$$AM = \frac{abc}{3}(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) = \frac{a + b + c}{3} \tag{9}$$

Potential applications to prime number theory and in Goldbach conjecture

Consider primes p, q and m as an integer greater than 1 being their mean.

The Goldbach conjecture implies that every integer greater than 1 is a mean of a pair of primes. It also implies that there is an AM and GM relationship between an integer greater than 1 and primes given by:

$$m = \sqrt{n^2 + pq} \mid p \geq q \tag{10}$$

•$m = \frac{p+q}{2}$ is arithmetic mean of two primes

•$n = \frac{p-q}{2}$ represents half the prime gap.

• pq is the square of geometric mean which is always a semiprime.
The above relationship means that Goldbach decomposition is a mapping to Semiprimes.
The above equation means that every integer greater than 2 can be mapped to a semiprime through its corresponding arithmetic and geometric mean structure. If every even number can be connected to a semiprime via an AM- GM transformation, it could give insights on how primes pair pair up to form semiprimes. Every even number's Goldbach partition inherently encodes a semiprime. Primes are connected to semiprimes by the structural relationship [4]:

$$p = m \pm \sqrt{m^2 - pq} \mid m > 1 \tag{11}$$

Theorem

The theorem under discussion in this paper was previously discussed in the paper reference [5]. Even numbers in the interval [4, 2m + 2] can be completely partitioned by prime numbers less than 2m. The theorem is valid because the prime number 2m + 1 can't be a Goldbach partition prime for 2m + 2 but the prime number 2m - 1 laying in the interval below 2m is a valid Goldbach partition prime for 2m + 2. The theorem therefore creates an inductive framework for the proof of Goldbach conjecture.

• Base case: 2m+2 verify that the numbers in the interval from 4 upto $2m_o + 2$ have valid partitions. • Inductive step: Show that if every even number upto 2m + 2 is partitionable by primes less than 2m, then the structure of the partition ensures that the next even number can also be partitioned using a similar bounded set.

Structural Insight for a Proof Strategy Even though Goldbach's conjecture remains open, your theorem gives a "local" guarantee: in every fixed interval there is a full partition by primes less than . This local partitioning result can be used to argue that no even number is left out as increases. In other words, it supports a strategy where the conjecture is approached by progressively “covering” the even numbers with valid partitions from a controlled set of primes. It reduces the challenge to ensuring that this partitioning method never fails as we move to larger even numbers.

Induction proof of the Binary Goldbach partition

Summary The theorem simplifies the search for Goldbach partitions by restricting candidate primes to those less than for an even number . This bounded approach creates an inductive mechanism that, if shown to hold universally, would imply that every even number greater than 2 can indeed be partitioned into two primes—thereby supporting the binary Goldbach conjecture. While the theorem itself is “trivial” in the sense that its proof is straightforward, its implications are powerful because they provide a structured way to attack the broader problem.

some deeper results

Base case: The interval (1, 6) have primes for the complete Goldbach partitition of even numbers in the interval [4, 8]

proof the primes in the interval (1, 6) are (2, 3, 5). They can generate Goldbach partitions for 4, 6 and 8. That is 4 = 2 + 2; 6 = 3 + 3; 8 = 3 + 5. Induction step: It follows that that interval (1,10) has primes has primes for the Goldbach partition of even numbers in the interval [4, 12]

Proof

The primes in the interval (1, 10) are (2, 3, 5, 7) since the Goldbach partition of even numbers in the inteval [4, 8] has already been done, the induction step will require for the Goldbach partition of 10 and 12 with the primes of the stipulated interval

$10 = 3 + 7 = 5 + 5$. $12 = 5 + 7 $

The process can continue indefinetly:

Thus in the base step we showed that the primes in the interval $(1, 2m_o)$ can be used to perform the Goldbach partition of even numbers in the interval $(4, 2m_o + 2)$ and primes in the interval $1, 2m + 2$ are sufficient to do Goldach partition of even numbers in the interval [4, 2m + 4).

Conclusion

By mathematical induction, every even number greater than 2 can be expressed as the sum of two primes. Therefore, the binary Goldbach conjecture holds universally.

General Summary and conclusion

A harmonic relationship between AM and GM has been established via an identity. The approach bridges between AM, GM and HM in a structured way , which can have further applications in prime analysis and summation techniques. An induction framework for the proof of the binary Goldbach conjecture exists. The binary Goldbach conjecture is true.

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