Symmetries of Natural Numbers

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Abstract. In present paper we prove an original theorem on the natural numbers. From this the theorem emerges a set of symmetries of the natural numbers which puts Number Theory on a new footing. One of the consequences of the theorem, but not the most important one, is the odd number factorization algorithm.

1. Introduction

The results of this paper are based on the following representation of odd numbers. If Π is an odd number and [x] is the integer part of x, then Π can be uniquely written in the form

$\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i$, $\nu \in \mathbb{N} \cup \{-1\}$,

where $\nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right]$, $\beta_i = \pm 1, i = 0,1,2,...,\nu - 1$ (Theorem 1). It is easy to show that in such a representation, the powers of 2 cannot be replaced by powers of any other integer. The main difference between this result and the known arithmetic systems (binary, decimal etc.) is that the coefficients of the linear combination can take negative values. This feature will allow us to define "the conjugate" (Definition 1), "the complementary" (Definition 2), "the L/R symmetry" (Definition 3), "the transpose" (Definition 4) and "the kernel" (Definition 9) of an odd number.

The mathematical objects that concern us primarily, namely the "octets of odd numbers", are defined through a combination of conjugates and transposes of odd numbers (Equation (49)). A second mathematical object, arising from symmetries, is the "chains" of odd numbers (Definition 10). Based on these concepts, we obtain a classification of odd numbers, and an algorithm for factoring composite odd numbers.

2. Odd numbers as linear combinations of consecutive powers of 2

In this Section, we derive the representation mentioned in the Introduction.

Theorem 1. *Let* Π *be an odd number, and* $\left[\frac{\ln \Pi}{\ln 2}\right]$ *is the integer part of* $\frac{\ln \Pi}{\ln 2} \in \mathbb{R}$. *Then* Π *can be uniquely written in the form*

$$\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i , \tag{1}$$

where $\nu \in \mathbb{N} \cup \{-1\}$, $\nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right]$, $\beta_i = \pm 1, i = 0,1,2,...,\nu - 1$.

Proof. If $\Pi = 1$ we have $\nu + 1 = \left[\frac{\ln 1}{\ln 2}\right] = 0 \Rightarrow \nu = -1$ and from Equation (1) we obtain

$\Pi = 1 \in \Omega_{-1} = \left[2^0, 2^1\right] = [1, 2]$.

If $\Pi = 3$ we have $\nu + 1 = \left[\frac{\ln 3}{\ln 2}\right] \Rightarrow \nu = 0$ and from Equation (1) we obtain

$\Pi = 3 = 2^1 + 2^0 \in \Omega_0 = \left[2^1, 2^2\right] = \left[2, 2^2\right]$.

We now examine the case where $\nu \in \mathbb{N}^* = \{1, 2, 3, \ldots\}$. The lowest odd value of Π in (1) is

$$\Pi_{\min} = \Pi(\nu) = 2^{\nu+1} + 2^{\nu} - 2^{\nu-1} - 2^{\nu-2} - \ldots - 2^1 - 1 = 2^{\nu+1} + 1. \tag{2}$$

The largest odd value of Π in (1) is

$$\Pi_{\max} = \Pi(\nu) = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + 2^{\nu-2} + \ldots + 2^1 + 1 = 2^{\nu+2} - 1. \tag{3}$$

From Equations (2) and (3) we get that for any odd $\Pi = \Pi(\nu, \beta_i)$ in Equation (1), the following inequality holds,

$$\Pi_{\min} = 2^{\nu+1} + 1 \le \Pi(\nu, \beta_i) \le 2^{\nu+2} - 1 = \Pi_{\max}. \tag{4}$$

The number $N(\Pi(\nu, \beta_i))$ of odd numbers in the closed interval $\left[2^{\nu+1} + 1, 2^{\nu+2} - 1\right]$ is

$$N(\Pi(\nu, \beta_i)) = \frac{\Pi_{\max} - \Pi_{\min}}{2} + 1 = \frac{\left(2^{\nu+2} - 1\right) - \left(2^{\nu+1} + 1\right)}{2} + 1 = 2^{\nu}. \tag{5}$$

The integers $\beta_i, i = 0, 1, 2, \ldots, \nu - 1$ in (1) can only take two values, namely $\beta_i = \pm 1$, and thus (1) gives exactly $2^{\nu} = N(\Pi(\nu, \beta_i))$ odd numbers. Considering also Equation (5), we conclude that for every $\nu \in \mathbb{N}^*$ Equation (1) gives all odd numbers in the interval $\Omega_{\nu} = \left[2^{\nu+1}, 2^{\nu+2}\right]$.

From Inequality (4), we obtain

$$2^{\nu+1} + 1 \le \Pi \le 2^{\nu+2} - 1$$

so we have $2^{\nu+1} < \Pi < 2^{\nu+2}$. Thus

$$(\nu + 1)\ln 2 < \ln \Pi < (\nu + 2)\ln 2$$

from which we get

$$\frac{\ln \Pi}{\ln 2} - 1 < \nu + 1 < \frac{\ln \Pi}{\ln 2}$$

and finally

$$\nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right]. \tag{6}$$

We prove now that every odd number $\Pi \ne 1$ can be uniquely written in the form of Equation (1). We write the odd Π as

$$\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i, \tag{7}$$

where $\nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right]$ and $\beta_i = \pm 1, i = 0, 1, 2, \ldots, \nu - 1$, and

$$\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \gamma_i 2^i, \tag{8}$$

where $\nu + 1 = \left[\frac{\ln \Pi}{\ln 2} \right]$ and $\gamma_i = \pm 1, i = 0,1,2,...,\nu - 1$.

From Equations (7) and (8) we get

$$(\beta_0 - \gamma_0) \cdot 2^0 + (\beta_1 - \gamma_1) \cdot 2^1 + (\beta_2 - \gamma_2) \cdot 2^2 + ... + (\beta_{\nu-1} - \gamma_{\nu-1}) \cdot 2^{\nu-1} = 0, \tag{9}$$

where

$\beta_i = \pm 1, i = 0,1,2,...,\nu - 1$, $i \in \{0,1,2,...,\nu - 1\}$ and $\gamma_i = \pm 1, i = 0,1,2,...,\nu - 1$, $i \in \{0,1,2,...,\nu - 1\}$.

If in Equation (9) there are $i \in \{0,1,2,...,\nu - 1\}$ such that $\beta_i \neq \gamma_i$, and let k is the smallest of them, then dividing by 2^{k+1}, we get an odd number equal to an even number. So, it follows that $\beta_i = \gamma_i \forall i = 0,1,2,...,\nu - 1$. □

In order to write an odd number $\Pi \neq 1,3$ in the form of Equation (1) we initially define the $\nu \in \mathbb{N}^*$ from Equation (6). Then, we calculate the sum

$2^{\nu+1} + 2^{\nu}$.

If $2^{\nu+1} + 2^{\nu} < \Pi$ we add $2^{\nu-1}$, whereas if $2^{\nu+1} + 2^{\nu} > \Pi$ then we subtract it. By repeating the process exactly ν times we write the odd number Π in the form of Equation (1). The number ν of steps needed in order to write the odd number Π in the form of Equation (1) is extremely low compared to the magnitude of the odd number Π, as derived from Inequality (4).

Example 1. For the odd number $\Pi = 23$ we obtain from Equation (6)

$$\nu + 1 = \left[\frac{\ln 23}{\ln 2} \right] \Rightarrow \nu = 3.$$

Then, we have

$2^{\nu+1} + 2^{\nu} = 2^4 + 2^3 = 24 > 23$ (thus 2^2 is subtracted)

$2^4 + 2^3 - 2^2 = 20 < 23$ (thus 2^1 is added)

$2^4 + 2^3 - 2^2 + 2^1 = 22 < 23$ (thus $2^0 = 1$ is added)

$2^4 + 2^3 - 2^2 + 2^1 + 1 = 23$.

Fermat numbers F_s can be written directly in the form of Equation (1), since they are of the form $\Pi_{\min}$,

$$F_s = 2^{2^s} + 1 = \Pi_{\min}(2^s - 1) = 2^{2^s} + 2^{2^s - 1} - 2^{2^s - 2} - 2^{2^s - 3} - ... - 2^1 - 1, \tag{10}$$

where $s \in \mathbb{N}$. Similarly, the Mersenne numbers M_p can be written directly in the form of Equation (1), since they are of the form $\Pi_{\max}$,

$$M_p = 2^p - 1 = \Pi_{\max}(p - 2) = 2^{p-1} + 2^{p-2} + 2^{p-3} + ... + 2^1 + 1, \tag{11}$$

where p is prime.

We now give the following definition.

Definition 1. Let Π be an odd number greater than 1, and consider the representation of Π as described in Theorem 1. Then *the conjugate* Π^* *of* Π is

$$\Pi^* = \Pi^*(\nu, \gamma_j) = 2^{\nu+1} + 2^{\nu} + \sum_{j=0}^{\nu-1} \gamma_j 2^j, \tag{12}$$

where $\gamma_j = -\beta_j$, $j = 0,1,2,...,\nu-1$.

Proposition 1. *If Π is odd, then the following hold.*

1. *We have that*

$$\left(\Pi^*\right)^* = \Pi. \tag{13}$$

2. *We have that*

$$\Pi^* = 3\cdot 2^{\nu+1} - \Pi. \tag{14}$$

3. *We have that Π is divisible by 3 if and only if Π^* is divisible by 3.*
4. *Two conjugate odd numbers cannot have common factors greater than 3.*
5. *Conjugates Π and Π^* are equidistant from the midpoint $3\cdot 2^{\nu}$ of the interval $\Omega_\nu = \left[2^{\nu+1}, 2^{\nu+2}\right]$.*

Proof. 1. This is an immediate consequence of Definition 1.

2. From Equations (1) and (12) we get

$$\Pi + \Pi^* = \left(2^{\nu+1} + 2^{\nu}\right) + \left(2^{\nu+1} + 2^{\nu}\right)$$

or equivalently

$$\Pi + \Pi^* = 3\cdot 2^{\nu+1}.$$

3. If the odd Π is divisible by 3 then it is written in the form $\Pi = 3x, x = odd$, and from Equation (14) we get $3x + \Pi^* = 3\cdot 2^{\nu+1}$, that is, $\Pi^* = 3\left(2^{\nu+1} - x\right)$. The converse can be proved similarly.

4. Let $\Pi = xy, \Pi^* = xz$, x, y, z are odd numbers. Then, Equation (14) implies that $x\left(y+z\right) = 3\cdot 2^{\nu+1}$, and consequently $x = 3$.

5. From equation (14) we obtain

$$\Pi - 3\cdot 2^{\nu} = 3\cdot 2^{\nu} - \Pi^*,$$

and thus

$$\left|\Pi - 3\cdot 2^{\nu}\right| = \left|3\cdot 2^{\nu} - \Pi^*\right|. \qquad \square$$

Proposition 1 implies that 3 is the only odd number which is equal to its conjugate; $3^* = 3\cdot 2^{0+1} - 3 = 3$. For the $\Pi = 1$, we define

$$1^* = 1. \tag{15}$$

It is easily proven that Theorem 1 is also valid for even numbers that are not powers of 2. In order to write an even number E that is not a power of 2 in the form of Equation (1), initially it is consecutively divided by 2 and it takes of the form of equation

$$E = 2^l \cdot \Pi, \tag{16}$$

where Π odd number, $\Pi \neq 1$, $l \in \mathbb{N}^*$. Then, we express Π as in Equation (1).

Example 2. By consecutively dividing the even number 368 by 2 we obtain

$$E = 368 = 2^4 \cdot 23.$$

Then, we write the odd number Π =23 in the form of Equation (1),

$$23 = 2^4 + 2^3 - 2^2 + 2^1 + 1$$

and we get

$$368 = 2^4 \cdot \left(2^4 + 2^3 - 2^2 + 2^1 + 1\right) = 2^8 + 2^7 - 2^6 + 2^5 + 2^4.$$

This equation gives the unique way in which the even number 368 can be written in the form of Equation (1). For even numbers the lowest power of two in Equation (1) is different from $1 = 2^0$.

The middle $3 \cdot 2^{\nu}$ of the interval $\Omega_{\nu} = \left[2^{\nu+1}, 2^{\nu+2}\right]$ is a center of symmetry of two conjugates numbers of the interval Ω_{ν}. Therefore, also for the even numbers E of Ω_{ν} the conjugate E^* is defined, and the Equation $E + E^* = 3 \cdot 2^{\nu+1}$ applies.

For the odd numbers of the interval Ω_{ν} the following applies.

Corollary 1.

If $\Pi \in \left[2^{\nu+1}, 3 \cdot 2^{\nu}\right] \subseteq \Omega_{\nu}$ *then there exists a unique* $\Pi' \in \left[3 \cdot 2^{\nu}, 2^{\nu+2}\right] \subseteq \Omega_{\nu}$ *such that* $\Pi' - \Pi = 2^{\nu}$.

Definition 2.

We define the odd numbers Π and Π' as "*complementary*".

For complementary numbers the following holds.

Corollary 2.

1. *We have that*

$\left(\Pi'\right)' = \Pi$.

2. *The complement and conjugate of an odd number* Π *are commutative,*

$\left(\Pi^*\right)' = \left(\Pi'\right)^*$.

3. The L/R symmetry

We now give the following definition.

Definition 3.

1. The odd number Π has *Left-symmetry* L when there exists an index L such that

$$\beta_{L-1} = +1, \beta_L = \beta_{L-2} = \ldots = \beta_1 = \beta_0 = -1, \quad (17)$$

where $L \in \{2, 3, 4, \ldots, \nu - 1\}$.

2. The odd number Π has *Right-symmetry* R when there exists an index R such that

$$\beta_{R-1} = -1, \beta_R = \beta_{R-2} = \ldots = \beta_1 = \beta_0 = +1, \quad (18)$$

where $R \in \{2, 3, 4, \ldots, \nu - 1\}$.

Sometimes, we will use the notation L(Π) and R(Π) for L and R, respectively, above.

Example 3. The prime number

Q=568630647535356955169033410940867804839360742060818433

is a factor of $F_{12} = 2^{4096} + 1$. From the Equation (6) we have $\nu + 1 = 178$, and then from Equation (1) we have

$$Q = 2^{178} + 2^{177} - 2^{176} + 2^{175} + 2^{174} + 2^{173} + 2^{172} - 2^{171} + 2^{170} + 2^{169} + 2^{168} + 2^{167} + 2^{166}$$
$$+2^{165} - 2^{164} + 2^{163} - 2^{162} - 2^{161} - 2^{160} - 2^{159} + 2^{158} + 2^{157} + 2^{156} - 2^{155} - 2^{154} - 2^{153} - 2^{152}$$
$$-2^{151} + 2^{150} - 2^{149} + 2^{148} - 2^{147} - 2^{146} + 2^{145} - 2^{144} + 2^{143} - 2^{142} - 2^{141} - 2^{140} + 2^{139} + 2^{138}$$
$$-2^{137} - 2^{136} + 2^{135} - 2^{134} - 2^{133} + 2^{132} - 2^{131} + 2^{130} - 2^{129} + 2^{128} - 2^{127} + 2^{126} - 2^{125} - 2^{124}$$
$$-2^{123} - 2^{122} - 2^{121} + 2^{120} - 2^{119} + 2^{118} - 2^{117} + 2^{116} - 2^{115} + 2^{114} - 2^{113} - 2^{112} - 2^{111} - 2^{110}$$
$$-2^{109} - 2^{108} + 2^{107} - 2^{106} + 2^{105} - 2^{104} + 2^{103} - 2^{102} + 2^{101} - 2^{100} + 2^{99} + 2^{98} - 2^{97} + 2^{96} - 2^{95}$$
$$-2^{94} + 2^{93} - 2^{92} + 2^{91} + 2^{90} - 2^{89} + 2^{88} - 2^{87} + 2^{86} + 2^{85} + 2^{84} - 2^{83} + 2^{82} - 2^{81} + 2^{80} + 2^{79}$$
$$-2^{78} - 2^{77} - 2^{76} - 2^{75} + 2^{74} + 2^{73} - 2^{72} - 2^{71} - 2^{70} + 2^{69} + 2^{68} + 2^{67} + 2^{66} + 2^{65} + 2^{64} - 2^{63}$$
$$-2^{62} + 2^{61} - 2^{60} - 2^{59} - 2^{58} - 2^{57} - 2^{56} + 2^{55} - 2^{54} - 2^{53} - 2^{52} - 2^{51} - 2^{50} - 2^{49} + 2^{48} + 2^{47}$$
$$-2^{46} + 2^{45} + 2^{44} + 2^{43} + 2^{42} - 2^{41} - 2^{40} + 2^{39} - 2^{38} - 2^{37} - 2^{36} + 2^{35} - 2^{34} - 2^{33} + 2^{32} + 2^{31}$$
$$-2^{30} + 2^{29} + 2^{28} + 2^{27} + 2^{26} + 2^{25} + 2^{24} - 2^{23} + 2^{22} + 2^{21} + 2^{20} - 2^{19} - 2^{18} - 2^{17} - 2^{16} + 2^{15}$$
$$+2^{14} - 2^{13} - 2^{12} - 2^{11} - 2^{10} - 2^{9} - 2^{8} - 2^{7} - 2^{6} - 2^{5} - 2^{4} - 2^{3} - 2^{2} - 2^{1} - 1$$

So the factor 568630647535356955169033410940867804839360742060818433 of F_{12} has Left-symmetry L (568630647535356955169033410940867804839360742060818433)=15.

We observe that if an odd number Q is $\beta_0 = -1$ in its representation (i.e., it has left symmetry), then

$$Q = 2^L \cdot K + 1, \tag{19}$$

where K is odd.

Similarly, if D is an odd number such that $\beta_0 = +1$, then

$$D = 2^R \cdot K - 1, \tag{20}$$

where K is odd.

Indeed, to prove (21) we note that

$$Q = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \ldots + \beta_{L+1}2^{L+1} + 2^L + 2^{L-1} - 2^{L-2} - \ldots - 2^2 - 2^1 - 1$$

where $\nu + 1 = \left[\frac{\ln Q}{\ln 2}\right]$. Then, we have the following sequence of equations

$$Q = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \ldots + \beta_{L+1}2^{L} + 2^{L-1} - \left(2^{L-2} + 2^{L-3} + \ldots + 2^2 + 2^1 + 1\right)$$

$$Q = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \ldots + \beta_{L}2^{L} + 2^{L-1} - \left(2^{L-1} - 1\right)$$

$$Q = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \ldots + \beta_{L}2^{L} + 1$$

$$Q = 2^L\left(2^{\nu+1-L} + 2^{\nu+1-L-1} + \beta_{\nu-1}2^{\nu+1-L-2} + \beta_{\nu-2}2^{\nu+1-L-3} \ldots + \beta_L\right) + 1$$

$$Q = 2^L K + 1$$

where

$$K = 2^{\nu+1-L} + 2^{\nu-L} + \beta_{\nu-1}2^{\nu-L-1} + \ldots + \beta_L\,. \tag{21}$$

Equation (20) is proved similarly. For the odd ones of form D we have,

$$K = 2^{\nu+1-R} + 2^{\nu-R} + \beta_{\nu-1}2^{\nu-R-1} + \ldots + \beta_R\,. \tag{22}$$

We give two examples.

Example 4. For odd number 18303 we have

$18303+1=2^7\times 143$.

Therefore $R(18303)=7$. Indeed, from Equation (6) we get $\nu=13$ and from Equation (1) we obtain

$18303=2^{14}+2^{13}-2^{12}-2^{11}-2^{10}+2^9+2^8+2^7-2^6+2^5+2^4+2^3+2^2+2^1+1$.

Example 5. For the number C1133 which is composite factor of F_{12} with 1133 digits, we have

$C1133-1=2^{14}\cdot K$.

Therefore, $L(C1133)=14$.

From Definition 1 and Equations (21), (22) we obtain the following.

Corollary 3.

The conjugate of $Q=2^L\cdot K+1$ *is* $Q^*=2^L\cdot K^*-1$, *and vice versa.*

It is easy to prove the following.

Proposition 2.

1. $Q_1Q_2=Q$.
2. $D_1D_2=Q$.
3. $Q_1D_1=D$.
4. $L(Q_1)<L(Q_2)\Rightarrow L(Q_1Q_2)=L(Q_1)$.
5. $L(Q)<R(D)\Rightarrow R(QD)=L(Q)$.
6. $R(D)<L(Q)\Rightarrow R(QD)=R(D)$.
7. $R(D_1)<R(D_2)\Rightarrow L(D_1D_2)=R(D_1)$.
8. $Symmetry(\Pi_1)=Symmetry(\Pi_2)\Rightarrow Symmetry(\Pi_1\Pi_2)>Symmetry(\Pi_1)=Symmetry(\Pi_2)$.

We give two examples:

Example 6. L (641)=7< L (114689)=14 => L (641×114689)=7.

Example 7. R (607)= 5< R (16633)=6 => L (607×16633)=5.

A consequence of Proposition 2 is the following.

Corollary 4. 1. Composite numbers of the form $C=2^{\nu+1}+1$ are written as

$C=2^{\nu+1}+1=(2^x\cdot K_1+1)(2^x\cdot K_2+1)$, $x\geq 3$,

where $x=1,2,3,..$ and K_1, K_2 are odd numbers.

2. Composite numbers of the form $C=2^{\nu+1}-1$ are written as

$C=2^{\nu+1}-1=(2^x\cdot K_1+1)(2^x\cdot K_2-1)$, $x\geq 4$,

where $x=1,2,3,..$ and K_1, K_2 are odd numbers.

From Definitions 1 and 2 we obtain the following.

Corollary 5. *In every conjugate pair* (Π,Π^*), *one number has left symmetry, and the other has right.*

4. Transpose of odd number. Categorization of odd numbers

We now give the following definition.

Definition 4. 1. We write the odd D in the form of Equation (1),

$$D = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \ldots + \beta_1 2^1 + 1, \tag{23}$$

where $\nu + 1 = \left[\frac{\ln D}{\ln 2}\right]$. We define *the transpose* $T(D)$ of D as

$$T(D) = \left(\frac{1}{2^{\nu+1}} + \frac{1}{2^{\nu}} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \ldots + \frac{\beta_1}{2^1} + 1\right)\cdot 2^{\nu+1} = 2^{\nu+1} + 3 + \sum_{k=1}^{\nu-1}\beta_k \cdot 2^{\nu+1-k}. \tag{24}$$

2. We write the odd Q in the form of Equation (1),

$$Q = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \ldots + \beta_1 2^1 - 1, \tag{25}$$

where $\nu + 1 = \left[\frac{\ln Q}{\ln 2}\right]$. We define *the transpose* $T(Q)$ *of* Q as

$$T(Q) = -\left(\frac{1}{2^{\nu+1}} + \frac{1}{2^{\nu}} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \ldots + \frac{\beta_1}{2^1} - 1\right)\cdot 2^{\nu+1} = 2^{\nu+1} - 3 - \sum_{k=1}^{\nu-1}\beta_k \cdot 2^{\nu+1-k}. \tag{26}$$

3. We set

$$T(1) = 1. \tag{27}$$

4. From Equations (24), (26), (27) we get the general equation

$$T(\Pi) = 2^{\nu+1} + \beta_0 \cdot \left(3 + \sum_{k=1}^{\nu-1}\beta_k \cdot 2^{\nu+1-k}\right), \tag{28}$$

where $\nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right]$.

Algorithm for the calculation of the transpose. Let Π be an odd number. We first calculate $\nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right]$ from Equation (6). Next, applying the algorithm described of Example 1, we write Π in the form of Equation (1), and we calculate $\beta_i = \pm 1, i = 1,2,3,\ldots,\nu-1$ and the transpose $T(\Pi)$ of Π from Equation (28).

We now prove a series of results regarding the transpose of an odd number.

Theorem 2.

1. *It holds that*

$$T(\Pi) = 1 \text{ if and only if } \Pi = 2^{\nu} - 3, \nu \geq 2, \nu \in \mathbb{N}. \tag{29}$$

2. *It holds that*

$$\left.\begin{array}{l} T(D) = D \\ \nu + 1 = \left[\frac{\ln D}{\ln 2}\right] \end{array}\right\} \Leftrightarrow \begin{cases} \beta_1 = 1 \\ \beta_{\nu-k} = \beta_{k+1} \\ k = 1,2,3,\ldots,\frac{\nu-2}{2}, \nu = even \\ k = 1,2,3,\ldots,\frac{\nu-1}{2}, \nu = odd \end{cases} \tag{30}$$

$$T(Q)=Q \quad \nu+1=\left[\frac{\ln Q}{\ln 2}\right] \Bigg\} \Leftrightarrow \begin{cases} \beta_1=-1 \\ \beta_{\nu-k}=-\beta_{k+1} \\ k=1,2,3,\ldots,\dfrac{\nu-2}{2}, \nu=even \\ k=1,2,3,\ldots,\dfrac{\nu-1}{2}, \nu=odd \end{cases} \cdot \tag{31}$$

Proof. 1. For $\Pi=2^{\nu}-3$ we get

$$\Pi=2^{\nu}-3=\left(2^{\nu}-1\right)-2=\left(2^{\nu-1}+2^{\nu-2}+2^{\nu-2}+\ldots+2^{1}+1\right)-2$$
$$=2^{\nu-1}+2^{\nu-2}+2^{\nu-2}+\ldots+2^{1}-1,$$

that is, Π has left symmetry, and thus Equation (26) implies $T(\Pi)=1$. Now, let $T(\Pi)=1$. The odd Π has either left or right symmetry. We only consider the former case, as the latter is similar. Then

$$\Pi=2^{n+1}+2^{n}+\beta_{n-1}2^{n-1}+\beta_{n-2}2^{n-2}+\ldots+\beta_2 2^2+\beta_1 2^1-1, \tag{32}$$

where $n+1=\left[\frac{\ln \Pi}{\ln 2}\right]$. Thus, from Equation (26), we get

$$T(\Pi)=-1-2^1-\beta_{n-1}2^2-\ldots-\beta_2 2^n+2^{n+1}.$$

Thus, we have the following sequence of equivalent equations

$$T(\Pi)=1$$
$$-1-2^1-\beta_{n-1}2^2-\ldots-\beta_2 2^n+2^{n+1}=1$$
$$-1-2^1-\beta_{n-1}2^2-\ldots-\beta_2 2^n+2^{n+1}+2^{n+2}=2^{n+2}+1$$
$$-1-2^1-\beta_{n-1}2^2-\ldots-\beta_2 2^n+2^{n+1}+2^{n+2}=2^{n+2}+2^n-2^{n-1}-2^{n-2}-\ldots-2^2-2^1-1$$

As the representation in Equation (1) is unique, we have

$$\beta_2=\beta_3=\beta_3=\ldots=\beta_{n-1}=+1.$$

Therefore, (32) implies

$$\Pi=2^{n+1}+2^{n}+2^{n-1}\ldots+2^{1}-1=2\left(2^{n}+2^{n-1}+2^{n-2}+\ldots+2^{1}+1\right)-1$$
$$=2\left(2^{n+1}-1\right)-1=2^{n+2}-3.$$

Setting $n+2=\nu$ we obtain $\Pi=2^{\nu}-3$.

2. This is proved similarly. We write D in the form of Equation (1), that is,

$$D=2^{\nu+1}+2^{\nu}+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+\ldots+\beta_1 2^1+1, \tag{33}$$

where $\nu+1=\left[\frac{\ln D}{\ln 2}\right]$. Equations (33) and (24) imply

$$T(D)=1+2+\beta_{\nu-1}2^2+\beta_{\nu-2}2^3+\ldots+\beta_2 2^{\nu}+2^{\nu+1}. \tag{34}$$

Then $T(D)=D$ if and only if

$$1+2+\beta_{\nu-1}2^2+\beta_{\nu-2}2^3+\ldots+\beta_2 2^{\nu}+2^{\nu+1}=2^{\nu+1}+2^{\nu}+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+\ldots+\beta_2 2^2+\beta_1 2^1+1.$$

As the representation in (1) is unique, the proof is complete. □

Theorem 3.

1. *If the odd number D has right symmetry, then*

$$T(D)-T(D^*)=6. \tag{35}$$

2. *If the odd number Q has left symmetry, then*

$$T(Q)-T(Q^*)=-6. \tag{36}$$

Proof. We only prove (35), as the proof of (36) is similar. From Equation (33), we get,

$$D^* = 2^{\nu+1}+2^{\nu}-\beta_{\nu-1}2^{\nu-1}-\beta_{\nu-2}2^{\nu-2}-...-\beta_1 2^1-1=Q. \tag{37}$$

Equation (26) implies

$$T(D^*)=-1-2+\beta_{\nu-1}2^2+\beta_{\nu-2}2^3+...+\beta_1 2^{\nu}+2^{\nu+1}. \tag{38}$$

Finally, from (34) and (38), we obtain $T(D)-T(D^*)=6$. □

Theorem 4.

For every odd Π, $\nu+1=\left[\frac{\ln\Pi}{\ln 2}\right]$, $\Pi\in\Omega_\nu=\left[2^{\nu+1},2^{\nu+2}\right]$, *then*

$$T(\Pi)<2^{\nu+2}. \tag{39}$$

Proof. Without loss of generality, we may assume that Π has right symmetry. From Equation (34), and taking into account that $\beta_i=\pm1, i=0,1,2,...,\nu-1$, we obtain

$$T(D)=1+2+\beta_{\nu-1}2^2+\beta_{\nu-2}2^3+...+\beta_2 2^{\nu}+2^{\nu+1}\le 1+2+2^2+...+2^{\nu}+2^{\nu+1}=2^{\nu+2}-1$$
$$T(D)\le 2^{\nu+2}-1<2^{\nu+2}.$$ □

This result implies that if an odd Π belongs to the interval $\Omega_\nu=\left[2^{\nu+1},2^{\nu+2}\right]$, its transpose $T(\Pi)$ can be found in intervals $\Omega_n, n\le\nu$.

Theorem 5.

If D, Q *belongs to the interval* $\Omega_\nu=[2^{\nu+1},2^{\nu+2}]$, $\nu+1=\left[\frac{\ln D}{\ln 2}\right]=\left[\frac{\ln Q}{\ln 2}\right]$, $\nu=4,5,6,...$, *then*

$$T(D-2)+T(D)=T(Q)+T(Q+2)=2^{\nu+2}. \tag{40}$$

Proof. The smallest odd number D with right symmetry in $\Omega_\nu=\left[2^{\nu+1},2^{\nu+2}\right]$ is $D_{\min}=2^{\nu+1}+3$. Thus, $D\in\Omega_\nu$ if and only if $(D-2)\in\Omega_\nu$ ($D>3$). The largest odd number with left symmetry in the interval Ω_ν is $Q_{\min}=2^{\nu+1}+1$. Thus, the following $Q\in\Omega_\nu$ if and only if $(Q+2)\in\Omega_\nu$ ($Q>3$). We do the proof of the equation $T(Q)+T(Q+2)=2^{\nu+2}$. The proof of the equation $T(D-2)+T(D)=2^{\nu+2}$ is similar. From Definition 3 we have

$$Q=2^{\nu+1}+2^{\nu}+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+\beta_{\nu-3}2^{\nu-3}+...+\beta_2 2^2-2-1$$

and

$$Q+2=2^{\nu+1}+2^{\nu}+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+\beta_{\nu-3}2^{\nu-3}+...+\beta_2 2^2-2+1.$$

From these Equations and (28) we get

$$T(Q)+T(Q+2)=2^{\nu+2}.$$ □

Theorem 6.

If D, Q belongs to the interval $\Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$, $\nu+1 = \left[\frac{\ln D}{\ln 2}\right] = \left[\frac{\ln Q}{\ln 2}\right]$, $\nu = 4,5,6,\ldots$, *then*

$$T(D) - T(D-4) = T(Q) - T(Q+4) = 2^{\nu+1}. \quad (41)$$

Proof. We do the proof of the equation $T(Q) - T(Q+4) = 2^{\nu+1}$. The proof of the equation $T(D) - T(D-4) = 2^{\nu+1}$ is similar. From Definition 3 we have

$$Q = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \beta_{\nu-3}2^{\nu-3} + \ldots + \beta_2 2^2 - 2 - 1$$

and

$$Q + 4 = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \beta_{\nu-3}2^{\nu-3} + \ldots + \beta_2 2^2 + 2 - 1.$$

From these Equations and (28) we get

$$T(Q) - T(Q+4) = 2^{\nu+1}.$$ □

Expunging $T(D)$ and $T(Q)$ from Equations (40) and (41) we obtain the following.

Corollary 6.

If D, Q belongs to the interval $\Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$, $\nu+1 = \left[\frac{\ln D}{\ln 2}\right] = \left[\frac{\ln Q}{\ln 2}\right]$, $\nu = 4,5,6,\ldots$, *then*

$$T(D-2) + T(D-4) = T(Q+2) + T(Q+4) = 2^{\nu+1}. \quad (42)$$

Theorem 7.

For every odd Π, then

$$T(2^n \cdot \Pi) = T(\Pi),$$

where $n \in \mathbb{N}$.

Proof. We prove the Theorem for odd D ones with left symmetry. The proof for odd Q ones with right symmetry is similar. We have

$$D = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \ldots + \beta_1 2^1 + 1,$$

where $\nu + 1 = \left[\frac{\ln D}{\ln 2}\right]$.

From this Equation and (24) we get the following equivalent equations,

$$2^n D = 2^{n+\nu+1} + 2^{n+\nu} + \beta_{\nu-1}2^{n+\nu-1} + \beta_{\nu-2}2^{n+\nu-2} + \ldots + \beta_1 2^{n+1} + 2^n$$

$$T(2^n D) = \left(\frac{1}{2^{n+\nu+1}} + \frac{1}{2^{n+\nu}} + \frac{\beta_{\nu-1}}{2^{n+\nu-1}} + \frac{\beta_{\nu-2}}{2^{n+\nu-2}} + \ldots + \frac{\beta_1}{2^{n+1}} + \frac{1}{2^n}\right) \cdot 2^{n+\nu+1}$$

$$= \left(\frac{1}{2^{\nu+1}} + \frac{1}{2^{\nu}} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \ldots + \frac{\beta_1}{2^1} + \frac{1}{2^0}\right) \cdot 2^{\nu+1} = T(D)$$ □

We now give the following definition.

Definition 5. Categorizing odd numbers. Let Π be an odd number.

1. We define as *symmetric* every odd Π for which

$$T(T(\Pi)) = \Pi. \quad (43)$$

2. We define as *asymmetric* every odd Π for which

$$T(T(\Pi)) \neq \Pi. \quad (44)$$

From the Euclidean division identity it follows that every odd number Π is written in one of the following forms,

$$Q = 8m+1 = 2^n K+1,\ n = 3,4,5,\dots, \tag{45}$$

$$V = 8m+3 = 2^2 K-1, \tag{46}$$

$$U = 8m+5 = 2^2 K+1, \tag{47}$$

$$D = 8m+7 = 2^n K-1,\ n = 3,4,5,\dots, \tag{48}$$

where K is odd and $m = 0,1,2,\dots$.

Numbers Q, U have left symmetry and V, D have right symmetry. From Equations (45) - (48) and (28) it follows that odd numbers Q, D are symmetric, and odd numbers V, U are asymmetric.

Proposition 3.

For the symmetric numbers of the interval Ω_ν *the following holds.*

1. *If odd number of the form* Q *have (left) symmetry* L, *then* $3 \le L \le \nu+1$. $Q = 2^{\nu+1}+1$ *is the unique odd number of* Ω_ν *with symmetry* $L = \nu+1$.

2. *If odd number of the form* D *have (right) symmetry* R, *then* $3 \le R \le \nu+1$. $D = 2^{\nu+1}+7$ *is the unique odd number of* Ω_ν *with symmetry* $R = \nu+1$.

Proof. We prove 1. The proof of 2 is similar. In Equation (45), if m is odd then $L = 3$, if m is even then $L > 3$. Of all the odd numbers in the interval Ω_ν, $Q = 2^{\nu+1}+1$ has the largest left symmetry, $L\left(2^{\nu+1}+1\right) = \nu+1$. □

From Equation (28) and Definition 5 we obtain the following.

Corollary 7.

1. *The odd number* $T(Q)$ *is of the form* Q.
2. *The odd number* $T(D)$ *is of the form* D.
3. *The odd number* $T(U)$ *is of the form* Q.
4. *The odd number* $T(V)$ *is of the form* D.

Theorem 8.

If an asymmetric odd number belongs to the interval Ω_ν, *then its transpose belongs to the interval* Ω_μ *with* $\mu < \nu$.

Proof. This is a direct consequence of Equations (46), (47) and (28). □

Equation (35) has been proved for all odd numbers with right symmetry. Equation (36) has been proved for all odd numbers with left symmetry. Thus from Theorem 3 we obtain the following.

Corollary 8.

1. For an asymmetric number of the form V,

$$T(V) - T\left(V^*\right) = 6.$$

2. For an asymmetric number of the form U,

$$T(U) - T\left(U^*\right) = -6.$$

From Definition 4 we obtain the following.

Corollary 9.

1. *Let a be an odd number*

$Q = 2^L \cdot K + 1,\ L \geq 4$,

K *is odd, belongs to the interval* Ω_ν, *then* $T(Q+4)$ *belongs to the interval* $\Omega_{\nu-1}$ *and* $T(Q+2)$ *belongs to the interval* $\Omega_{\nu-(L-2)}$.

2. *Let a be an odd number*

$D = 2^R \cdot K - 1,\ R \geq 4$,

K *is odd, belongs to the interval* Ω_ν, *then* $T(D-4)$ *belongs to the interval* $\Omega_{\nu-(R-2)}$ *and* $T(D-2)$ *belongs to the interval* $\Omega_{\nu-1}$.

From Equations (45) - (48) we obtain the following.

Corollary 10.

A. *1. The odd numbers of the form* U, $U \geq 21$, *have fixed symmetry* $L = 2$.

2. The odd numbers of the form V, $V \geq 11$, *have fixed symmetry* $R = 2$.

B. *1. The product* $Q \cdot U$ *is of the form* U.

2. The product $Q \cdot V$ *is of the form* V.

3. The product $D \cdot U$ *is of the form* V.

4. The product $D \cdot V$ *is of the form* U.

5. The product $U \cdot V$ *is of the form* D, *with symmetry* $R(D) \geq 4$.

6. The product $V_1 \cdot V_2$ *is of the form* Q, *with symmetry* $L = 3$.

7. The product $U_1 \cdot U_2$, *is of the form* Q, *with symmetry* $L = 3$.

From Proposition 2 and Corollary 10 we obtain the following.

Corollary 11.

A composite odd number has one of the following ten forms.

1. $Q = Q_1 Q_2$.
2. $Q = D_1 D_2$.
3. $Q = V_1 V_2$.
4. $Q = U_1 U_2$.
5. $V = Q_1 V_1$.
6. $V = D_1 U_1$.
7. $U = Q_1 U_1$.
8. $U = D_1 V_1$.
9. $D = Q_1 D_1$.
10. $D = U_1 V_1$.

From Proposition 2 and Corollaries 5 and 9 we obtain the following.

Corollary 12.

For the odd numbers Q, V, U, D *the following holds.*

1. *The product* $Q \cdot T(Q)$ *is of the form* Q.

2. *The product* $V \cdot T(V)$ *is of the form* U.

3. *The product* $U \cdot T(U)$ *is of the form* U.

4. *The product* $D \cdot T(D)$ *is of the form* Q.

From Definition 2 we obtain the following.

Corollary 13.

The complementary odd numbers Π *and* Π' *are of the same form.*

We now prove the following.

Proposition 4.

A. *1. If* $Q \in \Omega_\nu$ *is a symmetric number with left symmetry, then*

$T(Q) = T(3 \cdot 2^{\nu+1} - Q) - 6$.

2. If $U \in \Omega_\nu$ *is an asymmetric number with left symmetry, then*

$T(U) = T(3 \cdot 2^{\nu+1} - U) - 6$.

B. *1. If* $D \in \Omega_\nu$ *is a symmetric number with right symmetry, then*

$T(D) = T(3 \cdot 2^{\nu+1} - D) + 6$.

2. If $V \in \Omega_\nu$ *is an asymmetric number with right symmetry, then*

$T(V) = T(3 \cdot 2^{\nu+1} - V) + 6$.

Proof. The proposition is a consequence of Theorem 3 and Corollary 8. We prove A.1. A.2 and B.1, B.2 are proved similarly. From Equation (36) we get $T(Q^*) - T(Q) = 6$ and from Equation (14) we get

$T(3 \cdot 2^{\nu+1} - Q) - T(Q) = 6$

or equivalently

$T(Q) = T(3 \cdot 2^{\nu+1} - Q) - 6$. □

5. Octet of odd numbers

We now give the following definitions.

Definition 6. We define as *the octet* Φ of odd number Π the non ordered octet

$$\Phi = \left(\Pi, T(\Pi), (T(\Pi))^*, T\left((T(\Pi))^*\right), \Pi^*, T(\Pi^*), (T(\Pi^*))^*, T\left((T(\Pi^*))^*\right)\right). \quad (49)$$

Definition 7.

1. From Definitions 1 and 3 it follows that if Π is symmetric, then the numbers of the octet belong to the same interval Ω_ν. Π belongs to the octet, so $\nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right]$. We define this octet as *symmetric*.

2. From Definitions 1 and Theorem 8 it follows that if Π is asymmetric, then the numbers of the octet belong to the same interval Ω_ν. We define this octet as *asymmetric*.

We now give an example which also shows the ways in which we can write a symmetric octet.

Example 8. From Equation (49) we get the symmetric octet in which $\Pi = 889$ belongs, Φ=(889, 529, 1007, 895, 647, 535, 1001, 641). To distinguish the pairs of transposes and conjugates, we write the octet in the following form.

$$\begin{array}{ccccccc}
889 & \xleftrightarrow{T} & 529 & \xleftrightarrow{*} & 1007 & \xleftrightarrow{T} & 895 \\
{}^{*}\updownarrow & & & & & & \updownarrow^{*} \\
647 & \xleftrightarrow{T} & 535 & \xleftrightarrow{*} & 1001 & \xleftrightarrow{T} & 641
\end{array}$$

Because of Equation (13) $\left(\Pi^{*}\right)^{*} = \Pi$ two conjugates are always connected, in all octets, by the symbol $\xleftrightarrow{*}$, $\Pi \xleftrightarrow{*} \Pi^{*}$. With $\Pi_1 \xleftrightarrow{T} \Pi_2$ we denote that $T\left(\Pi_1\right) = \Pi_2$ and $T\left(\Pi_2\right) = \Pi_1$. If $T\left(\Pi_1\right) = \Pi_2$ and $T\left(\Pi_2\right) \neq \Pi_1$, we write $\Pi_1 \xrightarrow{T} \Pi_2$. We follow this notation when Π_1 is asymmetric. In our example the octet is symmetric. Therefore $\Pi_1 \xleftrightarrow{T} \Pi_2$ is valid for all of the octet numbers. The octet symmetries are easily seen when we place the numbers on the corners of a regular octagon.

$$\begin{array}{ccccccc}
 & & 889 & \xleftrightarrow{T} & 529 & & \\
 & {}^{*}\swarrow\!\nearrow & & & & \nwarrow\!\searrow^{*} & \\
647 & & & & & & 1007 \\
T\updownarrow & & & & & & \updownarrow T \\
535 & & & & & & 895 \\
 & \nwarrow\!\searrow^{*} & & & & {}^{*}\swarrow\!\nearrow & \\
 & & 1001 & \xleftrightarrow{T} & 641 & &
\end{array}$$

A symmetric octet can be composed of eight different numbers, like the one of the previous example, or of 4 different numbers or of 2 different numbers (with the exception of the degenerate octets $(1,1,1,1,1,1,1,1)$ of 1 and $(3,3,3,3,3,3,3,3)$ of 3). From the Definitions of the conjugate and the transpose, the following equations are easily proven

$$\begin{array}{l}
2^{\nu}+1 \xleftrightarrow{*} 2^{\nu+1}-1 \\
2^{\nu}+7 \xleftrightarrow{*} 2^{\nu+1}-7 \\
2^{\nu}+1 \xleftrightarrow{T} 2^{\nu+1}-7 \\
2^{\nu}+7 \xleftrightarrow{T} 2^{\nu}+7 \\
2^{\nu+1}-1 \xleftrightarrow{T} 2^{\nu+1}-1
\end{array} \quad . \tag{50}$$

Considering Equations (50) we get the symmetric octets

$$\left(2^{\nu+1}+1, 2^{\nu+2}-7, 2^{\nu+1}+7, 2^{\nu+1}+7, 2^{\nu+2}-1, 2^{\nu+2}-1, 2^{\nu+1}+1, 2^{\nu+2}-7\right), \tag{51}$$

where $\nu = 3,4,5,\ldots$.

The symmetric octets (51) consist of four different numbers. The Fermat numbers for $\nu+1=2^S, S\in\mathbb{N}$, and Mersenne numbers for $\nu+2=p=prime$ belong to these octets. The symmetric octet $(9,9,15,15,15,15,9,9)$ of conjugates $(\Pi,\Pi^*)=(9,15)$ consists of two numbers.

Asymmetric octets as generators of symmetric octets. If an odd number Π belongs to a symmetric octet, then its conjugate Π^* and its transpose $\mathrm{T}(\Pi)$ belong to the octet. Also, all the numbers in the symmetric octet belong to the same interval Ω_ν. The asymmetric octets result from a pair of conjugates (Π,Π^*) belonging to an interval Ω_ν and their transposes $(T(\Pi),T(\Pi^*))$ in another interval Ω_μ, $\mu<\nu$ (refer to Theorem 8). The octet of the pair $(T(\Pi),T(\Pi^*))$ is symmetric and we say that it is produced from the initial asymmetric octet.

We now present one example of an asymmetric octet in which one can see the way in which we can write it so that the asymmetry is evident and so are the symmetric octet that it produces.

Example 9. Let the pair of asymmetric conjugates $(U=10301,14275=V)$. We have $T(10301)=641$ and $T(14275)=895$. Thus $U=10301$ produces the symmetric octet to which $Q=641$ belongs, and $V=14275$ produces the (same) symmetric octet to which $D=895$ belongs.

$$
\begin{array}{lllllll}
10301 & & & & & & \\
\downarrow^{T} & & & & & & \\
641 & \xleftrightarrow{T} & 1001 & \xleftrightarrow{*} & 535 & \xleftrightarrow{T} & 647 \\
{}^{*}\updownarrow & & & & & & \updownarrow^{*} \\
895 & \xleftrightarrow{T} & 1007 & \xleftrightarrow{*} & 529 & \xleftrightarrow{T} & 889 \\
\uparrow^{T} & & & & & & \\
14275 & & & & & &
\end{array}
$$

The conjugates numbers of the interval Ω_ν express a simple symmetry, they have a center of symmetry at the middle $3\cdot 2^\nu$ of the interval Ω_ν. As a symmetry, the transpose of a symmetric odd number Π ($\Pi=Q$ or $\Pi=D$) can be expressed through the Octet of Odd Numbers. Starting from a symmetric number Π, taking conjugate- transpose or transpose-conjugate we return to the number Π. A geometric interpretation of "transpose" is given by Proposition 4. "The conjugate" and "the transpose" are the fundamental symmetries that emerge from Theorem 1. By using these symmetries we can find a set of symmetries of the natural numbers.

In the list below we have the transposes of eight odd numbers. The list shows the structure of the odd numbers related through the symmetry of the 'transpose'.

List of characteristic transposes.

$N \in \mathbb{N}, N \geq 3$

$U = 2^N - 3 \xrightarrow{T} 1$

$Q = 2^N + 1 \xleftrightarrow{T} 2^{N+1} - 7$

$V = 2^N + 3 \xrightarrow{T} 7$

$U = 2^N + 5 \xrightarrow{T} 2^N - 7$

$V = 2^N - 5 \xrightarrow{T} 2^{N-1} - 1$

$D = 2^N + 7 \xleftrightarrow{T} 2^N + 7$

$Q = 2^N - 7 \xleftrightarrow{T} 2^{N-1} + 1$

$V = 3 \cdot \left(2^N + 1\right) \xrightarrow{T} 15$

$U = 5 \cdot \left(2^N + 1\right) \xrightarrow{T} 2^{N+2} - 23$

$D = 7 \cdot \left(2^N + 1\right) \xleftrightarrow{T} 2^{N+2} + 31$

In the list the consequences of Theorem 8 for the asymmetric U and V are seen. Asymmetric $U = 2^N + 5$ has the smallest possible difference of an asymmetric with its transpose, $2^N + 5 - T\left(2^N + 5\right) = 2^N + 5 - \left(2^N - 7\right) = 5 - (-7) = 12$.

However, as a consequence of Theorem 8,
$U \in \Omega_{N-1} = \left[2^N, 2^{N+1}\right]$ and $T(U) \in \Omega_{N-2} = \left[2^{N-1}, 2^N\right]$.

6. Quadruples and pairs of odd numbers.

From the Definition 7 of symmetric octets and Theorem 3, we have that every symmetric octet consists of two ordered symmetric quadruples Θ of the form

$$\Theta = \left(Q, Q+6, Q^* - 6, Q^*\right), \tag{52}$$

so

$$\Phi = \left(Q_1, Q_1 + 6, Q_1^* - 6, Q_1^*, Q_2, Q_2 + 6, Q_2^* - 6, Q_2^*\right). \tag{53}$$

Quadruples (53) are symmetric, in the sense that they belong to symmetric octets. The differences of the corresponding numbers of two quadruples are the same. Thus we define the distance d of the quadruples of Equation (53) as

$$d = \left|Q_2 - Q_1\right|. \tag{54}$$

This equation also applies to all quadruplets, whether they belong to the same octet or not. If $d \neq 0$, then the symmetric octet consists of eight different numbers. If $d = 0$, then it consists of four different numbers. Such a quadruple is (51). In addition, if we use the equation $Q + Q^* = 3 \times 2^{\nu+1}$, $\nu \in \mathbb{N}$, we take the asymmetric quadruple

$$\left(Q = 3 \cdot 2^{\nu+1} - 3, Q + 6 = 3 \cdot 2^{\nu+1} + 3, Q^* - 6 = 3 \cdot 2^{\nu+1} - 3, Q^* = 3 \cdot 2^{\nu+1} + 3\right)$$

which consists of the pair

$$\left(3 \cdot 2^{\nu+1} - 3, 3 \cdot 2^{\nu+1} + 3\right). \tag{55}$$

We prove the following.

Proposition 5.
1. *If* $\nu = 1$, *the pair of* $\Omega_{\nu+1} = \Omega_2$ *is*
$(Q = 9, D = 15)$.
2. *If* $\nu \geq 2$, *the pair of the interval* $\Omega_{\nu+1}$ *is of the form (U,V),*
$(U = 3\cdot 2^{\nu+1} - 3, 3\cdot 2^{\nu+1} + 3 = V)$.
3. *The pair of asymmetrics* (U, V) *produces the symmetric pair* $(9,15)$ *for every* $\nu \geq 2$.
Proof. 1. If $\nu = 1$, from Equation (55) we get
$(3\cdot 2^2 - 3, 3\cdot 2^2 + 3) = (9,15)$.
2. We have,
$(3\cdot 2^{\nu+1} - 3) - 5 = 3\cdot 2^{\nu+1} - 8 = 8\cdot(2^{\nu-2} - 1) = 8m$,
$(3\cdot 2^{\nu+1} + 3) - 3 = 3\cdot 2^{\nu+1} = 8\cdot 3\cdot 2^{\nu-2} = 8m$.
3. We have,
$U = 3\cdot 2^{\nu+1} - 3 = 2^{\nu+2} + 2^{\nu+1} - 2^{\nu} + 2^{\nu-1} + 2^{\nu-2} + 2^{\nu-3} + \ldots + 2^1 - 1$,
and from Equation (28) we get $T(U) = 9$. Similarly we get $T(V) = 15$. □

If $Q = 8m + 1$, $m \in \mathbb{N}$, belongs to the interval $\Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$, $\nu = 3,4,5,\ldots$, then Q, $Q+6$ belong to the interval $[2^{\nu+1}, 3\cdot 2^{\nu}]$ and their conjugates Q^*, $Q^* - 6$ belong to the interval $[3\cdot 2^{\nu}, 2^{\nu+2}]$. Therefore, the different octets of the interval Ω_ν are given by the inequality
$2^{\nu+1} + 1 \leq 8m + 1 < 3\cdot 2^{\nu} + 1$
or equivalently
$2^{\nu-2} \leq m < 3\cdot 2^{\nu-3}$.
From this inequality we get
$$Q = 8(2^{\nu-2} + k) + 1,\ k = 0,1,2,\ldots,2^{\nu-3} - 1. \quad (56)$$
From Equation (56) it follows that the interval Ω_ν contains exactly $2^{\nu-3}$ different symmetric quadruples.

From Equation (49) we get $Q_2 = \left(T\left(Q_1^*\right)\right)^*$ and $Q_1 = \left(T\left(Q_2^*\right)\right)^*$. Therefore, the distance (54) of quadruplets of the same octet (53) is given by the equation
$$d = \left| Q - \left(T\left(Q^*\right)\right)^* \right|, \quad (57)$$
where $Q = Q_1$ or $Q = Q_2$.
From Equation (57) we obtain the following.
Corollary 14.
The symmetric odd number Π *(*$\Pi = Q$ *or* $\Pi = D$*) belongs to an octet that contains two same quadruples if and only if* $\left(T\left(\Pi^*\right)\right)^* = \Pi$.

Now let the odd numbers Q of the quadruple (52) with (left) symmetry $L \geq 4$,
$$Q = 2^L \cdot K + 1,\ L \geq 3, \quad (58)$$
where K is an odd number. Taking into account that the quadruples (52) belong to the interval

$\left[2^{\nu+1}+1, 3\cdot 2^{\nu}+1\right)$ we have $2^{\nu+1}+1 \le 2^{L}\cdot K+1 < 3\cdot 2^{\nu}+1$ or equivalently $2^{\nu+1-L} < K < 3\cdot 2^{\nu-L}$ and finally we obtain

$K = 2^{\nu+1-L}+1, 2^{\nu+1-L}+3, 2^{\nu+1-L}+5, \ldots, 3\cdot 2^{\nu-L}-1$.

From this Equation we get

$$K = 2^{\nu+1-L}+1+2\lambda,\ \lambda = 0,1,2,\ldots,2^{\nu-1-L}-1,\ \nu \ge L+1. \tag{59}$$

From Equations (58) and (59) we obtain,

$$Q_{\lambda} = 2^{L}\cdot\left(2^{\nu+1-L}+1+2\lambda\right)+1,\ L\ge 3,\ \nu\ge L+1,\ \lambda = 0,1,2,\ldots,2^{\nu-1-L}-1. \tag{60}$$

From Equation (60) it follows that if $\nu \ge L+1$, the interval Ω_{ν} contains exactly $N = 2^{\nu-1-L}$ different symmetric quadruples with symmetry L.

There are quadruples of odd numbers containing asymmetric numbers.

Definition 8.

We define as asymmetric the quadruples

$\left(U, U+4, U^{*}-4, U^{*}\right)$, $\left(V, V+4, V^{*}-4, V^{*}\right)$.

Numbers $U+4$ and $V^{*}-4$ are of the form Q. Numbers $U^{*}-4$ and $V+4$ are of form D. For the asymmetric numbers U and V we have,

$U \xrightarrow{T} Q$

and

$V \xrightarrow{T} D$.

We now prove the following.

Proposition 6.

1. If one of the equations

$U = 2^{n}\cdot(Q+3)-3 \xrightarrow{T} T(Q) \xleftrightarrow{T} Q$,

$V = 2^{n}\cdot(D-3)+3 \xrightarrow{T} T(D) \xleftrightarrow{T} D$

is valid, then the other is also valid, $n = 0,1,2,\ldots$.

2. If one of the equations

$U' = 2^{n}\cdot(D-3)-3 \xrightarrow{T} T\left(\left(T(D)\right)^{*}\right) \xleftrightarrow{T} \left(T(D)\right)^{*}$,

$V' = 2^{n}\cdot(Q+3)+3 \xrightarrow{T} T\left(\left(T(Q)\right)^{*}\right) \xleftrightarrow{T} \left(T(Q)\right)^{*}$

is valid, then the other is also valid, $n = 0,1,2,\ldots$.

Proof. We prove one of the combinations. The remaining combinations prove similarly. We assume that the third Equation is valid, we put $D = Q^{*}$ and we get

$V = 2^{n}\cdot\left(Q^{*}-3\right)+3 \xrightarrow{T} T\left(Q^{*}\right) \xleftrightarrow{T} Q^{*}$,

or equivalently, changing the symbolism,

$$T\left(2^{n}\cdot\left(Q^{*}-3\right)+3\right) = T\left(Q^{*}\right). \tag{61}$$

From Equation (6) it follows that if $Q \in \Omega_{\nu}$, then $U \in \Omega_{\nu+n}$. Thus, from Proposition 4 we get

$T(U) = T\left(3\cdot 2^{n+\nu+1}-U\right)-6$

or equivalently

$$T\left(U\right)=T\left(3\cdot 2^{n+\nu+1}-2^{n}\cdot\left(Q+3\right)+3\right)-6$$

or equivalently

$$T\left(U\right)=T\left(2^{n}\cdot\left(3\cdot 2^{\nu+1}-Q-3\right)+3\right)-6$$

or equivalently

$$T\left(U\right)=T\left(2^{n}\cdot\left(Q^{*}-3\right)+3\right)-6.$$

and with Equation (61) we get

$$T\left(U\right)=T\left(Q^{*}\right)-6$$

or equivalently, changing the symbolism,

$$U\xrightarrow{T}T\left(Q^{*}\right)-6$$

and considering Equation (52) for the quadruples of odd numbers ($D-6=Q$, where $D=T\left(Q^{*}\right)$) we obtain,

$$U\xrightarrow{T}T\left(Q^{*}\right)-6\xleftrightarrow{T}T\left(Q\right)\xleftrightarrow{T}Q.$$

□

From Equations (53) and (14) we get the following.

Corollary 15.

If $\Phi=\left(Q_{1},Q_{1}+6,Q_{1}^{*}-6,Q_{1}^{*},Q_{2},Q_{2}+6,Q_{2}^{*}-6,Q_{2}^{*}\right)$, $Q_{1}<Q_{2}$ *, is an octet of* Ω_{ν} *, then*

$$\Phi_{-}=\left(Q_{1}-2^{3},Q_{1}+6-2^{\nu-1},Q_{1}^{*}-6+2^{\nu-1},Q_{1}^{*}-2^{3},Q_{2}+2^{3},Q_{2}+6-2^{\nu-1},Q_{2}^{*}-6+2^{\nu-1},Q_{2}^{*}+2^{3}\right)\in\Omega_{\nu},$$

$$\Phi_{+}=\left(Q_{1}+2^{3},Q_{1}+6-2^{\nu},Q_{1}^{*}-6+2^{\nu},Q_{1}^{*}+2^{3},Q_{2}-2^{3},Q_{2}+6-2^{\nu},Q_{2}^{*}-6+2^{\nu},Q_{2}^{*}-2^{3}\right)\in\Omega_{\nu}$$

are also octets of odd numbers.

The Corollary 15 is true if $\Phi_{-}\in\Omega_{\nu}$, $\Phi_{+}\in\Omega_{\nu}$. If $\Phi_{-}\notin\Omega_{\nu}$, then Φ_{-} is not an octet of odd numbers. If $\Phi_{+}\notin\Omega_{\nu}$, then Φ_{+} is not an octet of odd numbers.

7. The kernel of odd numbers

In this Section we define the odd number kernel and present its basic properties.

Definition 9.

Let X_{m} be a subset of Ω_{ν} of the form

$$X_{m}=\left\{8m+1,8m+3,8m+5,8m+7\right\}=\left\{Q_{m},V_{m},U_{m},D_{m}\right\}\subseteq\Omega_{\nu}. \quad (62)$$

We define as "the kernel" $E=E_{m}$ of the elements of X_{m},

$$E=E_{m}=\frac{Q_{m}+3}{4}=\frac{V_{m}+1}{4}=\frac{U_{m}-1}{4}=\frac{D_{m}-3}{4}=2m+1\in\Omega_{\nu-2}. \quad (63)$$

We prove the following.

Proposition 7.

1. The set X_{m} is written in the form

$$X_{E}=\left\{4E-3,4E-1,4E+1,4E+3\right\}. \quad (64)$$

2. If the elements of the set X_{E} belong to the set Ω_{ν}, then for the kernel E the following applies,

$T\left(4E-1\right)+T\left(4E-3\right)=2^{\nu+2}$, (65)

$T\left(4E+1\right)+T\left(4E+3\right)=2^{\nu+2}$, (66)

$T\left(4E-3\right)-T\left(4E+1\right)=2^{\nu+1}$, (67)

$T\left(4E+3\right)-T\left(4E-1\right)=2^{\nu+1}$. (68)

3. For the kernel $E=E_m$ the following applies,

$T\left(4E-3\right)=T\left(3\cdot 2^{\nu+1}+3-4E\right)-6$,

$T\left(4E-1\right)=T\left(3\cdot 2^{\nu+1}+1-4E\right)+6$,

$T\left(4E+1\right)=T\left(3\cdot 2^{\nu+1}-1-4E\right)-6$,

$T\left(4E+3\right)=T\left(3\cdot 2^{\nu+1}-3-4E\right)+6$.

Proof. 1. From Equation (63) we get

$Q=4E-3,\ V=4E-1,\ U=4E+1,\ D=4E+3$. (69)

We substitute these Equations in (62) and we get (64).

2. We replace $Q=4E-3,\ V=4E-1,\ U=4E+1,\ D=4E+3$ of Equation (69) in (40), (41) and we get Equations (65) - (68).

3. We replace $Q=4E-3,\ V=4E-1,\ U=4E+1,\ D=4E+3$ of Equation (69) in Proposition 4 and we get Equations of 3 of Proposition 7.

From Equation (63) it follows that every odd E is a kernel of the odd ones of a set X. There are different sets X_m, $\mathrm{X}_{m'}$ with common kernel $E_m=E_{m'}$. In these cases X_m, $\mathrm{X}_{m'}$ belong to different sets Ω_ν, $\Omega_{\nu'}$. □

Proposition 8.

1. *The kernels of the pair* $\left(3\cdot 2^{\nu+1}-3,3\cdot 2^{\nu+1}+3\right)$ *of the interval* Ω_ν, $\nu\geq 2$, *produce the pair of numbers* $\left(3\cdot 2^{\nu-1}-1,3\cdot 2^{\nu-1}+1\right)$ *of interval* $\Omega_{\nu-2}$, *which belong to the quadruple* $\left(3\cdot 2^{\nu-1}-7,3\cdot 2^{\nu-1}-1,3\cdot 2^{\nu-1}+1,3\cdot 2^{\nu-1}+7\right)$.

2. *Every quadruple* $\left(Q,Q+6,Q^*-6,Q^*\right)$ *of set* Ω_ν *has two kernels*

$$E_1=\frac{Q+3}{4},$$

$$E_2=\frac{Q^*-3}{4},$$

with

$$E_1+E_2=3\cdot 2^{\nu-1}.$$

Proof. 1. For the pair $\left(3\cdot 2^{\nu+1}-3,3\cdot 2^{\nu+1}+3\right)$ of Ω_ν we have

$$E\left(U\right)=\frac{U-1}{4}=\frac{3\cdot 2^{\nu+1}-3-1}{4}=3\cdot 2^{\nu-1}-1\in\Omega_{\nu-2}$$

and

$E(V)=\frac{V+1}{4}=\frac{3\cdot 2^{\nu+1}+3+1}{4}=3\cdot 2^{\nu-1}+1\in\Omega_{\nu-2}$.

2. The quadruple $(Q,Q+6,Q^*-6,Q^*)$ of set Ω_ν has two sets of the form X, $X_1=\{Q,Q+2,Q+4,Q+6\}$ and $X_1=\{Q^*-6,Q^*-2,Q^*-4,Q^*\}$ with kernels $E_1=\frac{Q+3}{4}$ and $E_2=\frac{Q^*-3}{4}$ respectively. From these Equations we get $E_1+E_2=\frac{Q+Q^*}{4}$ and with Equation (14) we obtain $E_1+E_2=\frac{Q+Q^*}{4}=\frac{3\cdot 2^{\nu+1}}{4}=3\cdot 2^{\nu-1}$. □

From 1 of Proposition 8 it follows that a pair $(3\cdot 2^{\nu+1}-3,3\cdot 2^{\nu+1}+3)$ of interval Ω_ν produces two different quadruples. From 2 of Proposition 8 it follows that starting from a quadruple $\Theta_1=(Q,Q+6,Q^*-6,Q^*)$ of the interval Ω_ν, the kernels of

$\Theta_1=(Q,Q+6,Q^*-6,Q^*)$

constitute a new quadruple

$\Theta_2=(E_1,E_1+6,E_2-6,E_2)$

of the interval $\Omega_{\nu-2}$. Thus, starting from Θ_1 we obtain a sequence of quadruples $\Theta_1,\Theta_2,\Theta_3,\ldots$. To create the sequence $\Theta_1,\Theta_2,\Theta_3,\ldots$ it is necessary to check at each step the form Q, V, U or D of E_i, $i=1,2,3,\ldots$, taking into account that every quadruplet Θ is written in the form

$$\Theta=(Q,Q+6,D-6,D)=(V-2,V+4,U-4,U+2), \qquad (70)$$

where (Q,D) and (V,U) are pairs of conjugate numbers.

We present one examples.

Example 10.

For the quadruple

$\Theta_1=(6700417,6700423,5882489,5882495)$

we have

$E_1=\frac{6700417+3}{4}=1\ 675105=Q_1$,

$E_2=\frac{5882495-3}{4}=1\ 470623=D_1$.

Thus, from Equation (70) we get

$\Theta_2=(1675105,1675111,1470617,1470623)$.

We have

$E_3=\frac{1675105+3}{4}=418777=Q_2$,

$E_4=\frac{1470623-3}{4}=367655=D_2$.

Thus, from Equation (70) we get

$\Theta_3 = (418777, 418783, 367649, 367655)$.

We have

$E_4 = \dfrac{418777+3}{4} = 104695 = D_3$,

$E_5 = \dfrac{367655-3}{4} = 91913 = Q_3$.

Thus, from Equation (70) we get

$\Theta_4 = (91913, 91913, 104689, 104695)$.

We have

$E_6 = \dfrac{91913+3}{4} = 22979 = V_1$,

$E_7 = \dfrac{104695-3}{4} = 26173 = U_1$.

Thus, from Equation (70) we get

$\Theta_5 = (V_1 - 2, V_1 + 4, U_1 - 4, U_1 + 2) = (22977, 22983, 26169, 26175)$.

We have

$E_8 = \dfrac{22977+3}{4} = 5745 = Q_4$,

$E_9 = \dfrac{26175-3}{4} = 6543 = D_4$.

Thus, from Equation (70)we get

$\Theta_6 = (5745, 5761, 6537, 6543)$.

We have

$E_{10} = \dfrac{5745+3}{4} = 1437 = U_2$,

$E_{11} = \dfrac{6543-3}{4} = 1635 = V_2$.

Thus, from Equation (70) we get

$\Theta_7 = (V_2 - 2, V_2 + 4, U_2 - 4, U_2 + 2) = (1633, 1639, 1433, 1439)$.

we have

$E_{12} = \dfrac{1633+3}{4} = 409 = Q_5$,

$E_{13} = \dfrac{1439-3}{4} = 359 = D_5$.

Thus, from Equation (70) we get

$\Theta_8 = (409, 415, 353, 359)$.

We have

$E_{14} = \dfrac{409+3}{4} = 103 = D_6$,

$E_{15} = \frac{359-3}{4} = 89 = Q_6$.

Thus, from Equation (70) we get

$\Theta_9 = (89, 95, 97, 103)$.

We have

$E_{16} = \frac{89+3}{4} = 23 = D_7$,

$E_{17} = \frac{103-3}{4} = 25 = Q_7$.

Thus, from Equation (70) we get

$\Theta_{10} = (25, 31, 17, 23)$.

We have

$E_{16} = \frac{25+3}{4} = 7 = D_8 \in \Omega_1 = \left[2^2, 2^3\right]$,

$E_{17} = \frac{23-3}{4} = 5 = U_3 \in \Omega_1 = \left[2^2, 2^3\right]$.

The interval $\Omega_1 = \left[2^2, 2^3\right]$ cannot contain a quadruple. Thus the sequence has last term the quadruple Θ_{10}.

The numbers $Q_1 < Q_2 < Q_3 < Q_4$, $D_1 < D_2 < D_3 < D_4$ of a symmetric octet Φ belong to the set X_Φ,

$$X_\Phi = X_1 \cup X_2 \cup X_3 \cup X_4 = \{Q_1, V_1, U_1, D_1\} \cup \{Q_2, V_2, U_2, D_2\} \cup \{Q_3, V_3, U_3, D_3\} \cup \{Q_4, V_4, U_4, D_4\}. \quad (71)$$

The relationship of the numbers of the octet is given by Equation (49). If the octet consists of two same quadruples, we have $\Phi = \Theta$ and

$$X_\Theta = X_1 \cup X_2 = \{Q_1, V_1, U_1, D_1\} \cup \{Q_2, V_2, U_2, D_2\}. \quad (72)$$

For quadruple Θ of Equation (51) we get

$$X_\Theta = X_1 \cup X_2 = \left\{2^N+1, 2^N+3, 2^N+5, 2^N+7\right\} \cup \left\{2^{N+1}-7, 2^{N+1}-5, 2^{N+1}-3, 2^{N+1}-1\right\}. \quad (73)$$

To this set X_Θ belong the Fermat and Mersenne numbers.

8. Odd number chains

In this Section we define the chains of odd numbers.

Definition 10.

Let $\Pi \geq 15$ be an odd number. We obtain the odd numbers Π_1, Π_2, Π_3, Π_4 from the equations

$\Pi - 1 = 2^{N_1} \cdot \Pi_1$

$\Pi - 3 = 2^{N_2} \cdot \Pi_2$

$\Pi - 5 = 2^{N_3} \cdot \Pi_3$

$\Pi - 7 = 2^{N_4} \cdot \Pi_4$

where $N_1, N_2, N_3, N_4 \in \mathbb{N}^*$. We repeat the process for Π_1, Π_2, Π_3, Π_4, until we arrive at the odd numbers 1, 3, 5, 7. We define the terms of the resulting strictly decreasing sequence as *"the chain"* of the odd Π.

We give an example.

Example 11.

For $\Pi = 57$ we have

$$57-1=2^3 \cdot 7$$
$$57-3=2\cdot 27$$
$$57-5=2^2 \cdot 13$$
$$57-7=2\cdot 25$$

so

$$\Pi_1 = 7$$
$$\Pi_2 = 27$$
$$\Pi_3 = 13 \, .$$
$$\Pi_4 = 25$$

For $\Pi_2 = 27$ we have

$$27-1=2\cdot 13$$
$$27-3=2^3 \cdot 3$$
$$27-5=2\cdot 11$$
$$27-7=2^2 \cdot 5$$

so

$$\Pi_4 = 3$$
$$\Pi_5 = 11 \, .$$
$$\Pi_6 = 5$$

For $\Pi_3 = 13$ we have

$$13-1=2^2 \cdot 3$$
$$13-3=2\cdot 5$$
$$13-5=2^3 \cdot 1$$
$$13-7=2\cdot 3$$

so

$$\Pi_8 = 1 \, .$$

For $\Pi_5 = 11$ we have

$$11-1=2\cdot 5$$
$$11-3=2^3 \cdot 1$$
$$11-5=2\cdot 3 \, .$$
$$11-7=2^2 \cdot 1$$

For $\Pi_4 = 25$ we have

$25-1=2^3\cdot 3$

$25-3=2\cdot 11$

$25-5=2^2\cdot 5$

$25-7=2\cdot 9$

so

$\Pi_9=9$.

For $\Pi_9=9$ we have

$9-1=2^3\cdot 1$

$9-3=2\cdot 3$

$9-5=2^2\cdot 1$.

$9-7=2\cdot 1$

Thus we get chain A,

$\mathrm{A}=\{\Pi,\Pi_1,\Pi_2,\Pi_3,...,\Pi_9\}=\{57,27,25,13,11,9,7,5,3,1\}$,

arranging the elements of the chain in descending order.

Chains of odd numbers have a set of properties, some of which have been fully proven and some of which have not. Also, proving some of these properties is extremely time-consuming. Thus, in the remainder of the article we focus only on the presentation and application of the properties of chains.

First property of chains of odd numbers.

The odd numbers $D=8m-1$ and $Q=8m+1$, where $m=2,3,4,...$, give the same chain.

$D=8m-1$ and $Q=8m+1$ have successive kernels,

$$E(D)=\frac{8m-1-3}{4}=2m-1,$$

$$E(Q)=\frac{8m+1+3}{4}=2m+1.$$

Thus, the first property of chains relates the chains of odd numbers of the sets X_{2m-1} and X_{2m+1}.

We give an example.

Example 12.

We present the chains of odd numbers of the four sets X of the interval Ω_4.

33	35	37	39
15	15	17	19
13	9	15	17
7	7	9	9
5	5	7	7
3	3	5	5
1	1	3	3
		1	1

41	43	45	47
19	21	21	23
17	19	19	21
9	9	11	11
7	7	9	9
5	5	7	7
3	3	5	5
1	1	3	3
		1	1

49	51	53	55
23	25	25	27
21	23	23	25
11	11	13	13
9	9	11	11
7	7	9	9
5	5	7	7
3	3	5	5
1	1	3	3
		1	1

57	59	61	63
27	29	29	31
25	27	27	29
13	13	15	15
11	11	13	13
9	9	11	11
7	7	7	7
5	5	5	5
3	3	3	3
1	1	1	1

The matrices show the consequences of the first property of chains. From the first matrix, we also conclude that $D=31$ has the same chain as $Q=33$, and from the fourth matrix, that $Q=65$ has the same chain as $D=63$. Ultimately, the chains of odd numbers concern the sets X.

9. An algorithm for factoring odd numbers

The second property of chains concerns an algorithm for factoring odd numbers [1, 2, 3, 5, 8, 10]. Applying the algorithm for odd Π gives factors of both Π and Π'. Therefore the algorithm applies to a pair (Π, Π') of complementary odd numbers. We give the steps of algorithm for Π.

Step 1.

Let Π be an composite odd number. The first step of the algorithm depends on the form of Π.

1. If Π is of form Q, $\Pi = Q$, from equation

$Q - 9 = 2^3 \cdot M$

we obtain the odd number M.

2. If Π is of form V, $\Pi = V$, from equation

$V - 11 = 2^3 \cdot M$

we obtain the odd number M.

3. If Π is of form U, $\Pi = U$, from equation

$U - 13 = 2^3 \cdot M$

we obtain the odd number M.

4. If Π is of form D, $\Pi = D$, from equation

$D - 15 = 2^3 \cdot M$

we obtain the odd number M.

Step 2.

We calculate the chain of M.

Step 3.

For the chain of M, at least one of the following is true.

1. Π has common factors with the numbers of the chain.
2. The factors of the numbers in the chain include numbers that either belong to the same octet with factors of Π or produce (if they are asymmetric) octets to which factors of Π belong.

For the factorizations required by the algorithm, we can use the algorithm itself. This results in four sequences of factorizations, starting from Π_1, Π_2, Π_3 and Π_4 of the M chain.

There are cases in which the form of the factors of Π is known. One such case is the Fermat numbers, where their factors are of the form Q [4, 6, 7, 9]. Therefore, to factorize Fermat numbers or their composite factors, we apply the 1 of step 1.

We now give an example of applying the algorithm to an odd number whose factors we know. The aim of the example is to show how to apply the algorithm.

Example 13.

$\Pi = 62177 = 97 \cdot 641$ is of the form Q. From 1 of second step of the algorithm we get

$62177 - 9 = 2^3 \cdot 7771$.

Therefore it is

$M = 7771 = 19 \cdot 409$.

We gradually calculate the chain of M. We have

$$7771-1=2\cdot3\cdot5\cdot7\cdot37$$
$$7771-3=2^3\cdot971$$
$$7771-5=2\cdot11\cdot353$$
$$7771-7=2^2\cdot3\cdot647$$
so
$$\Pi_1=3\cdot5\cdot7\cdot37$$
$$\Pi_2=971$$
$$\Pi_3=11\cdot353 .$$
$$\Pi_4=3\cdot647$$
First, from Π_4, we take 647, which belongs to the same octet as the factor 641 of 62177.

$$\begin{array}{ccccccc} 889 & \overset{T}{\longleftrightarrow} & 529 & \overset{*}{\longleftrightarrow} & 1007 & \overset{T}{\longleftrightarrow} & 895 \\ {}^{*}\updownarrow & & & & & & \updownarrow^{*} \\ 647 & \overset{T}{\longleftrightarrow} & 535 & \overset{*}{\longleftrightarrow} & 1001 & \overset{T}{\longleftrightarrow} & 641 \end{array}$$

From $\Pi_1=3\cdot5\cdot7\cdot37$ we take
$$3\cdot5\cdot7\cdot37-1=2^2\cdot971$$
$$3\cdot5\cdot7\cdot37-3=2\cdot3\cdot647$$
$$3\cdot5\cdot7\cdot37-5=2^3\cdot5\cdot97 .$$
$$3\cdot5\cdot7\cdot37-7=2\cdot7\cdot277$$
We notice that the numbers 971 and $3\cdot647$ are repeated. Also, the factor 97 of 62177 appears for the first time, and new odd numbers enter the chain.

From $\Pi_2=971$ we get
$$971-1=2\cdot5\cdot97$$
$$971-3=2^3\cdot11^2$$
$$971-5=2\cdot3\cdot7\cdot23 .$$
$$971-7=2^2\cdot241$$
In the first equation, the factor 97 of 62177 appears, and indeed by repeating the product $5\cdot97$. Also, new odd numbers enter the chain.

From $\Pi_3=11\cdot353$ we get
$$11\cdot353-1=2\cdot3\cdot647$$
$$11\cdot353-3=2^3\cdot5\cdot97$$
$$11\cdot353-5=2\cdot7\cdot277 .$$
$$11\cdot353-7=2^2\cdot3\cdot17\cdot19$$
We notice that the numbers $3\cdot647$ and $5\cdot97$ are repeated. Also, new odd numbers enter the chain. For one of them, $3\cdot17\cdot19$, we get

$$3\cdot17\cdot19-1=2^3\cdot11^2$$
$$3\cdot17\cdot19-3=2\cdot3\cdot7\cdot23$$
$$3\cdot17\cdot19-5=2^2\cdot241$$
$$3\cdot17\cdot19-7=2\cdot13\cdot37 .$$

We now apply the algorithm for complementary of Π, $\Pi'=45793=11\cdot23\cdot181$. We have (refer to Corollary 13)

$45793-9=2^3\cdot5723$.

Therefore it is

$M'=5723=59\cdot97$.

We have

$$5723-1=2\cdot2861$$
$$5723-3=2^3\cdot5\cdot11\cdot13$$
$$5723-5=2\cdot3\cdot953$$
$$5723-7=2^2\cdot1429$$

so

$$\Pi_1'=2861$$
$$\Pi_2'=5\cdot11\cdot13$$
$$\Pi_3'=3\cdot953$$
$$\Pi_4'=1429 .$$

First, we notice that the factor 11 of 45793 appears in the factors of $\Pi_2'=5\cdot11\cdot13$. However, we apply the algorithm for factoring of $\Pi=62177$. We apply a part of the algorithm.

The factor 181 of $\Pi'=45793=11\cdot23\cdot181$ is asymmetric, and produces the symmetric number 73.

$U=181\xrightarrow{T}73$

The 73 belongs to the following octet.

$$\begin{array}{ccccccc} 73 & \xleftrightarrow{T} & 89 & \xleftrightarrow{*} & 103 & \xleftrightarrow{T} & 79 \\ {}^*\updownarrow & & & & & & \updownarrow^* \\ 119 & \xleftrightarrow{T} & 95 & \xleftrightarrow{*} & 97 & \xleftrightarrow{T} & 113 \end{array}$$

The factor 97 of $\Pi=62177$ belongs to the same octet.

In the factors of $M'=5723=59\cdot97$, the factor 97 of $\Pi=62177$ appears.

From $\Pi_4'=1429$ we get

$$1429-1=2^2\cdot3\cdot7\cdot17$$
$$1429-3=2\cdot23\cdot31$$
$$1429-5=2^4\cdot89$$
$$1429-7=2\cdot3^2\cdot79 .$$

The 89 belongs to the same octet as 97.

The 79 of the product $3^2\cdot79$ also belongs to the same octet as 97.

Regarding the factorization of Π', its factor 23 appears in the product $23\cdot31$.

10. Conclusion.

This article contains ten definitions, eight theorems, eight propositions, and fifteen corollaries, which give a set of symmetries of odd numbers. These symmetries put Number Theory on a new footing. In the previous Sections we have presented a large part of the consequences of the symmetries of natural numbers.

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