Possible Transition from Order into Chaos and vice versa.

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Abstract. *The specific model-case of the quadratic-iterator is an illuminating way of understanding the chaotic-behaviour. It is agreed that for the special-cases of iteration of transformations there are common characteristics of chaos: Sensitive dependence on initial conditions, mixing and dense, periodic points. Therefore discussion starts with an important metaphor in chaos-theory, kneading of dough, by 2 different uniform-processes performed iteratively each of them in unit-iterval:* [1] *Stretch the dough, fold it over in the middle and stretch it again (as often as required), and* [2] *stretch the dough, cut it in the middle, paste the 2 halfs together and stretch it again (as often as required). This processes guarantee that a pocket of spice inserted into the dough will be mixed thoroughly throughout the mass. Both kneading-processes were found to be compatible in view of their chaotic-characteristics. In a further step of discussion, equivalence could be shown between the 2 uniform kneading-processes and the non-uniform kneading of the quadratic-iterator* $y = a \cdot x(1-x)$, *where* $a = 4$ *were chosen, via simple coordinate-transformations of the unit-interval. Chaotic characteristics of all 3 iteration-transformations could also be proven as being equivalent to each other. Thus, further investigations were based now on quadratic-iterator. The range from states of order up to the complete chaotic dynamics of the quadratic-iterator can be divided into 3 distinct parts:* [1] *regime* $1 \leq a < (s_\infty =$ *FEIGENBAUM-point) were oscillations of the iterator will experience period-doublings,* [2] *an area* $s_\infty < a < 4$ *which can be looked as mirror-image of regime* [1], *and* [3] *the chaos-area for* $a = 4$. *Boarder between regime [1] and [2] is a CANTOR-set. The mirror-image-area of the quadratic-iterator's final-state-diagram is characterized by a complicated band-structure and therefore different orbit-dynamics can be expected for* $(a < s_\infty) \rightleftarrows (s_\infty < a)$. *In other words, transitions from order to chaos and vice-versa may occur but with respect to orbit-dynamics they happen differently every time.*

A^1 Paradigma of Chaos: The kneading of Dough.

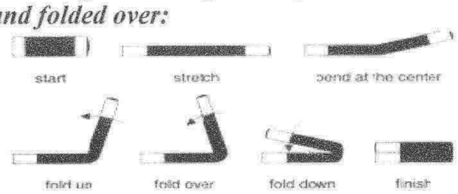
Mathematicians agree that for the special case of iteration of transformations 3 common characteristics of chaos will be fulfilled:

<1>	*Sensitive dependence on initial conditions.*
<2>	*Mixing from sequences of intervals by courses of orbits.*
<3>	*Dense periodic points.*

Kneading of dough provides an intuitive access to all these mathematical problems of chaos. Moreover, one will see that the kneading-process is closely related to the quadratic-iterator $x_{n+1} = ax_n(1-x_n)$ *with* $0 \leq x \leq 1$ *as iteration-variable and* a *as real parameter* $1 \leq a \leq 4$.

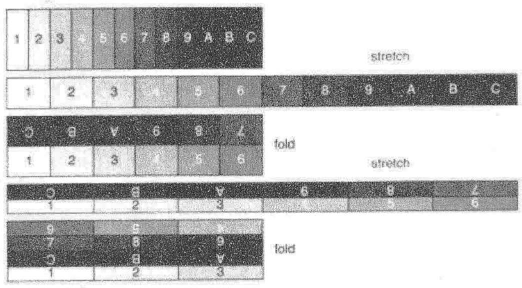
A^1_1 *Kneading is the process of stretching the dough and folding it over, repeated many times. The results have many in common with randomness.*

$A^1_{1.1}$ *The dough is homogeneously stretched to twice its length, then it is bent at the centre and folded over:*



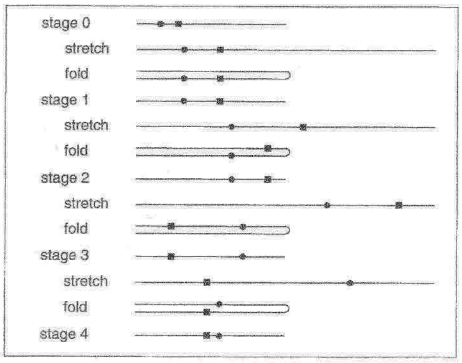
Uniform kneading by stretch-and-fold.

$A^1_{1.2}$ *In order to make more obvious how this kneading works on different parts of the dough, part of it is divided into* 12 *blocks which are processed then by* 2 *stretch-and-fold-operations:*



2 *operations of stretch-and-fold-kneading on* 12 *blocks of dough.*

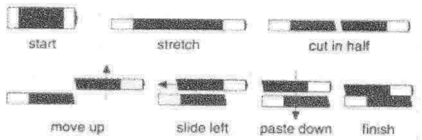
$A^1_{1.2.1}$ *The situation shall be idealized a little bit further. It is thought that folding these layers will not change their thickness and thus one may represent the dough by a infinitely-thin line-segment.*



2 *grains, symolized by a dot and a square, are subjected to* 4 *stages of stretch-and-fold. They are mixed throughout the dough.*

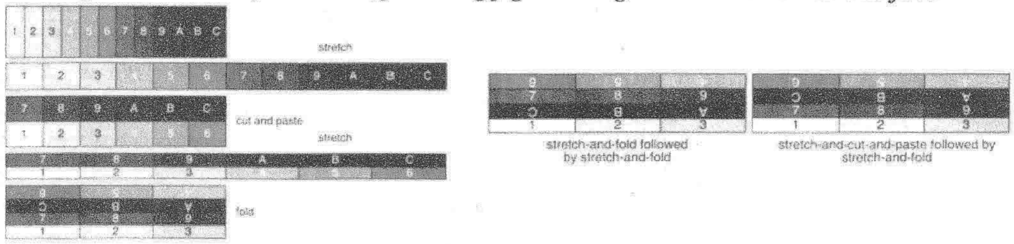
$A^1_{1.2.2}$ *The* 2 *grains are rather close together initially. But after a few kneadings it becomes obvious that one will find them in very different places. In fact, this will be a consequence due to mixing-properties of kneading. Thus, kneading destroys neighbourhood-relations, grains which are very close initially will likely not be close neighbours after a while. This is the effect of sensitive-dependence-on-initial-conditions. Small deviations in intial positions lead to large deviations in course of the process.*

A^1_2 *Next another kind of kneading-operation shall be discussed. Here again the dough again is stretched uniformly to twice of its length, but then it is cut at its centre into* 2 *parts and pasted afterwards on top of each other:*



Uniform stretch-cut-and-paste-kneading.

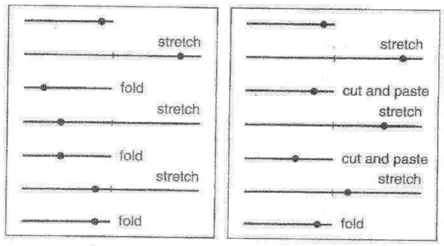
$A^1_{2.1}$ *When comparing the stretch-cut-and-paste-operation with the stretch-and-fold-operation, on* 1^{st} *glance it seems to be that both kneadings apparently mix particles around, but in very different manner generating quite distinct iterating-behaviours. However, suprisingly both kneadungs are essentially the same; following figure can give a* 1^{st} *idea about this fact:*



$A^1_{2.1.1}$ *Stretch-cut-and-paste followed by stretch-and-fold applied to* 12 *blocks of dough. The resulting horizontal order of the blocks is identical to the one obtained from the application of the* 2 *succeeding stretch-and-fold-operations in figure of* (A^1_1).

$A^1_{2.1.2}$ *Again the dough had been divided into* 12 *blocks. Then the stretch-cut-and-fold-operation was applied followed by* 1 *strech-and-fold-operation. The result is compared with the one obtained for* 2 *succeeding stretch-and-fold-operations in the bottom-part of the figure. One observes that they are identical when ignoring the vertical order of the pieces.*

$A^1_{2.1.3}$ *This gain suggests to neglect any thickness of the dough. Thus, from now on the dough is thought of being represented by a line-segment. This is the* 1^{st} *step towards a mathematical-model of the kneading-operations. By taking the interval* $(0,1)$ *as the original line-segment modeling the dough, one can now check how the* 2 *different kneading-operation act. The symbol* T *is used for the stretch-and-fold-operation and the symbol* S *for the stretch-cut-and-paste-operation.*

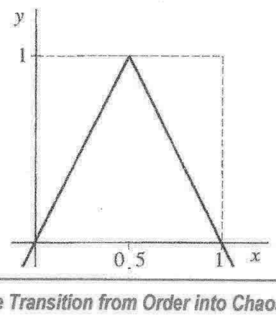


Tracing a particle in interval $(0,1)$ *for the* T*—operation (left) and* S*—operation (right). The particle starts in both experiments from the same initial-position* x_0.

$A^1_{2.1.3.1}$ *One observes in both experiments that the particle arrives after* $T(T(T(x_0)))$ *or* $T(S(S(x_0)))$ *exactly at the same position though the route in between is different. This means* $T(T(T(x_0))) = T(S(S(x_0)))$. *This experience along with the result of figure from* $(A^1_{2.1})$ *motivates one to conjecture an substitution-property of the* 2 *kneading-operations* $T^N = TS^{N-1}$ *where* $T^N = T(T(\ldots(N-2)-times))$ *and* $S^{N-1} = S(S(\ldots(N-3)-times))$.

A^1_3 *These mathematical-models for kneading of the* 1*—dimensional ideal of the dough are functions.*

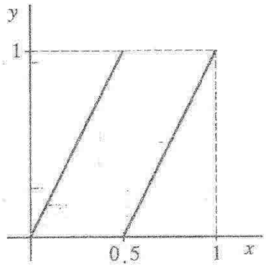
$A^1_{3.1}$ *The stretch-and-fold-kneading* T *is represented by following transformation* $T(x) = \{(2x \text{ if } x \leq ½) \wedge (-2x+2 \text{ if } x \geq ½)\}$, *following figure shows a graph of this transformation:*



Graph of the piecewise-linear tent-transformation T according to the equation above. The graph looks like a simplification of the parabola.

$A^1_{3.1.1}$ *A justification of this model is almost self-evident. The dough is modelled by the unit-interval* $(0,1)$. *The stretching-operation is taken care-of by the factor* 2 *in front of* x. *The* 1^{st} *half of interval* $(0,1)$ *is only stretched and not folded. Thus, the* 1^{st} *part from the definition of* T *is in order of* $T(x) = 2x$ if $x \leq ½$. *The* 2^{cd} *half-interval becomes* $(1,2)$ *after the stretching and must be folded over its left end-point. This equivalent to folding at* $x = 0$ *(multiplying with* (-1) *and shifting to the right by* 2 *units).*

$A^1_{3.2}$ *The model for the* 2^{cd} *procedure, the stretch-cut-and-paste kneading-operation is another elementary mathematical transformation, the saw-tooth-transformation* S, *defined for* $x \in (0,1)$: $S(x) = \{(2x \text{ if } x < ½) \wedge (2x-1 \text{ if } x \geq ½)\}$, *following figure shows its graph:*



Graph of saw-tooth-transformation S *according to the equation above. The graph justifies the name.*

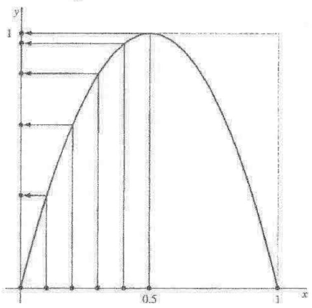
$A^1_{3.2.1}$ *There is a notation for the saw-tooth-function* S, *which differs from the above one. It uses a function which computes the fractional part* $\mathrm{Frac}(x)$ *of a number* x: $\mathrm{Frac}(x) = x-k$ if $k \leq x < k$, $k =$ *integer. With this notation the saw-tooth-transformation can be written as* $S(x) = \mathrm{Frac}(2x)$ for $0 \leq x < 1$. *Only for the point* $x = 1$ *the formula does not work. But this is not significant, because* $x = 1$ *is a fixed-point of the operator* S *and, moreover, there are no other points in the unit-interval which are transformed to this fixed-point. Thus it is no loss to neglect the fixed-point* $x = 1$.

$A^1_{3.2.1.1}$ *Starting with* $0 \leq x_0 < 1$ *one computes* $x_1 = \mathrm{Frac}(2x_0) \rightarrow x_2 = \mathrm{Frac}(2x_1)$ *and generalizes* $x_{k+1} = \mathrm{Frac}(2x_k)$, $k = 0,1,2,...$ *As one likes to know what* x_k *will be for some very large value of* k *in terms of* x_0, *one has to write* $x_k = \mathrm{Frac}(2^k x_0)$.

$A^1_{3.2.2}$ *If* x *is defined in binary-representation by* $0.a_1a_2a_3...$ *with* a_k *as binary digits, the transformation* $S(x)$ *must be:* $S(0.a_1a_2a_3...) = \mathrm{Frac}(2 \cdot 0.a_1a_2a_3...) = 0.a_2a_3...$, *because multiplication by* 2 *yields shifting all binary digits* 1 *place to the left followed by erasing the digit that is moved in front of the point. Due to the type of this almost mechanical procedure the* S*—transformation is also called the shift-operator when interpreted in context of binary-representations.*

A^1_4 *An argument shall be introduced which makes a connection between the feedback-system* $x \rightarrow 4x(1-x)$ *and the kneading of dough.*

$A^1_{4.1}$ *If one graphs the transformation* $y = 4x(1-x)$ *in* (x,y)*—coordinate-system one obtains a generic-parabola:*



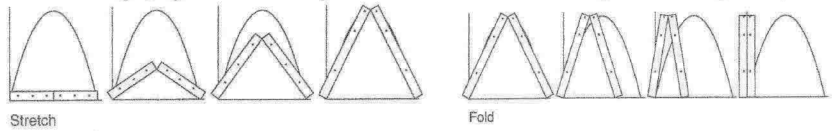
A generic-parabola are characterized by the fact that their graphs precisely fits into a square which has 1 *of its diagonals on the bisector of the* (x,y)*—coordinate-system.*

$A^1_{4.1.1}$ *Here one is interested only in* x*—values ranging* $0 \rightarrow 1$. *One will observe that the corresponding* y*—values also range from* $0 \rightarrow 1$, *and* y*—values monotonically increase for* $x < ½$ *and monotonically decrease in the range* $x > ½$. *One may notice that the interval* $(0,½)$ *on* x*—axis is stretched-out to interval* $(0,1)$

on y—axis, and the same is true for the interval $(½,1)$. In other words, the transformation $4x(1-x)$ stretches both intervals to twice of their lengths. The stretching, however, is non-uniform. In fact, small intervals (close to 0 and 1) are stretched a great deal, while intervals close to the midpoint $½$ are compressed.

$A^1_{4.2}$ *Here one now has reached a point where a connection can be made to to the former kneading-operations $T(x)$ and $S(x)$.*

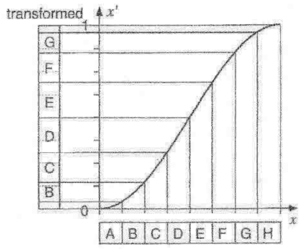
$A^1_{4.2.1}$ *It's known already that each half of the unit-interval is stretched to twice its length. Moreover, checking the end-points of the intervals, one finds $0 \to 0$, $½ \to 1$, 1*
$\to 0$. This means, the result of 1 application of the transformation $4x(1-x)$ to the interval $(0,1)$ can be interpreted as a combination of stretching and folding:



$A^1_{4.2.1.1}$ *In other words, the iteration $x \to 4x(1-x)$ is a relative of the uniform stretch-and-fold-kneading-operation.*

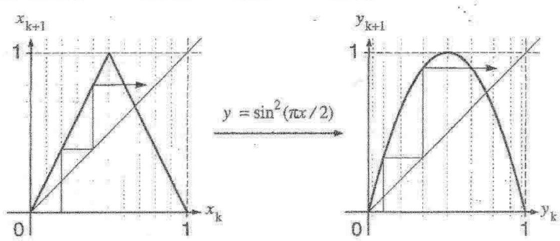
$A^1_{4.3}$ *All complex behaviour which one is able to show first for the shift-operator and then for the tent-transformation (uniform kneading-operators) can also be found in the non-uniform kneading due to the quadratic-iterator $x \to 4x(1-x)$.*

$A^1_{4.3.1}$ *The equivalence of the iteration of the tent-transformation $T(x)$ and the quadratic-parabola $4x(1-x)$ is established by a non-linear change of the coordinates given by $x' = h(x) = \sin^2(\pi x/2)$.*



Function $h(x)$ is the coordinate-transformation. Each x has its corresponding $x' = h(x)$ and vice-versa. Along the coordinate-axes, intervals do not retain their lengths when subjected to the transformation $h(x)$. One may note that the function h transforms $(0,1)$ to itself in a 1—to—1 fashion. For any $x' \in (0,1)$ there is exactly one $x \in (0,1)$ with $x' = h(x)$.

$A^1_{4.3.1.1}$ *When looking at an initial-point x_0 and its orbit x_1 x_2 x_3 ... under the tent-transformation the orbit becomes $x_1 = T(x_0)$ $x_2 = T^2(x_0)$ $x_3 = T^3(x_0)$ ... $x_k = T^k(x_0)$.... The h—transformed point x_0 is now $y_0 = x'_0 = h(x_0)$ the initial-point in new coordinates, those belonging to the iteration of the parabola. Computing now the iteration of $f(y) = 4y(1-y)$ using $y_0 = x'_0$ one will obtain $y_1 = f(y_0)$ $y_2 = f^2(y_0)$ ... $y_k = f^k(y_0)$....*



$A^1_{4.3.1.2}$ *Changing coordinates accordingly to the function $h(x)$ transforms graph of T—transformation to that of $f(y) = 4y(1-y)$. Iteration for T (left) is transformed into iteration for f (right) using h. Orbits are equivalent. Thus, iterating x_0 under T produces an orbit which is (after changing coordinates) the same as that of $y_0 = x'_0 = h(x_0)$ under the quadratic—f. In terms of the functions f and T this equivalence can be put in the form of functional-equivalence*
$f^k(h(x)) = h(T^k(x))$,
$k = 1,2,3,..$, *for all* $x \in (0,1)$.

$A^1_{4.3.2}$ *Now the mathematics behind the equivalence of iterations from the tent-transformation* T *and the quadratic-iterator* f *will be presented. All the tools one needs are* 2 *familiar trigonometric-identities* $\cos^2(\alpha) = 1-\sin^2(\alpha)$ *and* $\sin(2\cdot\alpha) = 2\cdot\sin(\alpha)\cdot\cos(\alpha)$.

$A^1_{4.3.2.1}$ *Iterating an initial-point* x_0 *under the tent-function and iterating the transformed-point* $x'_0 = \sin^2(x_0\pi/2)$ *under the parabola* $f(x) = 4x(1-x)$ *produces functions that correspond to each other by means of the transformation* $x' = h(x) = \sin^2(x\pi/2)$. *To establish this algebraically, one starts with* x_0 *for the parabola.*

$A^1_{4.3.2.2}$ *Thus,* $x_0x_1\ldots$ *is the iteration under the tent-function and* $y_0y_1\ldots$ *is the corresponding iteration under the parabola. One can show by induction that, in fact,* $y_k = x'_k = h(x)$ *for numbers* $k = 0,1,\ldots$, *proving the equivalence.*

$A^1_{4.3.2.3}$ *One starts with the transformation* $y_0 = \sin^2(x_0\pi/2)$, *where* $0 \leq x_0 \leq 1$. *One substitutes* y_0 *in the formula for the quadratic-iterator* $y_1 = 4y_0(1-y_0) = 4\sin^2(x_0\pi/2)\cdot(1-\sin^2(x_0\pi/2))$. *By using the trigonometric-identity* $\cos^2\alpha = 1-\sin^2\alpha$ *one obtains* $y_1 = 4\cdot\sin^2(x_0\pi/2)\cdot\cos^2(x_0\pi/2)$. *Simply using the double-angle-identity* $\sin(2\cdot\alpha) = 2\cdot\sin\alpha\cdot\cos\alpha$, *one obtains* $y_1 = \sin^2(x_0\pi)$.

$A^1_{4.3.2.4}$ *The* 1^{st} *iterate of* x_0 *under the tent-function is* $x_1 = T(x_0)$. *It can easily be shown that* y_1 *is in fact identical to* x_1 *after change of coordinates, i.e.,* $x'_1 = h(x_1) = y1$.

$A^1_{4.3.2.4.1}$ *Begining with case* $0 \leq x_0 \leq ½$, $x_1 = T(x_0) = 2x_0$ *and* $x'_1 = \sin^2(x_1\pi/2) = \sin^2(x_0\pi) = y_1$.

$A^1_{4.3.2.4.2}$ *Alternatively, for* $½ < x_0 \leq 1$. *One starts substituting* $x_1 = T(x_0) = 2-2x_0$. *Then* $x'_1 = \sin^2(x_1\pi/2) = \sin^2(\pi-x_0\pi)$. *By using initially* $\sin^2(\pi+\alpha) = \sin^2(\alpha)$ *and finally* $\sin^2(-\alpha) = \sin^2(\alpha)$ *one gets* $x'_1 = \sin^2(-x_0\pi) = \sin^2(x_0\pi) = y_1$.

$A^1_{4.3.2.5}$ *The result shows* $x'_1 = y_1$ *and the conclusion* $x'_k = y_k$ *for* $k = 0,1,2,\ldots$ *(followed by induction). Thus, since* $x'_k = h(T^k(x_0))$ *and* $y_k = f^k(h(x_0))$, *one has shown the functional-equation* $f^k(h(x)) = h(T^k(x)), k = 1,2\ldots$.

A^2 Paradigma of Chaos: Common Characteristcs of Kneading–Operations.

Central chaos-properties of kneading-operations, Sensitivity, Mixing and Dense-periodic-points, will be started with considering iterations of saw-tooth-transformation S. *The substitution-property allows one to carry-over these features to the iteration of tent-transformation* T. *In later steps one will conclude analysis of chaos by exploiting another equivalence, namely between tent-transformation and the quadratic-iterator.*

A^2_1 *One gets started with the saw-tooth-transformation* $S(x) = \mathrm{Frac}(2x)$ *for* $0 \leq x < 1$ *(see* $(A^1_{3.2})$ *and following) and reveals a new interpretation by passing to binary representation of the real-number* $0 \leq x < 1$. *Any real-number* x *from the unit-interval can be written as* $x = 0.a_1a_2a_3\ldots$, *where the* a_k *are binary-digits and* $x = a_12^{-1}+a_22^{-2}+a_32^{-3}+\ldots+a_k2^{-k}+\ldots$. *What does the iteration of the* S*–function in terms of binary-expansion mean? Multiplication by* 2 *means passing from* $0.a_1a_2a_3\ldots$ *to* $a_1.a_2a_3\ldots$. *Therefore,* 1 *application of the transformation is accomplished by first shifting all binary-digits* 1 *place to the left and then erasing the digit moved in front of the point,* $x = 0.a1a2a3\ldots \rightarrow S(x) = 0,a2a3a4\ldots$. *Because of the type of this procedure the transformation is called shift-operator too, when interpreted in the context of binary-expansion.*

$A^2_{1.1}$ *Under these aspects the motion of a spice-particle in dough (when kneading by the stretch-cut-and-paste-operation) can now be studied. One will turn to the* 3 *characteristics-of-chaos in iterated-transformations, Sensitivity, Mixing and Periodic-Points merged in everywhere.*

A^2_2 *In order to study sensitivity one may imagine an initial-number* $x_0 = 0.a_1a_2a_3\ldots$ *has been picked, but only specified up to* $N = 100$ *digits. Thus, the true number will differ from the specified one by at most at most* 2^{-100}, *a difference so small that it is considered not to matter at all. In any event one* *can consider this difference to be like an error-of-measurement. Since the digits $a_{101}a_{102}a_{103}$.... are not known, one may assume that in each step of the calculation somebody flips a coin and thereby determines those digits $a_{101}a_{102}a_{103}$.... Thus, one may say that the initial-number is only known up to some degree of uncertainty in the data which only affects the digits at position 101 and higher.*

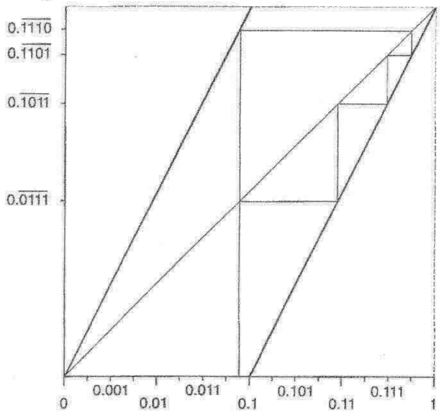
$A^2_{2.1}$ *If the iteration is executed as mentioned above, then the beginning behaves tamely. But if one continues iterating the noise creeps closer and closer to the decimal-point, and precisely after 100 iterations, the result will become perfectly random; this is called the phenomenon of sensitive-dependence-on-initial-conditions. It is an accurate and solid argument for the properties of the corresponding kneading-operation. Now an argument has been provided for the uniform distribution of spice in dough after kneading. If the spice originally comes in a clump and the coordinates of the particle are given as $0.a_1a_2a_3...a_ka_{k+1}...$, where the first k digits are the same for all particles, because they are clustered. The remaining digits are uniformly distributed modelling the random-mixing of the spice in the cluster. After k applications of the shift, the common coordinates are gone and only the random-digits are left, which yields a uniform distribution of spice throughout the entire dough.*

A^2_3 *If one takes a closer look on the sensitivity, a more precise definition of sensitivity will emerge: Given any point $x_0 \in (0,1)$ there exists a point y_0 arbitrarily close to x_0 such that the outcome of iteration started at x_0 and y_0 eventually will differ by certain threshold. This threshold must be the same for all points x_0 in the interval and is called sensitivity-constant. It should be noted that it is not required for all orbits started close to x_0 will develop this deviation exceeding this threshold. It should be argued that this definition of sensitivity holds for the iteration of the shift-operator. One claims that the threshold in this case can be as large as ½.*

$A^2_{3.1}$ *An arbitrary starting point in binary-representation may be picked by rolling a die writing 0 for any even roll and 1 for an odd roll. The result might be $x_0 = 0.0101101011100011$ 001.... Now its tried to find a starting-point y_0 close by, which should develop the difference to the orbit of x_0 reaching the threshold ½ at some point. For y_0 the same number as for x_0 may be picked except for 1 of the binary-digits, which is now changed. If one requires that y_0 has a distance to x_0 of at most 2^{-5}, then it is sufficient to flip the 6^{th} digit of x_0 and obtains $y_0 = 0.01011110111000110 01$.... After 5 iterations one gets: $x_5 = 0.010$ $111\ 00011\ 001... \leq ½$ and as required $y_5 = 0.11011100011001... = x_5 + ½$. Thus $|y_5 - x_5| = ½$.*

$A^2_{3.2}$ *Clearly, one can find points y_0 arbitrarily close to x_0 with the same property. All one needs to do is to just flip 1 of the binary-digits which must be of sufficiently high order. At 1 point in the iteration this digit will be the most significant one and the difference with the orbit of x_0 will be ½ again. One may note that all further iterates in both orbits are identical. A difference also in those iterations is not required by the definition. Of course one may devise other strategies for the choice of y_0 that produce an orbit which is different from that of x_0 in all iterations.*

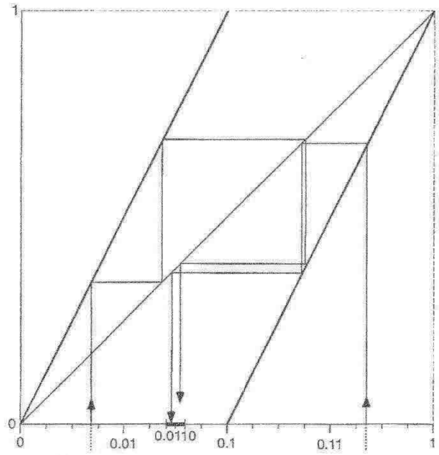
A^2_4 *To understand what the next chaos-phenomenon - dense periodic-points - means, an illustrative example in advance:*



Cycle of period 4: Point 0.[0111]... is a periodic-point. The binary-expression allows immediately to read-off the iterative behaviour (here visualized as graphical iteration).

$A^2_{4.1}$ *What happens if one specifies $x_0 = 0.[a_1a_2a_3...a_k]$..., if one has an infinite string of binary-digits which repeats after k digits? Running the iteration means that after k steps one will see x_0 again, and so forth. One will see a cycle of length k. Thereafter x_0 will be* *called periodic with respect to the binary-shift. Clearly, one can produce cycles of any length. But more importantly, for any given number x_0, number w_0 can be found arbitrarily close to x_0, which is periodic: If $x_0 = 0.a_1a_2a_3...$, $w_0 = 0.[a_1a_2a_3...a_k]...$ for some k can be chosen, then x_0 and w_0 differ by (at most) 2^{-k}, and w_0 makes the orbit to a periodic one. This means that periodic-points are dense.*

A^2_5 *The next chaos-phenomenon – mixing-gain shall first be demonstrated by an illustrative example:*



Mixing requires that any given interval J can be reached from any other interval. Here 2 examples are shown how one can reach a small interval at 0.0110.

$A^2_{5.1}$ *One may choose any 2 arbitrarily small sub-intervals I and J of the unit-interval. For the mixing one requires that a starting-point x_0 in I is found, whose orbit will enter the other interval J at some iteration. For the 2 sub-intervals I and J within unit-interval one defines $n > 0$ such that interval I has a length $> (½)^{n-1}$. It may be further: $0.a_1a_2a_3...$ the binary-representation of the midpoint from interval I. Moreover, $0.b_1b_2b_3...$ is the binary-representation of a point y from interval J.*

$A^2_{5.2}$ *Now the initial-point x_0 in I will be constructed in such a way that it after exactly n iterations of the shift-operator it will be equal to y. To define x_0 one copies the first n digits of the centre from I and afterwards appends all digits of the target-point y: $x_0 = 0.a_1...a_nb_1b_2b_3....$ Now, x_0 differs from the centre of I by at most 2^{-n} which is at most half the width of the interval I. Thus it must be contained in interval I. Secondly, after n iterations one gets $x_n = 0.b_1b2b_3... = y$. In the case of shift-operator one can hit any target-point in the interval J.*

A^2_6 *Closely related to mixing is ergodic-behaviour. Ergodicity means that if one picks a number x_0 at random in the unit-interval, then almost surely the results of the shift-operation will produce numbers which will get arbitrarily close to any number in the unit-interval. Numbers x_0 with a periodic pattern in their binary-expansion do not show such behaviour and in some way they are extremely scarcely populated in the unit-interval.*

A^3 Paradigma of Chaos: Transformation–specific Characteristics.

For the iteration of saw-tooth-transformation S (or shift-operation) or stretch-cut-and-paste-kneading it could be observed that it exhibits the 3 properties-of-chaos.

A^3_1 *Now the consideration will be continued with unfolding the chaos for the tent-transformation T (or the stretch-and-fold-kneading). By means of the substitution-property the iterations of T can be reduced to the iterations given by S–transformation. The k^{th} iterate x_k is obtained by $k-1$ binary-shifts followed by a single stretch-and-fold-operation T. Since the first part is just a shift by $k-1$ digits, one may easily carry-over all the complicated dynamic-behaviour (Sensitive-dependence, Denseness-of-periodic-points and Mixing) of the shift-transformation to the stretch-and-fold-transformation.*

$A^3_{1.1}$ *What are the periodic-points for the iteration of the tent-transformation T? Or more precisely, how can one find x_0 so that $x_n = x_0$ for a given integer n, where $x_j = T(x_{j-1})$ for $j = 1...n$. It should be asserted that, all one has to do now is to take a point w_0 which is periodic for the shift-transformation to obtain a periodic-point $x_0 = T(w_0)$ of T. Indeed, let $w_0 = S^n(w_0)$ be a periodic-point of S. Then one* *checks whether* $x_n = x_0$ *using the definition of* x_0 *and the substitution-property of the* 2 *kneading-transformations,* $x_n = T^n(x_0) = T^n(T(w_0)) = T^{n+1}(w_0) = T(S^n(w_0)) = T(w_0) = x_0$.

$A^3_{1.1.1}$ *Hence it is true that if* w_0 *is a periodic-point for the binary-shift. Then* $x_0 = T(w_0)$ *is a periodic-point for the stretch-and-fold-transformation with the same period. Using the result from above it is to reason that the periodic-points of* T *are dense in the unit-interval.*

$A^3_{1.1.2}$ *The tent-transformation is given by* $T(x) = \{(2x \text{ if } x \leq ½),(-2x+2 \text{ if } x > ½)\}$. *Let* $x = 0{,}a_1a_2a_3...$ *be a binary-expansion of* $x \in (0,1)$. *If* $x < ½$ *the tent- transformation is identical with the saw-tooth-transformation, thus* $T(x) = 0.a_2a_3a_4...$ if $x < ½$. *If* $x \geq ½$, *then* $S(x) = 2x-1$, *and* $T(x) = 2-2x = 1-(2x-1) = 1-S(x) = 1-0.a2a3a_4...$. *Introducing the dual-binary-digit* $a^* = \{(1 \text{ if } a = 0),(0 \text{ if } a = 1)\}$ *one will obtain in case of* $x \geq ½$, $T(x) = 0.a_2{}^*a_3{}^*a_4{}^*...$ *because* $0.a_2a_3a_4... + 0.a_2{}^*a_3{}^*a_4{}^*... = 1$. *The binary-representation of the tent-transformation is* $T(0.a_1a_2a_3...) = \{(0.a_2a_3a_4... \text{ if } a_1 = 0),(0.a_2{}^*a_3{}^*a_4{}^*... \text{ if } a_1 = 1)\}$. *To deal with the ambiguous-binary representations of rational-numbers one must require to use* 0.1 *for* ½. *The above binary-version of* T *works also for* $x = 1$ *when representing* 1 *as* 0.111....

$A^3_{1.1.3}$ *It had been shown already that a periodic-point* $w \in (0,1)$ *of the saw-tooth-function* S *with* $S^n(w) = w$ *induces a periodic-point* $x = T(w)$ *of* T *with* $T^n(x) = x$. *To show the denseness of these periodic-points of* T *one demonstrates that one can find periodic-points whose binary-expansion start with an arbitrary-sequence* $a_1...a_n$. *The point* $w = 0.[0a_1...a_n]... < ½$ *is periodic under* S *with period* $n+1$. *Because* T *is the shift-transformation when the* 1st *digit of the argument is* 0, *one will obtain* $x = T(w) = 0.a_1...a_n0$, *and* x *is periodic under* T *with period* $n+1$.

A^3_2 *It will be continued with the deviation of the mixing-property for the tent-transformation. Given are* 2 *open-intervals* I *and* J *in the unit-interval. It is always possible to choose* n *large enough and bits* $a_1...a_n$ *and* $b_1...b_n$ *so that all binary-numbers in* (0,1) *whose binary-expansion begin with* $a_1...a_n$, *are in interval* I *and all binaries starting with* $b_1...b_n$ *are in* J. *Now one specifies an initial-point* $x_0 \in I$ *such that the* n^{th} *iterate is in* J, $x_n = T^n(x_0) \in J$. *Again one treats the* 2 *cases* $a_n = 0$ *and* $a_n = 1$ *separately.*

<a = 0> *One chooses* $x_0 = 0.a_1...a_nb_1...b_n$ *and verifies:* $x_n = T^n(x_0) = T(S^{n-1}(x_0)) = T(0.a_nb_1...b_n) = T(0.0b_1...b_n) = 0.b_1...b_n \in J$.

<a = 1> *One chooses* $x_0 = 0.a_1...a_nb_1{}^*... b_n{}^*$ *and verifies:* $x_n = T^n(x_0) = T(S^{n-1}(x_0)) = T(0.a_nb_1{}^*...b_n{}^*) = T(0.1\, b_1{}^*...b_n{}^*) = 0.b_1{}^*...b_n{}^*11... \in J$.

A^3_3 *There is an elegant solution for the problem of deducing sensitivity which has been worked-out by a group of Australian-mathematicians [Ref. 1]. They showed in a theorem that the properties of mixing and dense-periodic-points are already suffice to show the* 3rd *property of chaos, sensitivity. In other words, if* f *is chaotic and* f *and* g *are equivalent via change* h *of coordinates, i.e.,* $f(h(x)) = h(g(x))$, *then* g *is chaotic too. The consequences for* f *and* g *are strong. For instance, when* f *is mixing then* g *is mixing, and likewise when* g *is mixing then* f *is mixing. Similarly, when* f *has dense-periodic-points then so has* g *and vice versa.*

$A^3_{3.1}$ *Thus, the knowledge of the shift-operation and its relation to the tent-transformation has provided the key to derive at least the* 2 *chaos-properties dense-periodic-points and mixing. It is quite natural to ask whether the* 3rd *chaos-property is in fact independent. That is, whether* 2 *of these conditions could imply the* 3rd *one or not. Another natural question is that of inheritance. Given that a mapping* f *is chaotic and that* g *is related to* f, *can one conclude that* g *is chaotic as well. After having analyzed the chaos-properties of saw-tooth-transformation* S *it was relatively easy to* *establish chaos-properties for the tent-transformation* T *when using the functional-equation* $\mathrm{TT} = \mathrm{TS}$. *The proper notions in connection with the open questions above are those of topological-conjugacy and topological-semi-conjugacy. What follows could be carried-out in very general situation but in the subsequent discussion it is preferred to stay with transformations of the real-line.*

$A^3_{3.2}$ *Let* X *and* Y *be* 2 *subsets of the real-line and* f *and* g *be* 2 *transformations* $\mathrm{f}\colon \mathrm{X}\to\mathrm{X}$ *and* $\mathrm{g}\colon \mathrm{Y}\to\mathrm{Y}$. *Then* f *and* g *are said to be topological-conjugate provided* f *and* g *are continuous and there is a homeomorphism* $\mathrm{h}\colon \mathrm{X}\to\mathrm{Y}$ *such that the functional-equation* $\mathrm{h(f(x))} = \mathrm{g(h(x))}$ *holds for all* $\mathrm{x} \in \mathrm{X}$.

<1> *The transformation* f *is said to be continuous provided for any* $\mathrm{x} \in \mathrm{X}$ *and any sequence* $\mathrm{x_1x_2x_3}\ldots$ *with limit* x *one has the sequence* $\mathrm{f(x_1)f(x_2)f(x_3)}\ldots$ *which a limit* $\mathrm{f(x)}$. *Alternatively and equivalently could be stated: The transformation* f *is continuous provided for any open-set* U *in* Y *the pre-image* $\mathrm{f^{-1}(U)} = \{\mathrm{x} \in \mathrm{X} \mid \mathrm{f(x)} \in \mathrm{U}\}$ *is open in* X.

<2> *A subset* V *of the real-line* $\mathbb{R}$ *is said to be open provided for any* $\mathrm{x} \in \mathrm{V}$ *there is an open interval* I *containing* x *which is entirely in* V. *A subset* U *of* X *is said to be open in* X *provided there is an open-subset of the real-line* $\mathbb{R}$, *say* V, *such that* $\mathrm{U} = \mathrm{X}\cap\mathrm{V}$.

<3> *A mapping* $\mathrm{h}\colon \mathrm{X}\to\mathrm{Y}$ *is said to be a homeomorphism provided* h *is continuous,* $1-\mathrm{to}-1$ *and onto, and the inverse-mapping* h^{-1} *is continuous too.*

<4> *It should be noted that topological-conjugacy is an equivalence-relation, thus the following* 3 *properties are true:*

<a> f *is topologically-conjugate to* f.

<b> *If* f *is topologically-conjugate to* g *then* g *is topologically-conjugate to* f.

<c> *If* f *is topologically-conjugate to* g *and* g *is topologically-conjugate to* h *then* f *is topologically-conjugate to* h.

A^3_4 *In the discussion of the saw-tooth- and tent-transformation the crucial-relation* $\mathrm{TT} = \mathrm{TS}$ *was established. In other words, if* $\mathrm{h} = \mathrm{T}$ *one gets a functional-relation of the form* $\mathrm{hS} = \mathrm{Th}$. *But there is a problem in using this for a topological-conjugacy between* T *and* S. *The transformation* T *is continuous and* S *is not. If* h *were a homeomorphism then also* S *would have to be continuous because of the functional-equation* $\mathrm{hS} = \mathrm{Th}$ *(which would be equivalent to* $\mathrm{S} = \mathrm{h^{-1}Th}$*), but this is not the case. The reason for that is* h *is not a homeomorphism. It is continuous and onto, but not* $1-\mathrm{to}-1$ *(each* $\mathrm{y} \neq 1$ *in* $(0{,}1)$ *has* 2 *pre-images* $\mathrm{x_1} \neq \mathrm{x_2}$*) such that* $\mathrm{h(x_1)} = \mathrm{y} = \mathrm{h(x_2)}$, *and there is no inverse-transformation for* h. *This situation leads to a very useful modification of the notion of topological-conjugacy the so called topological-semi-conjugacy.*

$A^3_{4.1}$ *Let* X *and* Y *be* 2 *subsets of the real-line and* f *and* g *be transformations* $\mathrm{f}\colon \mathrm{X}\to\mathrm{X}$ *and* $\mathrm{g}\colon \mathrm{Y}\to\mathrm{Y}$. *Then* g *is said to be topologically-semi-conjugate to* f *provided there is a continuous-and-onto-transformation* $\mathrm{h}\colon \mathrm{X}\to\mathrm{Y}$ *such that functional-equation* $\mathrm{h(f(x))} = \mathrm{g(h(x))}$ *holds for all* $\mathrm{x} \in \mathrm{X}$.

$A^3_{4.2}$ *It should be noted this is not a equivalence-relation, because as* g *is topologically-semi-conjugate to* f, f *may still not be topologically-semi-conjugate to* g. *Moreover, it should be noted there is no requirement for* f *or* g *to be continuous.*

$A^3_{4.3}$ *This is exactly the situation which was found for* T *and* S *previously. In other words,* T *is semi-conjugate to* S.

A^3_5 *This leads to important consequences:*

<1> *The functional-equation* $\mathrm{hf} = \mathrm{gh}$ *implies* $\mathrm{hf^n} = \mathrm{g^nh}$ *for any natural-number* n. *Indeed,* $\mathrm{hf} = \mathrm{gh}$ *implies* $\mathrm{ghf} = \mathrm{g^2h}$, *using* $\mathrm{gh} = \mathrm{hf}$ *this implies* $\mathrm{hf^2} = \mathrm{g^2h}$. *Likewise, one may obtain the general-case by induction.*

<2> *If* g *is assumed to be topological-semi-conjugate to* f *via a continuous and onto*

transformation h *and* f *may have the properties dense-periodic-points and mixing, then* g *has dense-periodic-points and mixing as well.*

<3> *This shows that the chaos-properties of* T *established above from that of* S *has to be seen in a very general background. Dense-periodic-points and mixing are inherited.*

<a> *In order to prove this, let* $f: X \to X$ *and* $g: Y \to Y$. *Periodic-points* g *are dense in* Y *and* $y \in Y$. *It will be shown that there is a sequence of periodic-points of* g *with limit* y. *As a pre-image of* y *under* h, x *will be chosen, i.e.* $h(x) = y$ *(because* h *is onto). Since periodic-points of* f *are dense in* X *one can find a sequence* $x_1 x_2 x_3 \ldots$ *with limit* x, *such that* $f^{n_e}(x_e) = x_e$. *In other words,* x_e *is a periodic-point of period* n_e. *One claims that the sequence* $\{y_e\}$ *with* $y_e = h(x_e)$ *has limit* y *and is a sequence of periodic-points of* g. *The* 1^{st} *claim is true because* h *is continuous. The* 2^{cd} *claim follows from the functional-equation* $hf^e = g^e h$. *Indeed:* $g^{n_e}(y_e) = g^{n_e}(h(x_e)) = h(f^{n_e}(x_e)) = h(x_e) = y_e$.

<b> *Secondly, the transformation* g *is mixing. Let* U *and* V *be* 2 *open-sets in* Y. *One should find* $y \in U$ *and a natural-number* n *such that* $g^n(y) \in V$. *Beginning with taking the pre-images* $A = h^{-1}(U)$ *and* $B = h^{-1}(V)$. *It should be noted that* A *and* B *are open, because* h *is continuous. Thus there exists a natural number* n *and* $x \in A$ *such that* $f^n(x) \in B$, *since* f *is mixing. Set* $y = h(x)$ *and by using the functional-equation one will obtain* $g^n(y) = g^n(h(x)) = h(f^n(x))$. *Since* $f^n(x) \in B$ *one can conclude that* $h(f^n(x)) = h(B) = V$.

A^3_6 *Finally one should come to a discussion of whether the* 3 *chaos-properties are independent from each other. In this context the following statement is true: If* X *is an arbitrary subset of the real-line and* $f: X \to X$ *is a continuous transformation which has the property of mixing and possesses dense-periodic-points. Then* f *also shows sensitivity-on-initial-conditions. In other words, if* f *is chaotic and* g *is topologically-semi-conjugate to* f, *then* g *is chaotic too. The proof of this fact is sketched subsequently:*

$A^3_{6.1}$ *One has to find* $d > 0$ *so that for any* $x \in X$ *and any open-subset* $J \subset X$ *containing* x *one has to find a point* $z \in J$ *and a natural number* n *so that* $|f^n(x) - f^n(z)| > d$. *The argument has* 2 *steps:*

$A^3_{6.1.1}$ *Firstly, there is a* $\delta_0 > 0$, *such that for any* $x \in X$ *there is a periodic-point* $p \in X$ *with the property that* $|f^k(p) - x| > \delta_0/2$, *for all* $k = 0,1,2,\ldots$. *Indeed, choosing* 2 *arbitrary periodic-points* r *and* s *with disjoint orbits, i.e., such that* $f^k(r) \neq f^j(s)$ *for all* k,j. *If* δ_0 *is the distance between these orbits* $\delta_0 = \min\{|f^k(r) - f^j(s)| \mid k,j \in \{0,1,2,\ldots\}\}$ *and* $x \in X$. *Then by triangle-inequality:* $\delta_0 \leq |f^k(r) - f^j(s)| = |f^k(r) - x + x - f^j(s)| = |f^k(r) - x| + |x - f^j(s)|$. *Thus, either* x *has a distance of at least* $\delta_0/2$ *to the orbit of* r *or the orbit of* s. *Otherwise if* $|f^k(r) - x| < \delta_0/2$ *and* $|x - f^j(s)| < \delta_0/2$ *for all* k,j, *one will arrive at a contradiction.*

$A^3_{6.1.2}$ *Secondly, it will be shown that* f *has sensitive-dependence-on-initial-conditions with sensitivity-constant* $\delta = \delta_0/8$. *There may be any point* $x \in X$ *and* J *may be any open subset of* X *containing* x. *Since periodic-points are dense in* X *one can find a periodic-point* p *in* $U = J \cap (x - \delta, x + \delta)$. *The period of* p *may be denoted by* n. *There may exist a periodic-point* $q \in X$ *whose orbit has at least a* 4δ*–distance from* x. *One may set:* $W_j = (f^j(q) - \delta, f^j(q) + \delta) \cap X$ *and* $V = f^{-1}(W_1) \cap f^{-2}(W_2) \cap \ldots \cap f^{-n}(W_n)$ *with* $j = 1 \ldots n$ *where* $f^{-j}(A) = \{z \in X \mid f^j(z) \in A\}$ *is the pre-image of the set* A *under* f. *It should be noted that* $q \in V$, *so that* V *is not empty. Moreover,* V *is an open set because* f *is continuous. Since* f *is also mixing one find a natural-number* k *and* $y \in U$ *such that* $f^k(y) \in V$. *Thus,* $1 \leq ni - k \leq n$. *By construction* *one will obtain that the point* $f^{ni}(y) = f^{ni-k}(f^k(y) \in f^{ni-k}(V)$ *is contained in the open set* W_{ni-k}. *On the other hand* $f^{ni}(p) = p$ *so that by triangle-inequality:* $|f^{ni}(p)-f^{ni}(y)| = |p-f^{ni}(y)| = |x-f^{ni-k}(q)+f^{ni-k}(q)-f^{ni}(y)+|p-x| \geq |x-f^{ni-k}(q)|-|f^{ni-k}(q)-f^{ni}(y)|-|p-x|$. *Therefore, since* $p \in (x-\delta,x+\delta)$ *and* $f^{ni}(y) \in (f^{ni-k}(q)-\delta,f^{ni-k}(q)+\delta) \cap X$ *one obtains:* $|f^{ni}(p)-f^{ni}(y)| > 4\delta-\delta-\delta = 2\delta$. *Thus one obtained either* $|f^{ni}(x)-f^{ni}(y)| > \delta$ or $|f^{ni}(x)-f^{ni}(p)| > \delta$. *Indeed, if both distances would be strictly less than* δ, *then by the triangle-inequality:* $|f^{ni}(p)-f^{ni}(y)| = |f^{ni}(p)-f^{ni}(x)+f^{ni}(x)-f^{ni}(y)| \leq |f^{ni}(p)-f^{ni}(x)|+|f^{ni}(x)-f^{ni}(y)| < 2\delta$, *which is a contradiction. Since* $(p, y) \in U \subset J$ *one gets that the* $ni-th$ *iterate of* f *at* p *or* y *is more than a distance* δ *from the* $ni-th$ *iterate of* f *at* x.

$A^3{}_7$ *By applying the formulas from* $(A^1{}_4)$ *(appropriate for the chance of coordinates) one can easily find a periodic-point of the quadratic-iterator. All one needs is a periodic-point for the tent-transformation, say* x_0. *Then one applies the equivalence transformation to obtain* $\sin^2(\pi x_0/2)$ *which is guaranteed to be periodic in the quadratic-iterator. Thus, the iteration of the tent-transformation and the parabola are totally equivalent. All the signs of chaos are found when iteration the quadratic-iterator* $f(x) = 4x(1-x)$.

<a> *Points that are periodic for the tent-transformation correspond to points that are periodic for the parabola.*

<b> *Points that show mixing by leading from one given sub-interval to another for the tent-transformation correspond to points that show the same behaviour for the parabola.*

<c> *Points that exhibit sensitivity for the tent-transformation correspond to points that show sensitivity for the parabola.*

However, it has to be remarked that these conclusions are not self-evident. In the following a proof for first 2 *properties is presented.*

$A^3{}_{7.1}$ *It may be: The tent-transformation denoted by* T, *the quadratic-transformation denoted by* $f(x) = 4x(1-x)$, *the transformation for the chance of coordinates denoted by* $h(x) = \sin^2(\pi x/2)$ *and the functional-equation as shown above* $f^k(h(x)) = h(T^k(x))$ *for* $k = 1, 2, \ldots$ *and* $x \in (0,1)$. *Furthermore, one knows that periodic points of* T *are dense in* $(0,1)$ *and* T *is mixing.*

$A^3{}_{7.2}$ *One claims that periodic-points of* f *are dense in* $(0,1)$ *with* $y \in (0,1)$. *It will be shown that there is a sequence of periodic-points from* f *with a limit* y. *One may choose* x *as pre-image of* y *under* h *(i.e.* $h(x) = y$*) because* h *is onto. Since periodic-points of* T *are dense in* $(0,1)$ *one can find a sequence of points* $x_1x_2\ldots$ *with a limit* x *and such that each point* x_e *is a periodic-point of* T *with some period (say* p_e*). Thus,* $T^{p_e}(x_e) = x_e$ *for* $e = 1,2,\ldots$. *One claims that the sequence of points* $y_1y_2\ldots$ *with* $y_e = h(x_e)$ *has a limit* y *and is a sequence of periodic-points from* f.

$A^3{}_{7.3}$ *The* 1st *claim is true because* h *is continuous. The* 2cd *claim follows from the functional-equation* $f^e h = hT^e$. *Indeed,* $f^{p_e}(y_e) = f^{p_e}(h(x_e)) = h(T^{p_e}(x_e)) = h(x_e) = y_e$.

$A^3{}_{7.4}$ *One further claims that the transformation* f *is mixing.* U *and* V *are supposed to be* 2 *open intervals in* $(0,1)$. *One must find a point* $y \in U$ *and a natural-number* e *so that* $f^e(y) \in V$.

$A^3{}_{7.4.1}$ *Starting by taking the pre-images* $A = h^{-1}(U) = \{x \in [0,1] \mid h(x) \in U\}$ *and* $B = h^{-1}(V)$. *It should be noted that* A *and* B *are open, because* h *is continuous. Thus there exists a natural-number* e *and* $x \in A$ *such that* $T^e(x) \in B$, *since* T *is mixing.*

$A^3{}_{7.4.2}$ *There may be* $y = h(x)$ *and now using the functional-equation one obtains:* $f^e(y) = f^e(h(x)) = h(f^e(x))$. *Since* $f^e(x) = B$ *one may conclude that* $h(f^e(x)) \in h(B) = V$. *Therefore* f *is mixing.*

$A^3{}_{7.5}$ *Dense-periodic-points for the tent-transformation and the equivalence of* T *and* f

yield that also f has dense-periodic-points. Mixing for T and the equivalence yield that f is mixing too. This approach does not work for the 3rd property-of-chaos, sensitivity. Sensitivity-on-initial-conditions is not generally inherited from one dynamical-system to another which has iterations that are equivalent by change-of-coordinates. In contrast, the properties of mixing and dense-periodic-points are passed-over to the equivalent-system.

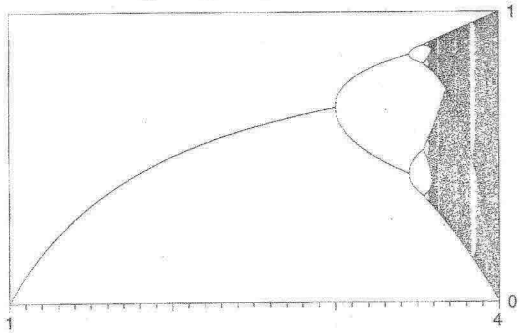
$A^3_{7.6}$ *But analogous to what had been expressed already in* (A^3_3, A^3_6) *with respect to the independence of chaos-properties from a transformation or inheritance of them between transformations can also be used here as a scheme for deducing sensitive-dependence-on-initial-conditions for the quadratic-iterator.*

B^1 **Analysis of Period–Doubling–Regime.**

Chaos and order have long been viewed as antagonistic-states in sciences. Special methods of investigation and theory have been designed for both. Natural laws like KEPLER's-law or NEWTON's-law represent the domain of order. Chaos was understood to belong to a different face of nature, seen as a higher degree of complexity or a more complex form of nature. One of the great surprises revealed through the studies of the quadratic-iterator $x_{n+1} = ax_n(1-x_n)$, *is that both antagonistic-states can be ruled by a single law. An even bigger surprise was the discovery that there is a very well-defined route which leads from one state (order) into the other state (chaos). Furthermore, it was recognized that this route is universal. Route means that there are abrupt qualitative-changes (called bifurcations) which mark the transition from order into chaos like schedules and universal means that these bifurcations can be found in many natural-systems both qualitatively and quantitatively.*

B^1_1 *In this context, the question is at forefront: what is the long term behaviour of the quadratic-iterator* $x_{n+1} = ax_n(1-x_n)$ *for all parameters* $1 \leq a \leq 4$. *Means, what happens to the iterates* x_n *when the dependence on the initial choice* x_0 *is diluted to almost* 0*?*

$B^1_{1.1}$ *Clearly, the iteration produces values* x_n *which remain in the interval* $(0,1)$ *as long as the intial-value* x_0 *is from that interval. For instance, with parameter* $a = 2$ *and a randomly chosen initial-value* x_0 *one would obtain a time-series* $x_0x_1x_2...$ *for this parameter. This poly-line (after a transient-phase of a few iterations) will settle-down at the fixed-point* ½ *which is called the final-state of this orbit. If one repeats this experiment for different initial-values* $x_0 \in (0,1)$ *one always will reach the same final-state. The complete set of final-states from quadratic-iterator for initial-values* $x_0 \in (0,1)$ *in dependence on the parameter-range* $1 \leq a \leq 4$ *is shown in the next diagram, called FEIGENBAUM-diagram.*



Final-state-diagram for the quadratic-iterator and parameters $1 \leq a \leq 4$.

$B^1_{1.1.1}$ *It should be noted that for* $a > 3$ *the final-state is not a mere point but a collection of* 2, 4 *or more points. For* $a = 4$ *one gets chaos as discussed in the form of* (A) *and the points of the final-states densely fill-up the complete unit-interval vertically.*

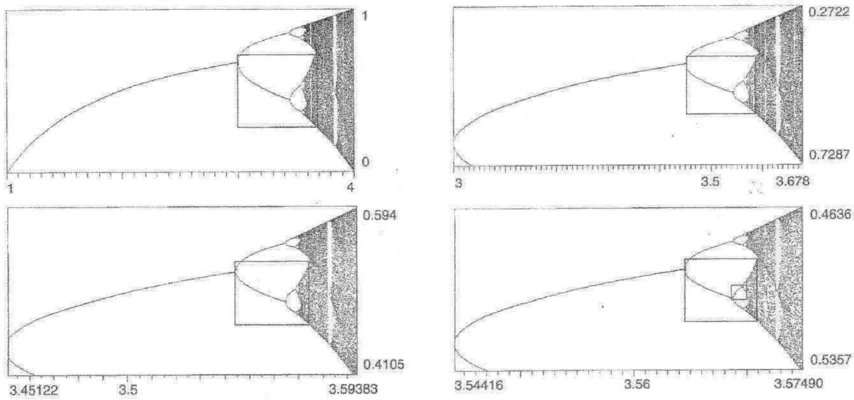
$B^1_{1.2}$ *One essential-structure seen in the FEIGENBAUM-diagram is that of a branching-tree which portrays the qualitative-changes in the dynamical-behaviour of the iterator* $x \rightarrow ax(1-x)$. *Out of a major stem one can see* 2 *branches bifurcating, and out of these branches another* 2 *branches bifurcate and so on in the same way. This is the period-doubling-regime of the scenario. Period-doubling crudely explained means:*

$B^1_{1.2.1}$ *Where one will see just* 1 *branch the long-term-behaviour of the system tends towards a fixed-state, which depends on the parameter-value* a *and will be* *reached (no matter at which initial-value $x_0 \in (0,1)$) an iteration will be started. As soon as 2 branches will be seen, this means that the long-term-behaviour of the system is now alternating between 2 different states, a lower one and an upper one. This is called periodic-behaviour and since there are 2 such states now, the situation describes a period—2—behaviour and compared to the former development the period has doubled. When one sees 4 branches instead of 2, all that has changed is that the period of the final-state-behaviour has increased from 2 to 4. This becomes the period-doubling-regime of the FEIGENBAUM-diagram: $1 \rightarrow 2 \rightarrow 4 \rightarrow 8 \rightarrow 16 \rightarrow \ldots \rightarrow 2^N \ldots$, where N is a natural number.*

$B^1_{1.2.2}$ *Beyond this period-doubling-cascade at the right-end of the figure a structure can be observed with a lot of detailed designs. Chaos has set-in and eventually at the parameter-value $a = 4$ it governs the unit-interval of the graph vertically.*

$B^1_{1.3}$ *The structure in figure $(B^1_{1.1})$ has self-similarity-properties, means that the route from order to chaos is one with infinite detail and complexity.*

$B^1_{1.3.1}$ *Next figures show a sequence of close-ups, starting with a reproduction of figure $(B^1_{1.1})$ and magnifying the rectangular window indicated in the initial-diagram, but showing it upside-down.*

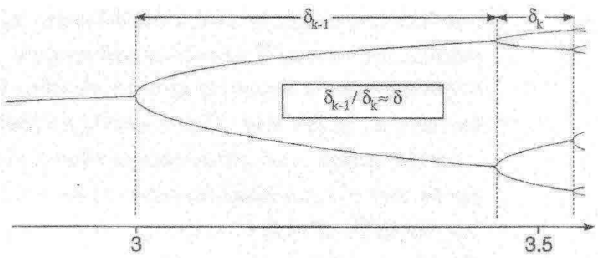


$B^1_{1.3.1.1}$ *A close-up-sequence of the final-state-diagram of the quadratic-iterator reveals its self-similarity. One should note that the vertical value in the upper-left and bottom-left magnifications have been reversed to reflect the fact that the previous diagram has been inverted. The upper-right magnification is also a vertical- inversion of the upper-left magnification the values, however, are in their normal relationship. Theoretically, one could go on infinitely often, in other words, the final-state-diagram is a self-similar-structure.*

$B^1_{1.4}$ *The branches in the period-doubling-regime become shorter and shorter as one looks from left to right. It's therefore a tempting thought to imagine that the lengths of the branches in direction of the a—axis might decrease relative to each other perhaps according to some geometric-law. This idea would lead to several consequences:*

$B^1_{1.4.1}$ *First of all, it would constitute a threshold, i.e., a parameter a beyond which the branches of the tree could never grow. This would mark the end of the period-doubling-regime. Indeed, there is such a threshold which became known as FEIGENBAUM-point $s_\infty = 3.5699456\ldots = a$. It is precisely the a—value at which the sequence-of-rectangles shown in $(B^1_{1.3.1})$ converge. The FEIGENBAUM-point splits the final-state-diagram into 2 very distinct parts, the period-doubling-tree on the left and the area governed by chaos on the right.*

$B^1_{1.4.2}$ *Secondly, if there is a rule that quantifies the way the period-doubling-tree approaches the FEIGENBAUM-point, one could try to compare it with the laws which might observe in related iterators. In fact, these ideas were carried-out by FEIGENBAUM himself and he found that a law could be isolated from the branching-behaviour and was exactly the same for many different iterators. In a very precise-sense the law can be captured in just 1 number which was measured to be $\delta = 4.6692\ldots$ and was called as FEIGENBAUM-constant.*

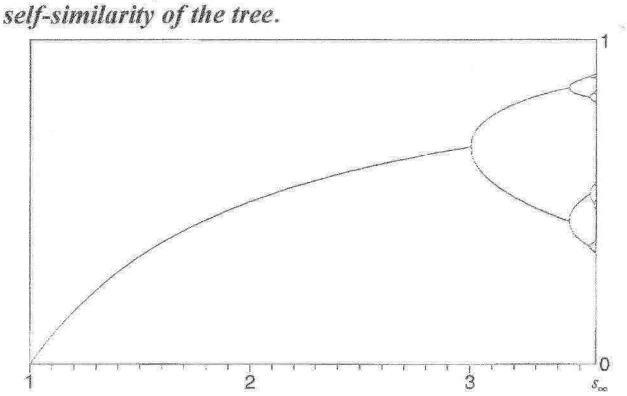


The ratio δ is the magnification-factor of succeeding branch-lengths.

Its appearance in many different systems was called universality. Roughly the meaning of constant δ is this: if one measures the lengths of 2 succeeding-branches (in a—axis-direction) then their ratio turns-out to be approximately δ.

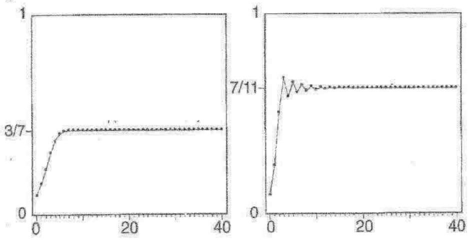
$B^1_{1.4.2.1}$ *Some years after FEIGENBAUM's-work physicists discovered that the scenario of period-doubling and the value of δ manifested themselves in real physical experiments, the meaning of universality was not mere covering very primitive mathematical models but real physical phenomena. In essence that means whenever a system behaves in a period-doubling fashion, then it is very likely that one will see the full structure of the FEIGENBAUM-diagram in it. In other words, although the quadratic-iterator in some sense is much to simple to carry any information about real systems, in a very striking and general sense it does carry the essential-information about how systems may develop chaotic-behaviour.*

B^1_2 *The portion of the final-state-diagram to the left of the FEIGENBAUM-point s_∞ is a self-similar fractal tree. It describes the period-doubling-scenario of the quadratic-iteraror, which leads from a very simple and orderly behaviour of the dynamics right to the beginning of chaos-region. One now should try to understand the mechanism lying at the base of its generation and leading to the self-similarity of the tree.*



The period-doubling-tree, 1st portion of the final-state-diagram.

$B^1_{2.1}$ *The discussion starts with the stem-of-the-tree, the part of $1 \leq a \leq 3$. This part represents a stable situation where the iteration is always led to 1 rest-point.*

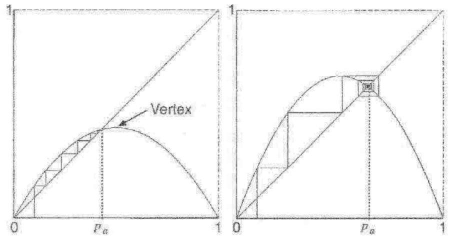


Plot shows time-series of initial-point $x_0 = 0.1$ for parameter-value $a = 1.75$, the iteration settles-down at $3/7$. Starting from $0 < x_0 \leq 1$ this value will always be approached. Right graph shows adequate situation for $a = 2.75$ with result $7/11$, course of iteration is not direct, it oscillates around final-state while settling-down.

$B^1_{2.1.1}$ *In both cases one has a situation, no matter where one chooses the initial-value $0 < x_0 \leq 1$, it can be expected that a long run will settle-down at the approptiate attractor $A(a)$.*

$B^1_{2.1.2}$ *When starting the iteration of the quadratic-iterator $f_a = ax(1-x)$ exactly from attractor $x_0 = A(a)$, one gets for all iterates $x_0 = x_1 = x_2 = \dots = x_0$ the same value. In other words x_0 is a fixed-point of f_a. Thus x_0 solves the equation $ax(1-x) = x$ and deliver the 2 solutions $(a-1)/a = p_a$ and $p_0 =$*

0. *Moreover, it can be noted that if* $x_0 = 1$ *then* $x_1 = x_2 = \ldots = 0$. *In other words can be said* 1 *is a pre-image of* $p_0 = 0$. *But* $x_0 = 0$ *and* $x_0 = 1$ *are the only initial-values which lead to* 0, *all other values are attracted by* $p_a = (a-1)/a$.

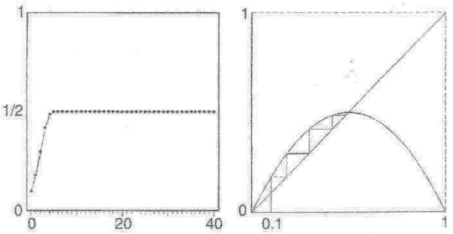


Graphical-iteration near stable fixed-point. Iteration is performed for $a = 1.75$ *(left) and* $a = 2.75$ *(right). Both cases were started from* $x_0 = 0.1$. *The iteration settles at* $p_a = (a-1)/a$ *which is* $3/7$ *(left) and* $7/11$ *(right).*

In other words, iteration is pushed away from rest-point $p_0 = 0$. *One says,* p_0 *is a repeller or unstable-fixed-point. On the other hand* $p_a = (a-1)/a$ *is a stable –fixed-point or attractive-fixed-point of all parameters* a *between* 1 *and* 3. *One can verify these facts using graphical-iteration from above.*

$B^1_{2.1.2.1}$ *The iteration is represented here by a path with horizontal and vertical steps called a poly-line. For the situation of* $a = 1.75$ *the parabola* $ax(1-x)$ *insects the bisector at the fixed-points* $p_0 = 0$ *and* $p_a = (a-1)/a$. *Between these values the parabola lies above the bisector, but its vertex lies beyond the intersection. Thus the iteration must be repelled away from* 0. *On the other hand the poly-line is trapped between the parabola and the bisector and thus is led directly to the* 2^{cd} *intersection-point at* $(a-1)/a$.

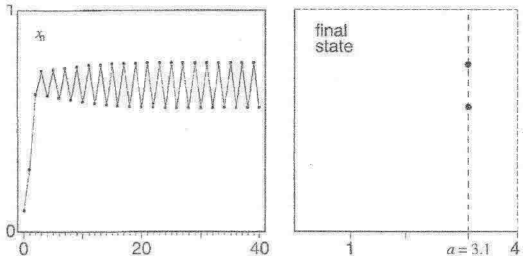
$B^1_{2.1.2.2}$ *The right graph in figure* $(B^1_{2.1.2})$ *shows the situation for* $a = 2.75$. *In this case the bisector intersets the parabola beyond its vertex. Thus the poly-line begins to spiral around the point of intersection. The spiraling is directed inwards, the process again settles-down at* $(a-1)/a$. *In other words, the fixed-point is still attractive although the local-behaviour (the way orbits are attracted) has changed. Spiraling sets in as soon as the vertex of the parabola surpasses the right intersection-point of the parabola and the bisector. In other words, it has to be searched for the case where the intersection-point and the vertex of the parabola come together. Since the parabola has its maximum at* $x_{max} = ½$, *one has to solve the equation* $½ = a/2(1-½)$ *for it, the solution is* 2, *next figure shows the appropriate situation:*



For $a = 2$ *super-attractive-situation to be bserved. The graphical-iteration shows orbit rushes into fixed-point.*

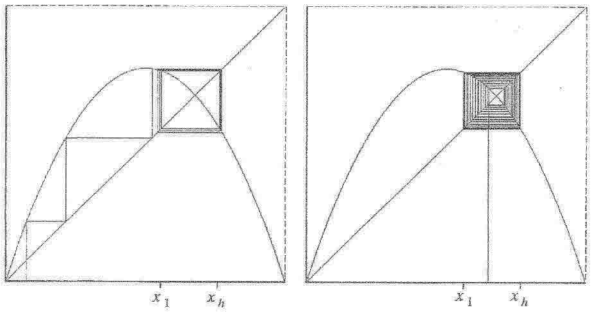
$B^1_{2.1.2.2.1}$ *Left graph is a plot of the time-series for the initial-value* 0.1. *Compared with the time-series for* $a = 2.75$ *in figure* $(B^1_{2.1})$ *it can be remarked that for* $a = 2$ *the attractive fixed-point is approached much faster. Indeed, this point is called super-attractive. It is very special in the interval* $1 \leq a \leq 3$ *and occurs only at point* $a = s_1 = 2$.

B^1_3 *Having studied the dynamics of the quadratic-iterator* f_a *in detail for the parameters* $1 \leq a \leq 3$, *now discussion will be continued by increasing* a *beyond* 3. *For those parameter-values the fixed-point* $p_a = (a-1)/a$ *will lose its stability and become a repeller. The question arises now, is there a different attractor that takes over the role of* p_a*? To start answering here, next figure shows a case for* $a = 3.1$.



Time-series for parameter $a = 3.1$ where the orbit started from initial-values $x_0 = 0.1$. The iteration leads to a final state which consists of 2 points $x_l(a)$ and $x_h(a)$.

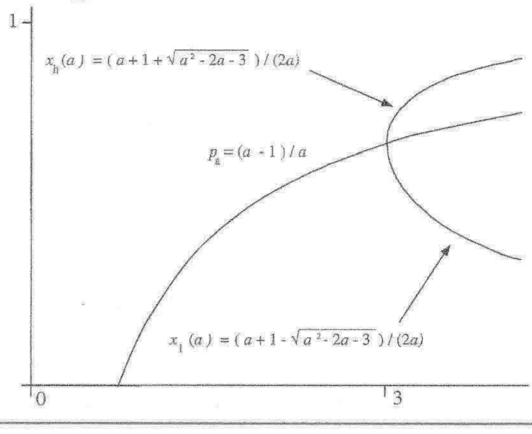
$B^1_{3.1}$ *In the above picture a time-series is obtained which exhibits an entirely new behaviour. There is oscillation as in case of $2 \leq a \leq 3$, but it does not finally settle-down to 1 single point. Rather, it stabilizes in oscillating between 2 values, a low one $x_l(a)$ and a high one $x_h(a)$. Thus, in the final state-diagram one obtains just these 2 points at parameter $a = 3.1$:*



Graphical-iteration for $a = 3.1$. The periodic-cycle $\{x_l(a), x_h(a)\}$ is theattractor for the quadratic-iterator. On the left initial-point for the orbit is $x_0 = 0.075$, while onthe right it is $x_0 = 0.65$.

$B^1_{3.1.1}$ *In left-graph one first notices a familiar staircase. But then the poly-line turns into an inward-spiral which slowly runs into a repeating loop. In the right plot initial-value x_0 is close to the unstable fixed-point p_a. Its orbit spirals-outwards the same loop as seen on the left. In other words, while for $a < 3$ the fixed-point p_a attracts all iterations, it turns into a repeller when $a > 3$. Close to p_a, iteration will be pushed away. Fixed-point p_a loses its stability as a crosses the border $b_1 = 3$. This particular-parameter-value is called a bifurcation-point. Findings for the final-state-diagram shall be summurized in the next figure.*

$B^1_{3.1.2}$ *$p_a = (a-1)/a$ is attractor for all iterations starting in interval $(0,1)$ at parameter-values $1 < a < 3$. Formally the attractor is a single point $A(a) = \{p_a\}$ for $1 < a < 3$. For $a > 3.$, p_a still exists. Thus, an iteration started precisely at this point and remains there forever, $p_a = x_0 = x_1 = ...$ However, p_a is a repeller and therefore not part of the final-state $A(a)$. The attractiveness has been taken over by the loop which oscillates between 2 values $x_l(a)$ and $x_h(a)$. Thus the final-state is the attractor made of 2 points $A(a) = \{x_l(a), x_h(a)\}$ for $a > 3$. The pair is called a $2-$cycle or orbit of period 2. It's characterized by the fact that x_l is transformed into x_h and vice-versa. This periodic-orbit exists for all parameters $3 < a < 4$. However, at fixed-point p_a loses stability at $b_1 = 3$, also the $2-$cycle loses stability at a certain parameter-value $b_2 > 3$, this will be discussed next.*



Within this bifurcation-diagram the fixed-point p_a and 2 cycleiterations $\{x_l(a), x_h(a)\}$ are shown.

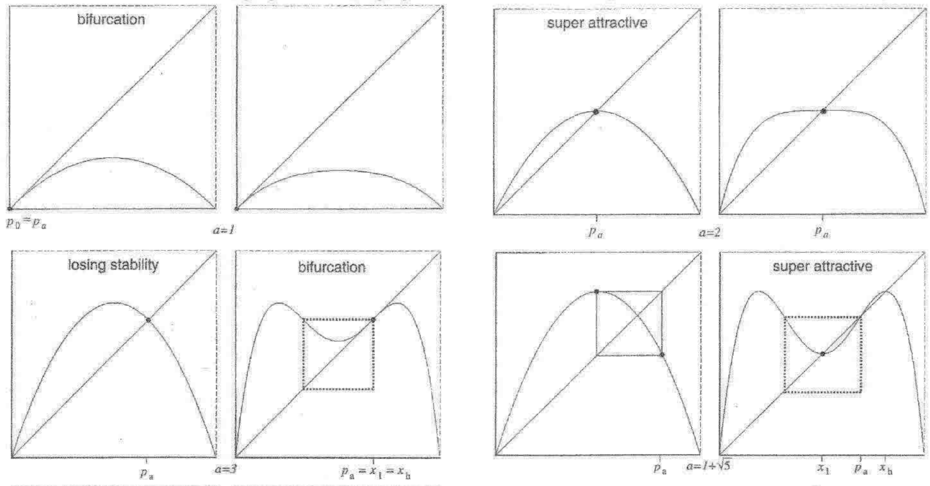
$B^1_{3.2}$ *Writing again* $f_a = ax(1-x)$ *for the quadratic-iterator, one needs to find the solutions of the* 4^{th} *order equation* $f_a(f_a(x)) = -a^3x^4+2a^3x^3-(a^2+a^3)x^2+(a^2-1)x = 0$.

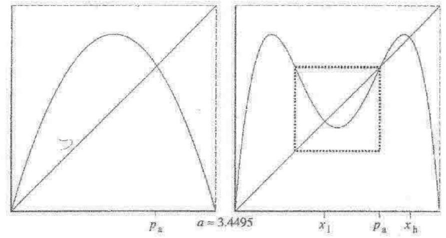
$B^1_{3.2.1}$ *One already knows the fixed-points of the iteration which solve* $f_a(x) = x$ *(i.e.,* 0 *and* $(a-1)/a$*). The solution* $x = 0$ *allows to simplify the equation to a* 3^{rd} *order equation* $-a^3x^3+2a^3x^2-(a^2+a^3)x+(a^2-1) = 0$. *Knowing the* 2^{cd} *solution* $(a-1)/a$, *one divides the* 3^{rd} *order equation by* $x = (a-1)/a$, *which leads to* $-a^3x^2+ (a^2+a^3)x-(a^2+a) = 0$ *or further dividing by* $-a^3$ *gives* $x^2-(a+1)x/a+ (a+1)/a^2 = 0$. *The roots of this quadratic-equation are* $x_h(a) = (a+1+\surd[a^2-2a-3])/2a$ *and* $x_l(a) = (a+1-\surd[a^2-2a-3])/2a$.

$B^1_{3.2.2}$

Considering only parameters $0 \le a \le 4$, *one will note that these solutions are defined only for* $a \ge 3$. *Moreover, at* $a = 3$ *one gets* $x_l(a) = x_h(a) = (a-1)/a$, *the* 2 *solutions bifurcate from the fixed-point* p_a. *Figure* ($B^1_{3.1.2}$) *shows the bifurcation-diagram of the explicitly calculated solutions. One should note that in the final-state-diagram one cannot see the solution of* $p_a = (a-1)/a$ *for* $a > 3$. *This corresponds to the fact that although the fixed-point continues to exist, it has become unstable (i.e., the iteration is pushed away from this point).*

$B^1_{3.3}$ *In case of quadratic-iterator the notation* $f_a(x) = ax(1-x)$ *had been introduced so far as a symbol for the set of typical-orbits with initial-values* $0 \le x_0 \le 1$ *and parameter-values* $1 \le a \le 3$. *The* 2^{cd}*-order-composition* $f_a(f_a(x)) = f_a^{\,2}(x)$ *called* 2^{cd}*-iterate of* f_a *now considered too. The graph of* $f_a^{\,2}(x)$ *is given by the* 4^{th} *-degree-polynomial* $f_a^{\,2}(x) = a^2x(1-x)(1-ax(1-x)) = (-a^3x^4)+2a^3x^3-(a^2+a^3)x^2+a^2x$. *It has* 4 *fixed-points* $\{0, p_a, x_l(a), x_h(a)\}$, *these are the* 2 *fixed-points of* f_a *and* 2 *elements of the* 2*-cycle. The graphs of* $f_a(x)$ *and* $f_a^{\,2}(x)$ *shall now be compared more systematically.*





Comparing f_a*(top-left) and* $f_a^{\,2}$*(top-right) for* $a = (1 \wedge 2)$.
Comparing f_a*(middle-left,* $a = 3$*) and* $f_a^{\,2}$*(middle-right,* $a = 1+\surd5$*).*
Comparing f_a*(bottom-left,* $a = 3$*) and* $f_a^{\,2}$*(bottom-right,* $a = 3.4496$*).*

$B^1_{3.3.1}$ *It will be started at* $a = 1$, *which is the parameter where the fixed-point* $p_0 = 0$ *of* $f_a(x)$ *becomes unstable and a new fixed-point* $p_a = (a-1)/a$ *begins to exist (for* $a > 1$*). Here the graph of* $f_a^{\,2}(x)$ *looks a bit lower and the fixed-points are identical to those of* $f_a(x)$. *Then at* $a = s_1 = 2$ *one reaches the super-attractive-case for* $f_a(x)$. *The new fixed-point* p_a *and the critical-point* $x_{max} = ½$ *coincide. The graph of* $f_a^{\,2}(x)$ *has now reached the same height, but its top is almost flat.*

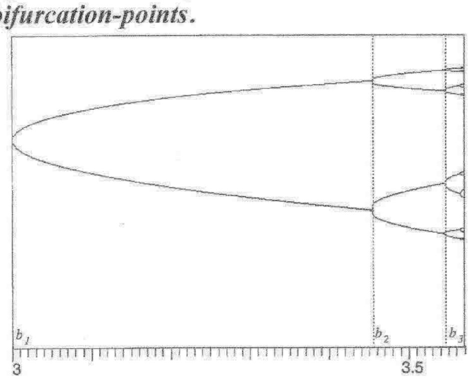
$B^1_{3.3.2}$ *At* $a = b_1 = 3$ *the fixed-point* p_a *for* $f_a(x)$ *looses its stability. Also for* $f_a^{\,2}(x)$

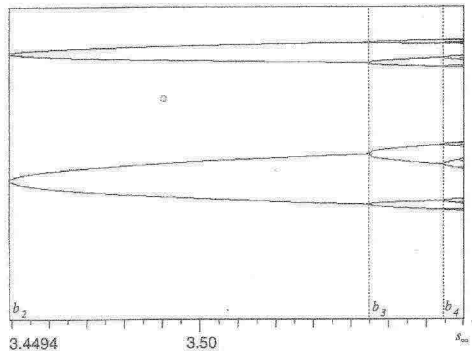
the fixed-point p_a *looses its stability, but here* 2 *new, additional fixed-points begin to exist (for* $a > 3$*),* $x_l(a)$ *and* $x_h(a)$*. Note that the portion of the graph which is enclosed be the dashed square looks like the graph of* $f_a(x)$ *for* $a = 1$*. The bifurcation at* $b_1 = 3$ *is called a period-doubling-bifucation, a fixed-point becomes unstable and gives birth to a* 2—*cycle.*

$B^1_{3.3.3}$ *Then at* $a = s_2 = 1+\sqrt{5}$ *one obtains the super-attractive-case for* $f_a^2(x)$*, the fixed-point* $x_l(a)$ *and the critical-point* $x_{crit} = ½$ *are identical. For* $f_a(x)$ *does this mean that iterating* x_{crit} *for* 2 *steps leads back to* x_{crit} *again. If one increases the parameter further to* $a = b_2 = 3.4495$*, the fixed-point* $x_l(a)$ *of* $f_a^2(x)$ *becomes unstable too. In other words, all the changes which one observes for* $f_a(x)$ *while varying* $1 \le a \le 3$ *can also be found for* $f_a^2(x)$ *in parameter-range a from* $a > 3$ *to* $a = b_2$*.*

$B^1_{3.3.4}$ *Now one can guess what happens if one increases* a *beyond* b_2*. One will find fixed-points of* $f_a^2(f_a^2(x))$ *which bifurcate-off from* $x_l(a)$ *and* $x_h(a)$*. This composition of* $f_a^2(x)$ *is nothing else but the* 4^{th}—*iterate of* $f_a(x)$*, i.e.,* $f_a(f_a(f_a(f_a(x))))$*, and will be written as* $f_a^4(x)$*. The new stable fixed-points of* f_a^4 *are equivalent to the birth of a stable-cycle of period—*4 *for* f_a*. If one increases the parameter* a *further, stability is lost again, which marks the birth of a period—*8*—cycle for* f_a *and so on. Again, the bifurcations are called period-doubling-bifurcations. The periods of the attractive-cycles are* 1, 2, 4, 8, 16, 32, … ,2^n *for* $n = 0, 1, 2, \ldots$.

$B^1_{3.4}$ *The process establishes* 2 *sequences of important parameters. The parameters* $s_1, s_2, s_3, ..$ *for which super-attractiveness of* $f_a, f_a^2, f_a^4, \ldots$ *is to be obtained. For these the critical-point* $x_{crit} = ½$ *is a fixed-point of* $f_a, f_a^2, f_a^4, ..$ *ect. The sequence* $b_1, b_2, b_3, \ldots$ *of parameter-values for which one has a period-doubling-bifurcation. It has been marked that* $s_1 = 2$, $s_2 = 3.236\ldots$, $b_1 = 3$ *and* $b_2 = 3.44949\ldots$*, where these sequences lead to the FEIGENBAUM-point* s_∞*. The sequence of period-doubling-bifurcations will now be discussed a bit further. It seems that the distance* d_k *between* 2 *successive-bifurcation-points* $d_k = b_{k+1} - b_k$, $k = 1,2,3,\ldots$ *decreases rather rapidly. This is also visible in the next figures where the period-doubling-tree has been enlarged to show some more of its bifurcation-points.*





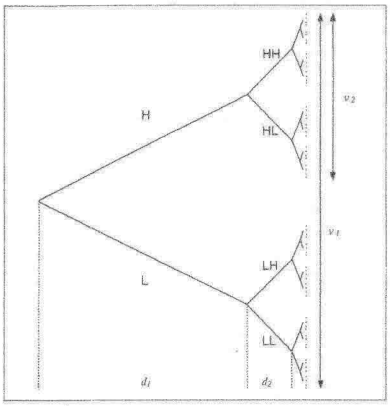
Period-doubling-tree. Positions $b_1, \ldots, b_4$ *indicate that* $(b_2-b_1)/(b_3-b_2) \approx (b_3-b_2)/(b_4-b_3)$.

$B^1_{3.4.1}$ *A* 1^{st} *guess would be that the decrease is geometric, i.e. that* $d_k/d_{k+1} = \delta$*. In this case bifurcations* b_k *would form a converging-sequence* $b_k = b_k + d_1(1 + 1/\delta + \ldots + 1/\delta^{k-2})$, k=2, 3, … *with limit* $b_\infty = \delta_1 + d_1\delta/(\delta-1)$*. Unfortunately things are not that easy. The decrease in* d_k *from bifurcation to bifurcation is not exactly geometric but only approximately geometric (i.e. the ratio* $\delta_k = d_k/d_{k+1}$ *converges with increasing* k*), computations by FEIGENBAUM resulted in* $\lim_{k\to\infty}(\delta_k) = \delta = 4.6692016091029\ldots$

B^2 Analysis at FEIGENBAUM—Point and its surroundind Neighbourhood.

Self-similarity-features in the final-state-diagram of the quadratic-iterator had previously been discussed in *connection with the figures $(B^1{}_{1.3.1})$. This kind of self-similarity is already contained in the period-doubling-tree, ranging from $a = 1$ to $a = s_\infty$. However, the self-similarity in either case is not strict; although the branches look like small copies of the whole tree, there are parts, like the stem of the tree, which clearly do not. Moreover, even the branches of the tree are not exact copies of the entire tree. Here one has to use the term self-similarity in a more intuitive-sense without being precise.*

$B^2{}_1$ *For the period-doubling-tree is to be observed that the sequence of differences d_k between the parameters of the bifurcation-points is not precise-geometric. In other words, when one makes close-ups as in figures $(B^1{}_{1.3.1})$, scaling-factor slightly changes among the close-ups approaching $\delta = 4.669....$ But this is only true for the scaling in horizontal-direction of parameter a. With respect to the vertical-direction one has to scale (in limit) with approximately 2.3.*



Schematic-representation of period-doubling-tree with scaling-factors $d_1/d_2 = 4.669$. The vertical-range of the complete tree is ν_1 and the range of upper main-branch ν_2. The ratio is $\nu_1/\nu_2 = 2.3$. It is to be noted that the leaves of the tree form a strictly self-similar CANTOR-set.

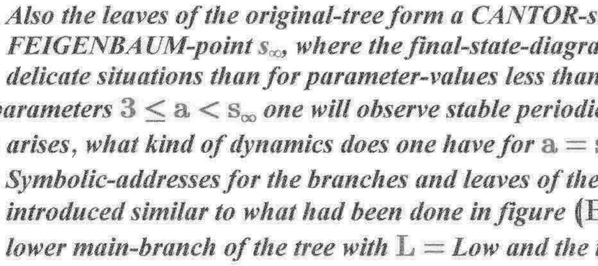
$B^2{}_{1.1}$ *When comparing this tree with the origininal bifurcation-tree, the non-linear-distortion becomes apparent. Here branches of the same stage are exactly the same. In the original period-doubling-tree, branches have different sizes. Nevertheless, one can identify corresponding branches.*

$B^2{}_{1.2}$ *Also the leaves of the original-tree form a CANTOR-set. This happens right at the FEIGENBAUM-point s_∞, where the final-state-diagram reaches a new stage with more delicate situations than for parameter-values less than s_∞.*

$B^2{}_2$ *For all parameters $3 \leq a < s_\infty$ one will observe stable periodic-orbits as final-states. Now the question arises, what kind of dynamics does one have for $a = s_\infty$?*

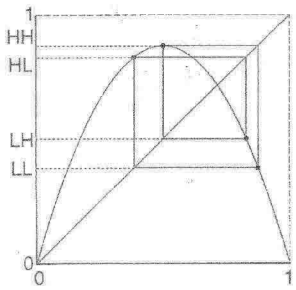
$B^2{}_{2.1}$ *Symbolic-addresses for the branches and leaves of the period-doubling-tree shall now be introduced similar to what had been done in figure $(B^2{}_1)$ already. First one labels the lower main-branch of the tree with $L =$ Low and the top-branch with $H =$ High. When the 2 branches split into 4 one labels the upper 2 parts with HH and HL and the lower 2 parts with LH and LL. This is the 2^{cd} stage of the addressing-hierarchy. The branches of the 3^{rd} stage will obtain the labels HHH, HHL, HLH, HLL, LHH, LHL, LLH, LLL. In general one will obtain 2^k—sub-branches labelled with k—letter-addresses for stage k. One can start now the dynamics of orbits on the CANTOR-set in terms of these addresses.*

$B^2{}_{2.2}$ *Now, one can start discussing the dynamics of orbits on the CANTOR-set at the FEIGENBAUM-point in terms of addresses. For all parameters a between $b_1 = 3$ and $b_2 = 3.4495$ there is a stable periodic-oscillation of period—2. This is the range of parameters where 1—letter-addresses are sufficient. Oscillations occur between H—banches and L—branches:*



Period—2—dynamics stage—1: H is mapped into L and L is mapped into H. Left diagram shows in form of CANTOR-construction corresponding mapping of addresses on the 2 axes.

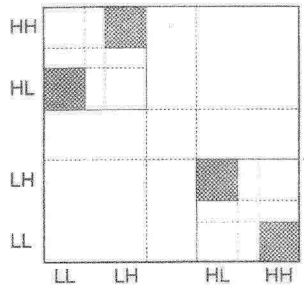
$B^2{}_{2.3}$ *Next stage $b_2 < a < b_3$ which represents the oscillation between 4 different values. 2—letter-addresses are needed:*



Period—4—dynamics stage—2: $HH \rightarrow LL \rightarrow HL \rightarrow LH \rightarrow HH$.

B^2_{3} *The same kind of arguments allows one to determine the address-sequences which describe orbits of period* $8, 16, 32, \ldots$ *each one bifurcating from a* 2^k*—cycle (with* $k = 2, 3, 4, \ldots$*). Infinite address-sequences show transformations from points in the CANTOR-set (at FEIGENBAUM-point) to other points in the CANTOR-set. It describes the symbolic dynamics of the quadratic-iterator at the FEIGENBAUM-point.*

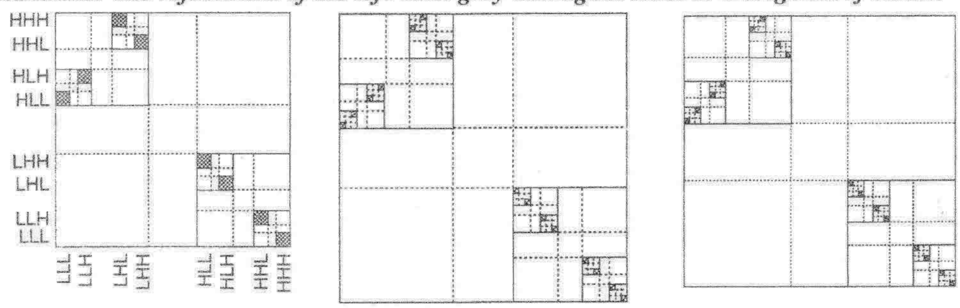
$B^2_{3.1}$ A_k *may be the set of all* k*—letter-addresses (formed by* H *and* L*) and* A_∞ *the set of all infinite addresses. The dynamics of a periodic-orbit with respect to point-addresses is described by a transformation* $f_k: A_k \rightarrow A_k$*; for instance the* 4*—cycle can be described by:* $f_2(HH) = LL$, $f_2(LL) = HL$, $f_2(HL) = LH$, $f_2(LH) = HH$. *This transformation can be visualized as a stage—*2*—transformation-diagram in next figure:*



The axes of this transformation-diagram are divided as in a typical CANTOR-set-construction. In limit one will obtain a diagram visualizing the transformation $f_\infty: A_\infty \rightarrow A_\infty$ *as transformation of points of a CANTOR-set.*

$B^2_{3.2}$ *For each stage, the axes of the transformation-diagrams are divided as in a typical CANTOR-set-construction. Thus, in limit one will obtain a diagram which visualizes the transformation* $f_\infty: A_\infty \rightarrow A_\infty$ *as a transformation of points of a CANTOR-set. The refinement of the lower-right grey box of the stage—*1*—diagram (of* $(B^2_{2.2})$ *left) shall now be discussed. In stage* 3 *it becomes apparent that here a diagonal of boxes begin to form. Thus, for the corresponding addresses starting with* H *one get the transformation-rule:* $f_\infty(HX_2X_3X_4\ldots) = L(X_2)^T(X_3)^T(X_4)^T\ldots$, *where* $X^T = \{(H \text{ if } X = L),(L \text{ if } X = H)\}$.

$B^2_{3.3}$ *Next, the refinement of the* 2 *upper boxes from stage—*2*—diagram* $(B^2_{3.1})$ *shall be examined. The refinement of the left-most grey-box again leads to a diagonal of boxes:*



Graphs from above show the CANTOR-diagams for period—8 *(left) for period—*16 *(middle) and period—*32 *(right).*

Again one can write-down a transformation-rule, now for the addresses which start with LL: $f_\infty(LLX_3X_4X_5\ldots) = HLX_3X_4X_5\ldots$. *The refinement of the top-most grey-box is not simple, but even more striking since here self-similarity is built-in. In fact, the refinement of this box shown in stage—*3 *is just a scaled-down copy of the stage—*1*—diagram, and in general at stage—*k *this is a scaled-down copy of the complete stage for the addresses which start with diagram at stage—*$(k-2)$. *In limit this leads to the self-similarity of the transformation-diagram for* f_∞*: the graph of* f_∞ *for the addresses which*

$B^2_{3.4}$ To compute the transformation of an address beginning with $LHX_3X_4X_5\ldots$ one first writes-down HH, then one drops the first 2 letters of the original address and applies f_∞ to the remaining letters, thus one obtains $f_\infty(X_3\ X_4\ X_5\ldots)$. Finally one appends the result of this evaluation to the initial letters HH.

start with LH is a scaled-down-copy of the complete graph.

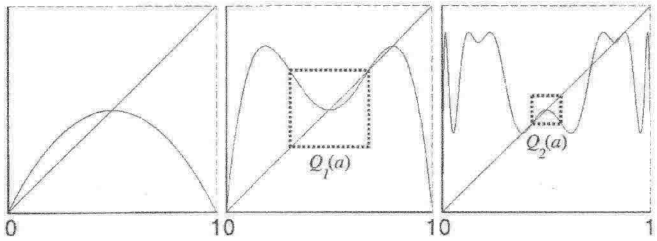
$B^2_{3.5}$ *At FEIGENBAUM-point the final-state of the iterator is given by an infinitely long non-periodic orbit in a CANTOR-set which gets arbitrarily-close to every point of the CANTOR-set.*

B^2_4 *Now it's shall be turned to the self-similarity-features related to the change of dynamics as the parameter a increases. For $1 \leq a \leq 3$ one has just 1 attractive-fixed-point at $p_a = (a-1/a)$ and all orbits belonging to initial values between 0 and 1 converge to this attractor.*

$B^2_{4.1}$ *The way initial-values are attracted changes at $a = s_1 = 2$ which marks the supper-attractive case. For parameters below s_1 initial-values are attracted directly monotonically while for parameters above s_1 the orbit spirals around the fixed-point.*

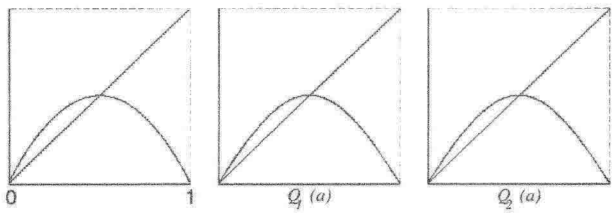
$B^2_{4.2}$ *At $a = b_1 = 3$ the fixed-point p_a becomes unstable, and an attractive$-2-$cycle is born. The old fixed-point p_a continues to exist, but now has become a repeller. The $2-$cycle undergoes all changes which one could see for the fixed-point. Especially to be noted that there is again the super-attractive case at $a = s_2$. The $2-$cycle finally becomes unstable at $a = b_2$, and everything is repeated for a $4-$cycle and so on.*

B^2_5 *At each period-doubling-bifurcation the dynamics of the iteration becomes more complex, though the mechanism is always the same. This is related to the similarity of the graph $f_a(x) = ax(1-x)$ (parabola) to sections of graphs of the iterated transformations $f_a^{\ 2}(x)$, $f_a^{\ 4}(x)$, $f_a^{\ 8}(x)$ and so on at higher parameters a.*



Singularity of parabola: $f_{s_1}(x)$ and parts of the graph $f_{s_2}^{\ 2}(x)$ and $f_{s_3}^{\ 4}(x)$ at super-attractive-parameters s_1, s_2 and s_3.

$B^2_{5.1}$ *One can make the similarity of the graphs even more apperent if one makes a close-up of the squares outlined in the figures above. One enlarges the squares such that they match the unit-square which encloses the whole graph.*

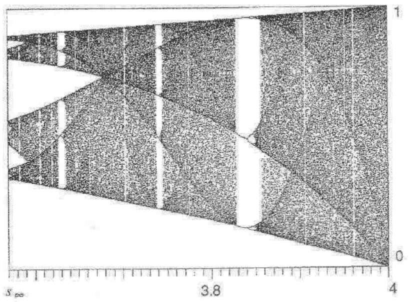


Enlargements: similarity between the graphs $f_{s_1}(x)$ and close-ups of $f^2_{\ s_2}(x)$ and $f^4_{\ s_3}(x)$ (dotted squares in figure above).

$B^2_{5.1.1}$ *Only the left graph is a parabola, the 2 others represent $4^{th}-$ and $8^{th}-$order polynomials. It should be noted that when magnifying $f^2s_2(x)$ the graph has become also been flipped horizontally and vertically. If one does this for all values s_x and the corresponding f_{s_x} one will obtain a sequence of close-ups settling-down on the graph of a new function f_∞.*

C <u>From Order to Chaos, the Mirror–Image.</u>

It concerns the second part of the final-state-diagram, the parameter-range between the FEIGENBAUM-point $a = s_\infty$ and $a = 4$. This part of the diagram is called the chaotic mirror-image of periodic-doubling -tree. There are features of period-doubling of the 1^{st} part of the final.-state-diagram (though in reverse order). However, while in 1^{st} part of the diagram for each parameter a one exactly can predict what the dynamics are, here (where chaos reigns) it becomes more complicated.



Final-state-diagram of the quadratic-iterator: Part—2.

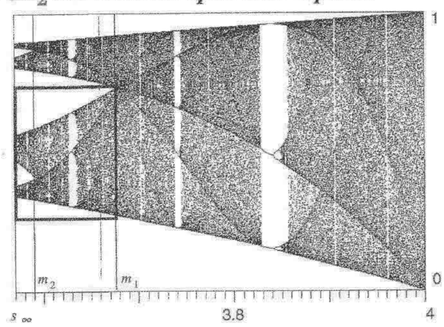
C_1 *The situation for parameter-value $a = 4$ had already been investigated previously in chapter (A). This is region where the graph of $f_a(x) = ax(1-x)$ spans the unit-square and chaos can be observed in the whole unit-interval. In final-state-diagram this is represented by the randon-looking distribution of dots which vertically span the range between 0 and 1.*

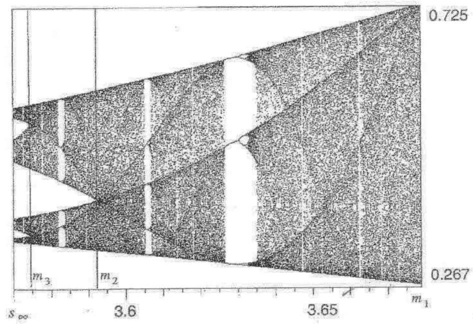
C_2 *This kind of chaotic-dynamics is not present for all parameters in the 2^{cd}—part of the diagram.*

$C_{2.1}$ *Chaos seems to be interrupted by windows of order where the final-state again collapses to only a few points, corresponding to attractive-periodic orbits.*

$C_{2.2}$ *Additionally there seems to be an underlying-structure of bands resulting from points not being uniformly-distributed in each vertical-line. Points seem to condense at certain lines which border bands that encapsulate the chaotic-dynamics.*

$C_{2.3}$ *For $a = 4$ there is only 1 band spanning the whole unit-interval. As parameter a decreases this band slowly narrows. Then at a parameter labelled with m_1, it splits-into 2 parts; and at $a = m_2$ the 2 bands split-into 4 parts:*

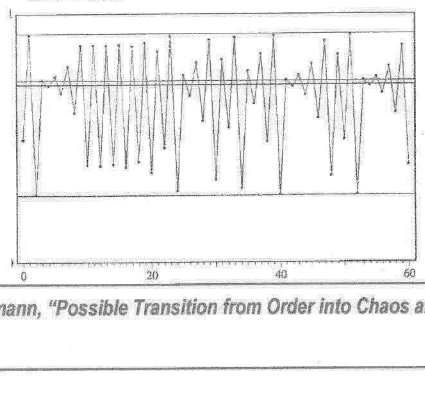


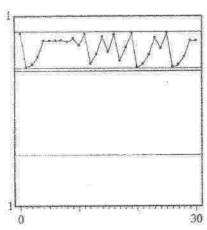


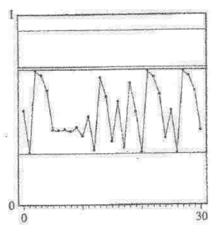
$C_{2.4}$ *If one magnifies the diagram between the parameters s_∞ and m_1 at the window shown in figure ($C_{2.3}$), more band-splitting-points can be found. In fact, there is an infinite, decreasing sequence of parameters m_1, m_2, m_3, ... at which one observes the splitting into 2, 4, 8, ... (in general 2^k) bands. This can be interpreted as another consequence of the self-similarity of final-state-diagram at the FEIGENBAUM-point, because this sequence leads exactly to the FEIGENBAUM-point $s_\infty = m_\infty$.*

$C_{2.4.1}$ *As shown in [Ref.2] the band-merging-points obey a growth-law $d_k = m_{k+1} - m_k$ similar to what had already been found for the sequence of the super-attractive-parameters and parameters of the period-doubling-bifurcation-points in the 1^{st} part of final-state-diagram. Furthermore, it could be confirmed that the ratio $dk/dk{+}1$ converges to the unversial-constant $\delta = 4.669...$ as the number k decreases.*

C_3 *What kind of change lies hidden behind the band-splitting (or band-merging)? The following graphs show a typical time-series of an orbit f_a for the parameter $a = 3.67$, which is slightly below $a = m_1 = 3.6785$.*

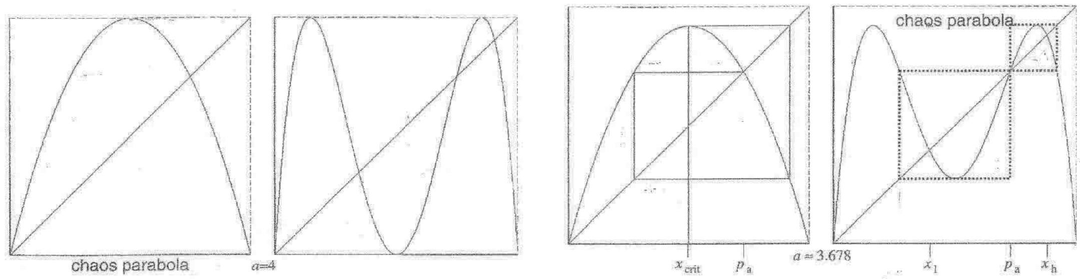






$C_{3.1}$ *Although the dynamics behave chaotically, it oscillates from step to step back and forth between 2 distinct bands (left graph). In other words, if one looks at the dynamics of $f_a^{\,2}$ one sees points only moving chaotically either in the upper or in the lower band (right graph). In summary, the 1^{st} band-splitting is also a kind of period-doubling-bifurcation.*

C_4 *Again f_a and $f_a^{\,2}$ shall be compared by using graphical-iteration. Next figure shows the result for $a = 4$ (left) and $a = m_1 = 3.6785$ (right):*

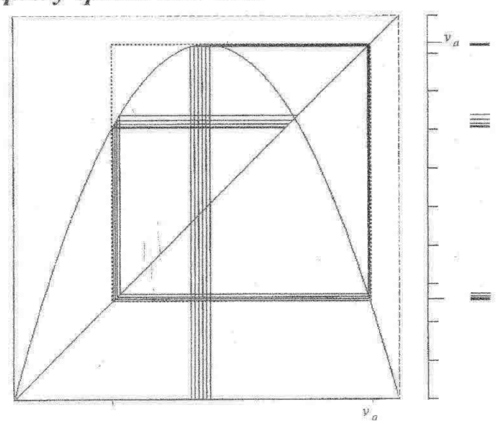


$C_{4.1}$ *The parabola in the left-left graph $4x(1-x)$ is called generic-parabola. Generic-parabolas are characterized by the fact that their graphs precisely fits into a square which has one of its diagonals on the bisector of the $(x,y)-$coordinate-system.*

$C_{4.2}$ *It should be noted that for $a = m_1 = 3.6785$ a generic-parabola in $f_a^{\,2}$ could also be found (the right-right graph above). However, this is not quite correct, because $f_a^{\,2}$ is not really a parabola, but rather a $4^{th}-$degree polynomial having a graph that only looks parabolic in the outlined region enclosed by the dashed square.*

$C_{4.2.1}$ *Once the iteration of $f_{m_1}^{\,2}$ has led into this region it is trapped, and it is expect to see chaotic-behaviour which spans the interval $(1/a, (a-1)/a)$. This corresponds to the lower-band visible in the final-state-diagram right at $a = m_1$. The upper-band corresponds to the part of the graph of $f_a^{\,2}$ enclosed by the small dotted square. Also in this region the iteration is trapped and spans the interval $((a-1)/a, a/4)$.*

$C_{4.3}$ *Now it can be guessed what the situation for all other parameters $a = m_e$ will be. In all these case one will find a generic-parabola (i.e. in the graphs of $f_{m_2}^{\,4}, f_{m_3}^{\,8}, \ldots, f_{m_e}^{\,2^i} \ldots, (i = e = 2, 3, 4, \ldots)$. Of course this explains what one sees at the special-parameters $a = m_e$. On the other hand it seems possible to trace these bands in between. Somehow they shine through the whole 2^{cd} part of the final-state-diagram: there is a mechanism behind this observation. Next figure shows the graphical-iteration of a few initial values which had been chosen to be equally-spaced near 0.5:*



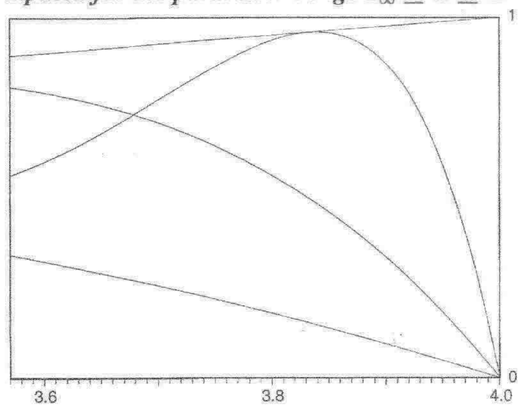
Initial-values near 0.5: graphical-iteration of some equally-spaced initial-values near 0.5. The first 3 iterates condense at $v_a = f_a(0.5)$, $f_a(v_a)$ and $f_a^{\,2}(v_a)$.

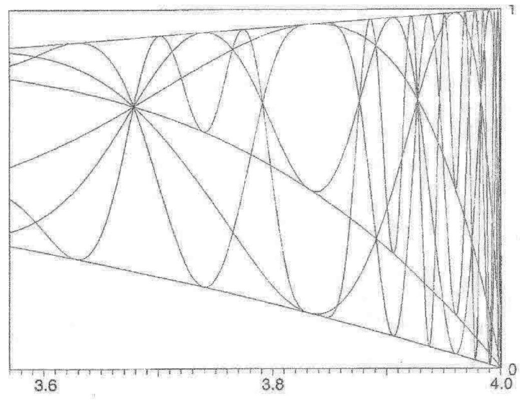
$C_{4.3.1}$ *For each initial-value 3 iterations have been performed and the corresponding outcome has been drawn on the right side of the graph. It can be noted that the iterations never leaves the outlined square (i.e. the points of the final-state-diagram have to be within the interval between the critical-value $v_a = f_a(0.5)$ and $f_a(v_a)$). Furthermore, is to be observed that the values of the iteration condense a bit at these points. This happens because the parabola has its vertex at 0.5 which squeezes nearby orbits together. In the histogram, one thus can expect a spike at $v_a = f_a(0.5)$. Moreover, there should be another spike at the next iterate $f_a(v_a) = f_a^{\,2}(0.5)$, and also at the following one, $f_a^{\,2}(v_a)$, and so on. For $a = 4$, however,*

$v_a = 1$ *and all further iterates are* 0. *Thus, it is reasonable to expect only the* 2 *spikes at* 0 *and* 1. *For* $a = m_1$, *on the other hand, one has* $f_a^{\ 2}(v_a) = p_a = (a-1)/a$, *the fixed-point of* f_a *and all futher iterates are the same. Therefore, there should be* 3 *spikes at* v_a, $f_a(v_a)$ *and* $f_a^{\ 2}(v_a) = p_a$.

$C_{4.3.2}$ *In summary, this leads to the conjecture that what one sees shinning through as lines of condensation in figures* (C) *and* $(C_{2.3})$ *could be the trace of the iterates of the critical-value* v_a.

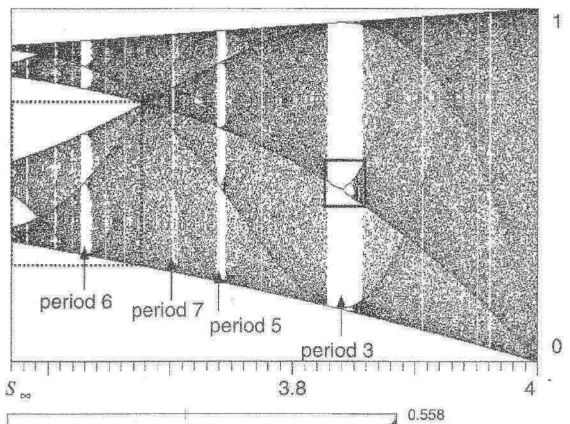
C_5 *Next figure shows an experiment which confirms the conjecture. The first* 8 *iterates of* 0.5 *have been computed for the parameter-range* $s_\infty \leq a \leq 4$.



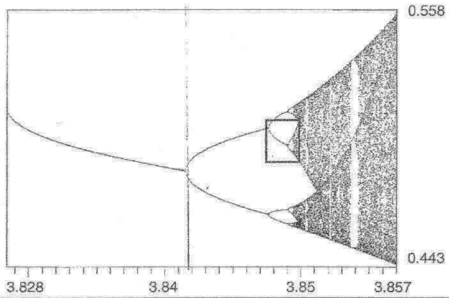


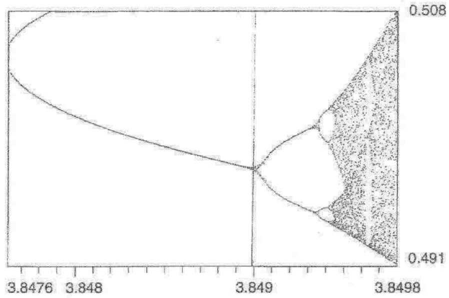
$C_{5.1}$ *The left plot shows the first* 4 *iterares (i.e.,* v_a *to* $f_a^{\ 3}(v_a)$*). These lines apparently correspond to the main-bands which shine through the final-stare-diagram in figures* (C) *and* $(C_{2.3})$. *The right plot shows all* 8 *iterates exhibiting more of the relation to finer band-structures. Although these critical-lines (i.e., the iteration* $f_a^{\ k}(v_a)$ *of the critical-value* v_a*) explain the perception of the band-structure in the final-state-diagram, this dores not mean that the complete lines as shown are part of the final-state. The final-states are bound by these lines, but one can see that from a certain parameter-value (about* $a = 3.82843$*) the final-states consist of a stable attracting periodic-cycle of which only* 1 *point is shown in next bow-up.*

C_6 *In fact, this blow-up shows a small part of the white windows which interrupt the chaotic-region of the final-state-diagram. There are an infinite number of such windows, which all correspond to stable periodic-cycles. The one between* $a \approx 3.828$ *and* $a \approx 3.857$ *is the most prominent one; it is the so-called period*$-3-$*window. Not only this window has been indicated in next figure but also the windows of period*-5, *period*-7 *and period*-6:



Windows of periodicity: Starting with the period$-3-$*window at* $a \approx 1+\sqrt{8} = 3.8284$ *down to period*$-6-$*window.*

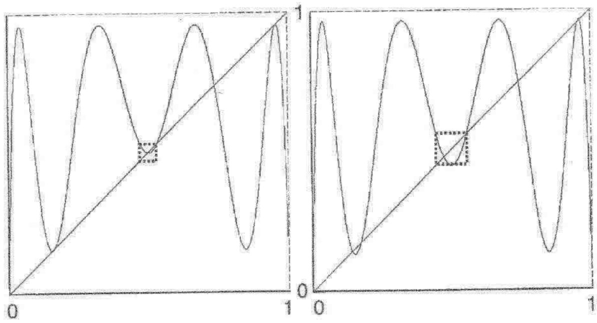




$C_{6.1}$ *The period-3—window shall first be examined a bit further. The bottom-part of the above figure shows 2 successive close-ups of the part which is marked by the black-frame. Again one will discover self-similarity. One sees smaller and smaller copies of the whole final-state-diagram. And indeed, one can find the complete scenario of the period-doubling-chaos and band-splitting again, however on a much smaller scale. And again, the mechanisms behind this are the same as before. There is only one important difference, instead of $f_a(x)$, everything is based now on $f_a^{\,3}(x)$.*

$C_{6.2}$ *Period-doubling begins when the 3 fixed-points of $f_a^{\,3}(x)$ loose their stability and 6 new fixed-points of $f_a^{\,6}(x)$ are born (i.e., for f_a one has a 6—cycle). As a increases further, each of the stable-fixed-points of $f_a^{\,6}$ will undergo a period-doubling-bifurcation (i.e., for f_a one will obtain attracting-cycles of lenghth $3{\cdot}2^2$), and so on. The relative-length of the interval for which these stable-cycles exist will be governed once more by universal-number $\delta =$ 4.669.... At the end of this period-doubling-scenario, near $a = 3.8415...$, there will again be a transition to chaotic behaviour very much like that at the FEIGENBAUM-point s_∞.*

$C_{6.3}$ *By taking a close look at some graphs of $f_a^{\,3}$, the following figure for the super-attractive-case can be drawn:*



Graph of the 3rd iterate $f_a^{\,3}$ of f_a. Left super-attractive-case. Right fully developed chaos.

$C_{6.3.1}$ *At the centre again one observes a segment which looks like a small parabola. Indeed, the changes of this small part are responsible for the complete scenario of the period-doubling which ends at $a \approx 3.857$ in fully developed chaos as it is shown in the centre-part of figure (C_6). The corresponding graph of $f_a^{\,3}$ is shown on the right hand side of figure $(C_{6,3})$. And indeed, at the centre a generic-parabola is visible.*

$C_{6.3.2}$ *If one magnifies any of the other periodic-windows, one will make exactly the same findings, but everything will be on an even smaller scale. In fact, between the period—3—window and the band-merging-point m_1 there are an infinite number of windows for all odd integers 3, 5, 7, 9, 11, ... which can be found in reversed order. But as the period increases the size of these windows rapidly decreases.*

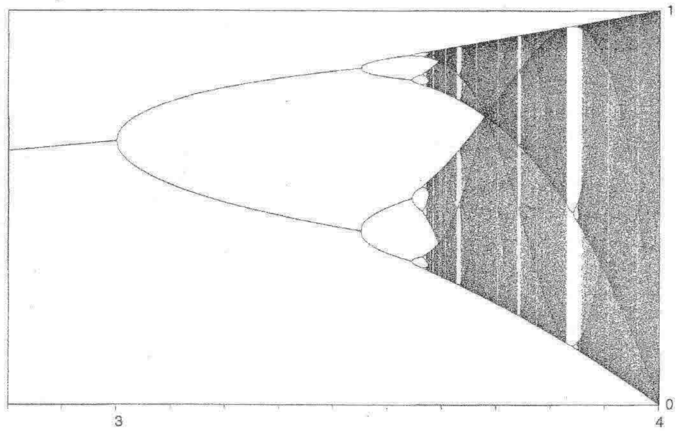
$C_{6.4}$ *At left part of figure $(C_6(\text{top}))$ enclosed in a dashed rectangle, one can find everything once again, but now with a double period. In other words, it starts with a period—6—window, then one finds a period—10—window, ect. In general, in this step one will find windows of period—$(2{\cdot}k)$, for all odd integers $k \geq 3$.*

$C_{6.5}$ *All windows mentioned under (C_6) are contained as parts in bands resulting from splittings at points $a = m_1, m_2, m_3, ...$ But only these latter bands are crucial in a discussion about a continous, geometrical transition between 1st part of the final-state-diagram and its mirrored image at FEIGENBAUM-point s_∞. The various windows contained in these bands will only have an influence on the dynamics within the bands.*

C_7 *The decreasing sequence of band-splitting-points $m_1, m_2, m_3, ...$ (mentioned in $(C_{2.4})$ and $(C_{2.4.1})$ above) and the increasing sequence of bifucation-points $b_1, b_2, b_3, ...$ (described in $(B^1_{3.4})$ and $(B^1_{3.4.1})$ during 1st part of the final-state-diagram) converge in an identical manner from both sides towards the FEIGENBAUM-point s_∞. From this fact one may guess that the CANTOR-set at s_∞ will enable geometrical continuity $(a \leq s_\infty) \rightleftarrows (s_\infty \geq a)$.*

$C_{7.1}$ *The band organating from the chaos-region of the quadratic-iterator at $a = 4$ splits into 2 sub-bands at $a = m_1$. The upper one of these sub-bands may be named H (for high) and the lower one may be called L (for low). Following this naming-convention further on, newly generated bands at $a = m_2$ will have to be designated then by HH, HL, LH and LL (from* *top to bottom). As one finally symbolically addressed all bands due to band-splitting-points* m_k *with* $k = 1, 2, 3, \ldots$ *this way, the sequence* m_k *will lead one exactly to the limit* $m_\infty = s_\infty$. *Moreover in* $[Ref.2]$ *it could be verified that the ratio* d_k/d_{k+1} *converges to the universal-constant* $\delta = 4.669\ldots$ *as the number* k *increaes. Thus, in horizontal direction one obtains same conditions as under* $(B^2{}_1)$ *and* $(B^2{}_{2.1})$.

$C_{7.2}$ *As shown in next graph, there will be no vertical discontinuity at* s_∞ *for the parameter-variation* $(a \leq s_\infty) \rightleftarrows (s_\infty \geq a)$. *This means, also the scaling with respect to the vertical direction one has to scale (in limit) with approximately* 2.3, *similar as under* $(B^2{}_1)$ *and* $(B^2{}_{2.1})$.



Final-state-diagram for the quadratic-iterator and parameter a *between* 2.8 *and* 4.

$C_{7.3}$ *Thus, geometrically speaking it doesn't matter which direction* $(a \leq s_\infty) \rightleftarrows (s_\infty \geq a)$ *one will cross the FEIGENBAUM-point. But the dynamics is highly dependend on the cross-over-direction, because in the* 1^{st} *part of the final-states are only effected by period-doublings, while in the mirror-image-part of the diagram complicated interactions between various band-types take place.*

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