A formulation of vacuum Euclidean relativity on a general class of metric spaces.

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February 4, 2025

Abstract

A general geometric apparatus extending the standard calculus on Riemannian manifolds in a coordinate independent way is developed. It is, moreover, obvious that all quantities involved give rise to the correct limits in such case when convergence is subtle enough. We finish by giving the correct equivalent of the Einstein tensor which approximately is covariantly conserved on a certain deformation scale.

1 Introduction.

In these days, it is of general interest to generalize the notions of curvature and torsion to a setting where no coordinates are given, but merely an abstract path metric is present. To the best knowledge of this author, nobody has achieved such feature so far; in that regard does this paper develop novel ways of looking at those quantities without the use of tensor calculus. The latter is emergent only in the case of differentiable manifolds and metric tensors thereupon. Concretely, we shall generalize the notions of vectors, find an appropriate substitute for the covariant derivative and study two distinct definitions of a generalization of the Riemann tensor. Discrete Lorentzian or Riemannian structures often emerge in some approaches to quantum gravity, ranging from Lorentzian simplicial spacetimes, causal sets to abstract Lorentz spaces as studied by this author [1, 2, 3] in the past. First of all, it is key to understand that one cannot rely upon generalizations of notions of differentiability as the reader may easily check that the Leibniz rule and linearity of the differential cannot be extended to the general class of continuous functions. So, we have to give up on vectorfields, the Lie bracket, the Lie derivative and all that. Nevertheless, the notion of a vector and the one of parallel transport can be generalized and it is well known that parallel transport fully defines the connection even in the presence of torsion. We shall discuss that in somewhat more detail from different points of view later on; so our first task is to generalize vectors, as well as a global and local addition of vectors using the parallel transporter. The idea that parallel transport preserves the metric generalizes trivially in our setting and the notion of torsion

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gets several distinct meanings here. Open problems regard the existence and uniqueness of a torsionless transporter given a certain path metric; we will leave these issues open for the future.

2 Topological differentials.

Let X be any topological space (we do not insist upon it being metrical yet) and consider an equivalence relation $R \subset X \times X$ which is topologically open. R defines vectors, that is $(x, y) \in R$ is a vector connecting x with y; the correspondance to the usual vectors on a manifold being that (x, y) has to be thought of as the vector at x such that thenimage of the exponential map equals y, so they defined in a way relative to a meric and not a coordinate system. As said in the introduction, the notion of transport can easily be generalized and is defined by means of the following

$$\nabla_X : \{(x, y, z) : y, z \in R(x, \cdot)\} \to X \times X : (x, y, z) \to \nabla_{(x,y)}(x, z) = (y, w)$$

is called the transported relation regarding (x, z) over (x, y) from x to y and as such it indicates a preffered path or geodesic at least locally. ∇_X should obey the following further properties: (a) for any x, there exists an open O around it, such that $\{x\} \times O \subset R$ and such that for any $y, z \in O$ holds that $\nabla_{(x,y)}(x, z) \in R$, allowing one to define the composition of two transporters (b) ∇_X is continuous in the product topology (c) $\nabla_X(x, x, z) = (x, z)$, $\nabla_X(x, y, x) = (y, y)$ indicating that transport over the zero vector is the identity map and the zero vector gets transported into the zero vector. Before we proceed, it is useful to defientwo projections $\pi_1 : R \to X : (x, y) \to x$ and $\pi_2 : R \to X : (x, y) \to y$. We shall impose a furter condition on R which is that for any x and sufficiently small neighborhood O around it, that for any y, z, p, q it holds that $(\pi_2(\nabla_{(x,y)}(x, z)), \pi_2(\nabla_{(x,p)}(x, q))) \in R$ meaning that for sufficiently small vectors sufficiently small vectors around a point, the resulting endpoints of the parallel transport again constitute a vector. Another, useful operation is the reversion P which maps (x, y) into (y, x), something which has to do with the linear structure of vectors. To localize, the reversion, we define $\tilde{P}(x, y)$ as $\nabla_{P(x,y)}(P(x, y)) \in R(x)$, so again, taking the minus sign is a geometrical operation. On R, it is now possible to define two kinds of (non-commutative) sums; the first one is mere composition, that is

$$(x, y) \circ (y, z) = (x, z)$$

being non local operation and the second one

$$(x, y) \oplus (x, z) = \nabla_{(x,y)}(x, z) \circ (x, y)$$

being a local operation. The reader notices that the reversion also defines a minus operation

$$(x, y) \ominus (x, z) = (x, y) \oplus \tilde{P}(x, z).$$

So, the reader understands that the local notion of a sum is a geometrical one and not one which merely originates from the manifold structure. Now, we can easily define the torsion functor

$$T : X \times R(x) \times R(x) \to R(x) : (x, y, z) \to ((x, y) \oplus (x, z)) \ominus ((x, z) \oplus (x, y))$$

and we shall prove that in a way this coincides with the usual definition in case y, z converge to x at the same rate. The Riemann function may be defined in a sufficiently small neighborhood of x as

$$R(x,p,q,r) = ((x,p) \oplus ((x,q) \oplus (x,r))) \ominus ((x,q) \oplus ((x,p) \oplus (x,r))).$$

The reader notices here that we did not include the commutator in this definition as we have no natural substitute for a vectorfield, neither commutator and all draggings are supposed to define commuting vectorfields anyway. We shall investigate these two definitions in further detail in the next section. There is no meaningful topological way to define this, you need a metric for that. Finally, we may consider functions between two metrical spaces $(X, d_X), (Y, d_Y)$ with vector structures R, T and transporters ∇_X, ∇_Y defined upon it: we then say that $F : X \to Y$ is differentiable in a surrounding of $x \in X$ in case for any open $\mathcal{V} \subset T(F(x))$ there exists an open neighborhood $\mathcal{O} \subset R(x)$ such that the canonical bicontinuous mapping $DF(w,v) : (w,v) \in \mathcal{O}^2 \to \mathcal{V}^2$, $v, w \in \mathcal{O}$ defined by $(F(v), F(w)) = DF(v,w)$ satisfing

$$DF(((x,y) \oplus (x,w)) = DF(\nabla_{(x,y)}(x,w)) \circ DF(x,y)$$

also obeys

$$\frac{d_2(DF\,((x,y) \oplus (x,w)) \ominus (DF(x,y) \oplus DF(x,w)))}{\epsilon} \to 0$$

in case $d_1(x,y) = \epsilon a$, $d_1(x,w) = \epsilon b$, where $a, b > 0$ constants, which is the linearity condition. To define the torsion and Riemann "tensor", we need additional information. A connection is called weakly metric compatible if and only if

$$d(\nabla_{(xy)}(xz)) = d((xz))$$

which is, by itself insufficient to select for an "integrable" class of connections; for example, consider $\mathbb{R}^2$ with the standard Euclidean metric and define the connection $\nabla_{(x,y)}(x,z) = (y, y + R(z - x))$ where R is the rotation over the minimum of the angle θ between the vector $y - x$ and $z - x$ and $\pi - \theta$ in opposite orientation to the one defined by $z - x$ and $y - x$. Then the reader convinces himself that the angle is not preserved and that the torsion function vanishes identically. So, we must insist upon a stronger metric compatibility which says that the angles are preserved. For doing this, we need a path metric defined by the property that for any $x, y \in X$ it holds that there exists a $z \in X$ such that

$$d(x,z) = d(y,z) = \frac{d(x,y)}{2}.$$

The latter is equivalent to stating that there exists a curve, called a geodesic, $\gamma : [0,1] \to X$ which minimizes the length functional L for paths with endpoints x, y and, moreover, $L(\gamma) = d(x,y)$. The latter is defined by

$$L(\gamma) = \sup_{0=t_0<t_1\ldots<t_n=1, n>0} \sum_{j=0}^{n-1} d(\gamma(t_j), \gamma(t_{j+1}))$$

and γ can be parametrized in arc-length parametrization by means of the Radon Nikodym derivative. Furthermore, this only makes sense if the geodesic connecting two points x, y close enough to one and another exists and is unique so that we can associate vectors to geodesics. Consider a point $x \in X$ and take a sequence of points y_n, z_n placed on two half geodesics emanating from x converging in the limit for n to infinity towards x. In case the limit

$$\lim_{n \to \infty} \frac{d(x, y_n)^2 + d(x, z_n)^2 - d(y_n, z_n)^2}{2d(x, y_n)d(x, z_n)}$$

exists, we define the angle $\theta_x(y, z)$ between both geodesics by equating the latter expression to $\cos(\theta_x(y, z))$. So, we must also require that ∇_X preserves angles; in short, $\theta_x(y, z) = \theta_p(\pi_2(\nabla_{(x,p)}(x, y)), \pi_2(\nabla_{(x,p)}(x, z)))$ for x, p, y, z sufficiently close to one and another. Obviously, this is still not enough given that one may consider the connection $\nabla_{(x,y)}(x, z) = (y, y-(z-x))$ and notice that $(x, y) \oplus (x, y) = (x, x) = 0$. The reader sees immediately that angles as well as distances are preserved and that the torsion vanishes since $(x, y) \oplus (x, z) = (x, y-(z-x))$ and $(x, z) \oplus (x, y) = (x, z-(y-x))$ so that

$$((x, y) \oplus (x, z)) \ominus ((x, y) \oplus (x, z)) = (x, y-(z-x)) \oplus \tilde{P}(x, z-(y-x)) =$$

$$(x, y-(z-x)) \oplus (x, y-(z-x)) = (x, x) = 0$$

since $\tilde{P}(x, z-(y-x)) = \nabla_{(z-(y-x),x)}(z-(y-x), x) = (x, x-(z-y)) = (x, y-(z-x))$. So, therefore we need to impose the strongest form, which amounts to an integrability condition which is that the d geodesics are auto-parallel curves meaning that for any geodesic γ from x to y in arclength parametrization, it holds that

$$\nabla_{(\gamma(t),\gamma(s))}(\gamma(t), \gamma(s)) = (\gamma(s), \gamma(2s-t))$$

for $s > t$ sufficiently small. In that case, we find back the ordinary Levi-Civita connection with vanishing torsion in case for metrics on a manifold. To allow for torsion, one may impose that for any vector x, y sufficiently small, there exists a unique curve γ from x to y in arclength parametrization such that for $t < s$ sufficiently small, the above condition holds. We shall henceforth insist upon the last integrability condition. To give a nontrivial example of our construction, take two manifolds glued together at a point p, with identified induced metrics on both meaning there exist two orthonormal basis at p which are identified by means of a linear mapping $T : T\mathcal{M}_p \to T\mathcal{N}_p : v \to T(v)$ and T^{-1} of course for the opposite directions. Then, for general vectors $a \in \mathcal{M}_p$ corresponding to a unique vector (p, x) and $b \in T\mathcal{N}_p$ corresponding to a unique vector (p, y), one can define $a \oplus b \equiv a \oplus_{\mathcal{M}} T^{-1}(b)$ in $\mathcal{M}$ resultng in a vector (p, z) and vice versa for $b \oplus a \equiv b \oplus_{\mathcal{N}} T(a)$. So, usually, the torsion function does not vanish, but it does so for infinitesimal vectors $a = \epsilon a'$, $b = \epsilon b'$ keeping a' and b' fixed. In the limit for ϵ to zero (as we shall show in full detail below) will $a \oplus T^{-1}(b)$ reduce to $\epsilon(a' + T^{-1}b') + O(\epsilon^2)$ so that in first order of ϵ, we have that

$$(a \oplus T^{-1}(b)) \ominus (b \oplus T(a)) = \epsilon(a' + T^{-1}(b') - T^{-1}(b' + T(a'))+O(\epsilon^2) = O(\epsilon^2)$$

and we will show below that even the second order term in ϵ vanishes in case the torsion tensors are anti-podal. Notice that differentiability is a priori a metric dependent concept but as the reader may verify, this is not the case for smooth metrics and general metric compatible connections defined by scalar products on a manifold. Here, the metric locally trivializes and the connection gives subleading corrections so that the sum reduces to the ordinary one. Let us work this out in full detail here so that the reader understands that the usual manifold definitions follow from ours. Given a metric tensor, $g_{\mu\nu}$ the reader verifies that the general connection is given by

$$\widehat{\Gamma}^{\delta}{}_{\mu\nu} = \Gamma^{\delta}{}_{\mu\nu} - \frac{1}{2}\left(T_{\mu\nu}{}^{\delta} + T_{\nu\mu}{}^{\delta} - T^{\delta}{}_{\mu\nu}\right)$$

where

$$T^{\delta}{}_{\mu\nu}$$

is the Torsion tensor which is anti-symmetric in $\mu\nu$ and in the previous expression, lowering and raising of indices has been done by means of the metric tensor. Now, take two vectors V, W at x, take $\epsilon > 0$ and consider the exponential map defined by ϵV, equivalent to (x, y) and ϵW, equivlent to (x, z) respectively. Up to second order in ϵ those are given by

$$y = x + \epsilon V - \frac{\epsilon^2}{2}\widehat{\Gamma}(V, V)$$

and likewise for W. Parallel transport of ϵW along ϵV gives

$$W(y) = \epsilon W - \epsilon^2 \widehat{\Gamma}(V, W)$$

and likewise for V, W interchanged. Hence,

$$\nabla_{(x,y)}(x, z) = \left(y, x + \epsilon V - \frac{\epsilon^2}{2}\widehat{\Gamma}(V, V) + \epsilon W - \epsilon^2\widehat{\Gamma}(V, W) - \frac{\epsilon^2}{2}\widehat{\Gamma}(W, W)\right)$$

and likewise for V, W interchanged. The reader notices that $\widehat{\Gamma}(V, V)$ can be retrieved from the geodesic equation and therefore $\widehat{\Gamma}(V, W)$ from the transport equation, both in order ϵ^2. We shall make this now precise. One sees now that

$$(x, y) \oplus (x, z) = \left(x, x + \epsilon(V + W) - \frac{\epsilon^2}{2}\left(\widehat{\Gamma}(V + W, V + W) + T(V, W)\right)\right)$$

implying that

$$\pi_2\left((x, z) \oplus (x, y)\right) = x + \epsilon\left(W + V - \frac{\epsilon}{2}T(W, V)\right)$$
$$-\frac{\epsilon^2}{2}\widehat{\Gamma}\left(W + V + \frac{\epsilon}{2}T(W, V), W + V + \frac{\epsilon}{2}T(W, V)\right).$$

Hence,

$$((x, y) \oplus (x, z)) \ominus ((x, z) \oplus (x, y)) = (x, x + \epsilon^2 T(W, V) + O(\epsilon^3))$$

so, as promised, the torsion tensor emerges in leading order ϵ^2. To make this precise in our setting, consider the generalized geodesics γ_y, γ_z in arclength parametrization representing with $\gamma_y(0) = x$, $\gamma_y(1) = y$ and likewise for γ_z. Furthermore, choose any reference direction γ_q then we have that with

$$\widehat{T} := T(s) := \pi_2(T(x, \gamma_y(s), \gamma_z(s)))$$

that

$$\theta(\widehat{T}, \gamma_q), \;\; \lim_{s \to 0} \frac{d(x, T(s))}{s^2}$$

are well defined and fully capture the Torsion tensor without coordinates. In order to find the Riemann tensor, we need to be a bit more careful and expand terms up to the third power of ϵ; more in particular,

$$(x,y) := \left(x, x+\epsilon V - \frac{\epsilon^2}{2}\widehat{\Gamma}(V,V) - \frac{\epsilon^3}{6}\left((V\widehat{\Gamma})(V,V) - \widehat{\Gamma}(\widehat{\Gamma}(V,V),V) - \widehat{\Gamma}(V,\widehat{\Gamma}(V,V))\right)\right)$$

and

$$W(y) = W - \epsilon\widehat{\Gamma}(V,W) - \frac{\epsilon^2}{2}\left((V(\widehat{\Gamma}))(V,W) - \widehat{\Gamma}(\widehat{\Gamma}(V,V),W) - \widehat{\Gamma}(V,\widehat{\Gamma}(V,W))\right)$$

so that

$$\begin{aligned}\pi_2\left((x,y)\oplus(x,z)\right) = x+\epsilon V - \frac{\epsilon^2}{2}\widehat{\Gamma}(V,V) - \frac{\epsilon^3}{6}\left((V\widehat{\Gamma})(V,V) - \widehat{\Gamma}(\widehat{\Gamma}(V,V),V) - \widehat{\Gamma}(V,\widehat{\Gamma}(V,V))\right) + \\ \epsilon\left(W - \epsilon\widehat{\Gamma}(V,W) - \frac{\epsilon^2}{2}\left((V\widehat{\Gamma})(V,W) - \widehat{\Gamma}(\widehat{\Gamma}(V,V),W) - \widehat{\Gamma}(V,\widehat{\Gamma}(V,W))\right)\right) \\ -\frac{\epsilon^2}{2}\left(\widehat{\Gamma}(W,W) - \epsilon\left(\widehat{\Gamma}(\widehat{\Gamma}(V,W),W) + \widehat{\Gamma}(W,\widehat{\Gamma}(V,W)) - (V\widehat{\Gamma})(W,W)\right)\right) \\ -\frac{\epsilon^3}{6}\left((W\widehat{\Gamma})(W,W) - \widehat{\Gamma}(\widehat{\Gamma}(W,W),W) - \widehat{\Gamma}(W,\widehat{\Gamma}(W,W))\right).\end{aligned}$$

We seek now for the associated geodesic of time ϵ which maps to this endpoint; that is we have to solve for

$$Z(V,W,\epsilon) = V + W - \frac{\epsilon}{2}T(V,W) + \frac{\epsilon^2}{6}K(V,W)$$

such that

$$x + \epsilon Z - \frac{\epsilon^2}{2}\widehat{\Gamma}(Z,Z) - \frac{\epsilon^3}{6}\left((Z\widehat{\Gamma})(Z,Z) - \widehat{\Gamma}(\widehat{\Gamma}(Z,Z),Z) - \widehat{\Gamma}(Z,\widehat{\Gamma}(Z,Z))\right)$$

equals the previous expression up to third order in ϵ. This leads to

$$\begin{aligned}(W\widehat{\Gamma})(W,W) - \widehat{\Gamma}(\widehat{\Gamma}(W,W),W) - \widehat{\Gamma}(W,\widehat{\Gamma}(W,W) + (V\widehat{\Gamma})(V,V) - \widehat{\Gamma}(\widehat{\Gamma}(V,V),V) - \\ \widehat{\Gamma}(V,\widehat{\Gamma}(V,V)) + 3(V\widehat{\Gamma})(W,W) - 3\left(\widehat{\Gamma}(\widehat{\Gamma}(V,W),W) + \widehat{\Gamma}(W,\widehat{\Gamma}(V,W))\right) + \\ 3\left((V\widehat{\Gamma})(V,W) - \widehat{\Gamma}(\widehat{\Gamma}(V,V),W) - \widehat{\Gamma}(V,\widehat{\Gamma}(V,W))\right)\end{aligned}$$

must be equal to

$$\begin{aligned}-K(V,W) - \frac{3}{2}\left(\widehat{\Gamma}(V+W,T(V,W)) + \widehat{\Gamma}(T(V,W),V+W)\right) + ((V+W)\widehat{\Gamma})(V+W,V+W) - \\ \widehat{\Gamma}(\widehat{\Gamma}(V+W,V+W),V+W) - \widehat{\Gamma}(V+W,\widehat{\Gamma}(V+W,V+W))\end{aligned}$$

which leads to

$$\begin{aligned}K(V,W) = (W\widehat{\Gamma})(V,V) + (W\widehat{\Gamma})(V,W) + (W\widehat{\Gamma})(W,V) + (V\widehat{\Gamma})(W,V) - 2(V\widehat{\Gamma})(V,W) - 2(V\widehat{\Gamma})(W,W) \\ -\frac{5}{2}\widehat{\Gamma}(\widehat{\Gamma}(V,W),V) + 2\widehat{\Gamma}(\widehat{\Gamma}(V,V),W) + \frac{1}{2}\widehat{\Gamma}(\widehat{\Gamma}(W,V),V) + \\ \frac{1}{2}\widehat{\Gamma}(\widehat{\Gamma}(W,V),W) + \frac{1}{2}\widehat{\Gamma}(\widehat{\Gamma}(V,W),W) - \widehat{\Gamma}(\widehat{\Gamma}(W,W),V) + \frac{1}{2}\widehat{\Gamma}(V,\widehat{\Gamma}(V,W)) + \\ \frac{1}{2}\widehat{\Gamma}(W,\widehat{\Gamma}(V,W)) + \frac{1}{2}\widehat{\Gamma}(V,\widehat{\Gamma}(W,V)) + \\ \frac{1}{2}\widehat{\Gamma}(W,\widehat{\Gamma}(W,V)) - \widehat{\Gamma}(V,\widehat{\Gamma}(W,W)) - \widehat{\Gamma}(W,\widehat{\Gamma}(V,V)).\end{aligned}$$

The kinetic term can be rewritten as

$$2\left((W\widehat{\Gamma})(V,W)-(V\widehat{\Gamma})(W,W)\right)+\left((W\widehat{\Gamma})(V,V)-(V\widehat{\Gamma})(W,V)\right)+2\left((V\widehat{\Gamma})(W,V)-(V\widehat{\Gamma})(V,W)\right)$$
$$+\left((W\widehat{\Gamma})(W,V)-(W\widehat{\Gamma})(V,W)\right)$$

which suggests for two distinct Riemann tensors and two derivatives of torsion tensors. Further computation yields that

$$K(V,W)=2\widehat{R}(W,V)W+\widehat{R}(W,V)V+2\widehat{\nabla}_V T(W,V)+\widehat{\nabla}_W T(W,V)+\frac{1}{2}T(V,T(V,W))+\frac{1}{2}T(W,T(W,V)).$$

The reader must note here that we used the following defition of the Riemann tensor [4]

$$\widehat{R}(X,Y)Z=\widehat{\nabla}_X\widehat{\nabla}_Y Z-\widehat{\nabla}_Y\widehat{\nabla}_X Z-\widehat{\nabla}_{[X,Y]}Z;$$

Note also that $K(V,\lambda V)=0$ and the reader immediately calculates that

$$Z(S,Z(V,W,\epsilon),\epsilon)=S+V+W-\frac{\epsilon}{2}\left(T(V,W)+T(S,V)+T(S,W)\right)+$$
$$\frac{\epsilon^2}{6}(K(S,V+W)+K(V,W)+3T(S,T(V,W)))$$

and therefore

$$D(S,V,W,\epsilon):=Z(Z(S,Z(V,W,\epsilon),\epsilon),-Z(V,Z(S,W,\epsilon),\epsilon))=-\epsilon T(S,V)+$$
$$\frac{\epsilon^2}{6}\left(K(V,W)+K(S,V+W)-K(V,S+W)-K(S,W)\right)+$$
$$\frac{\epsilon^2}{6}\left(3T(S,T(V,W))-3T(V,T(S,W))+3T(S+V+W,T(S,V))\right)$$

and the expression of order $\frac{\epsilon^2}{6}$ reduces to

$$2\left(\widehat{R}(S,V)W+\widehat{R}(W,S)V+\widehat{R}(V,W)S\right)+6\widehat{R}(V,S)W+3\widehat{R}(V,S)V+3\widehat{R}(V,S)S+3\widehat{\nabla}_S T(V,S)+3\widehat{\nabla}_V T(V,S)$$

$$+\widehat{\nabla}_V T(W,S)+2\widehat{\nabla}_W T(V,S)-\widehat{\nabla}_S T(W,V)+3T(S,T(V,W))-3T(V,T(S,W))+3T(S+V+W,T(S,V))$$

In the absence of torsion, our vectorfield reduces to

$$\frac{\epsilon^2}{2}(2\widehat{R}(V,S)W+\widehat{R}(V,S)V+\widehat{R}(V,S)S).$$

In general, the reader may enjoy observing that $D(S,V,W,\epsilon)=-D(V,S,W,\epsilon)$; in order to eliminate the quadratic terms in the above expression, it is useful to consider

$$E(S,V,W,\epsilon):=D(S,V,W,\epsilon)-D(S,V,-W,\epsilon)=$$

$$\epsilon^2\left(\frac{2}{3}\left(\widehat{R}(S,V)W+\widehat{R}(W,S)V+\widehat{R}(V,W)S\right)-2\widehat{R}(S,V)W+\frac{1}{3}\widehat{\nabla}_V T(W,S)\right)$$
$$+\epsilon^2\left(-\frac{2}{3}\widehat{\nabla}_W T(S,V)+\frac{1}{3}\widehat{\nabla}_S T(V,W)+T(S,T(V,W))+T(V,T(W,S))+T(W,T(S,V))\right)$$

so that we now have a tensor! The reader immediately notices that in the absence of torsion this expresion reduces to

$$-2\epsilon^2\widehat{R}(S,V)W$$

by means of the first Bianchi identity, so we would have isolated the Riemann curvature. In general, the first Bianchi identity reads

$$\widehat{R}(S,V)W + \widehat{R}(W,S)V + \widehat{R}(V,W)S =$$

$$T(T(S,V),W)+T(T(W,S),V)+T(T(V,W),S)+\widehat{\nabla}_S T(V,W)+\widehat{\nabla}_W T(S,V)+\widehat{\nabla}_V T(W,S)$$

so that the above expression reduces to

$$\epsilon^2 \left(-2\widehat{R}(S,V)W + \widehat{\nabla}_S T(V,W) + \widehat{\nabla}_V T(W,S) + \frac{1}{3}\left(T(S,T(V,W)) + T(V,T(W,S)) + T(W,T(S,V))\right) \right).$$

In order to get rid of the torsion terms, the reader may verify that

$$\frac{1}{3}\left(E(S,V,W,\epsilon) + E(W,S,V,\epsilon) + E(V,W,S,\epsilon)\right) =$$

$$\epsilon^2 \left(T(S,T(V,W)) + T(V,T(W,S)) + T(W,T(S,V))\right)$$

using the first Bianchi identity again. So, therefore

$$\frac{8}{9}E(S,V,W,\epsilon) - \frac{1}{9}E(V,W,S,\epsilon) - \frac{1}{9}E(W,S,V,\epsilon) = \epsilon^2 \left(-2\widehat{R}(S,V)W + \widehat{\nabla}_S T(V,W) + \widehat{\nabla}_V T(W,S) \right)$$

There is no way to further reduce this and eliminate the remaining derivatives of the Torsion tensor and the reader is invited to play a bit around and consider different sum operations in order to extract those. Finally, we return to the case without torsion, which is considerably easier and we now turn the prescription into our novel language; the reader may verify that to third order in ϵ our definition of $E(S,V,W,\epsilon)$ coincides with

$$E(x,p,q,r) := [((x,p) \oplus ((x,q) \oplus (x,r))) \ominus ((x,q) \oplus ((x,p) \oplus (x,r)))] \ominus$$

$$\left[\left((x,p) \oplus ((x,q) \oplus \tilde{P}(x,r)) \right) \ominus \left((x,q) \oplus ((x,p) \oplus \tilde{P}(x,r)) \right) \right]$$

and we have applied the same limiting procedure as we did tor the torsion tensor previously. The reader may repeat that exercise and define $E(x,p,q,r)(s)$ with $s \in \mathbb{R}_+$ and show that

$$d(E(x,p,q,r)(s)) \sim 2s^3 ||\widehat{R}(S,V)W||.$$

Considering the angle with a reference direction, the entire Riemann tensor may be retrieved in a coordinate independent way. Note also that we have a very nice "arithmetic" interpretation of torsion and curvature; that is, they express the failure of $\oplus$ to be commutative and perhaps associative to some extend. In the next section, we shall abandon the case with torsion and give an entirely different prescription for the Riemann tensor. This treatment shall be more basic and rough, which may not be a bad thing given the connections constructed so far are extremely subtle. We now finish this section by some comments upon differentiability and how the usual bundle apparatus of differential geometry may be generalized to our setting.

Given that we dispose of a local notion of a (non-commutative) sum whos infinitesimal version may very well become commutative and associative as explained previously and moreover, we have a natural notion of scalar multiplication by means of our generalized exponential map which associates to a vector (x,y) a unique geodesic γ in arclength parametrization such that $\gamma(0) = x$ and $\gamma(s) = y$, then we define for any sufficiently small positive real number λ,

$$\lambda(x, y) = (x, \gamma(\lambda s))$$

and in case λ is negative we suggest

$$\lambda(x, y) := (-\lambda)\tilde{P}(x, y)$$

and the reader immediately verifies that these definitions induce the usual ones on the tangent bundle of a manifold. The reader should understand therefore, that it is natural to speak of directions at x defined by means of the geodesics (with respect tot the connection, so they don't need to be the geodesics of the metric) and that also in our general context of a non-commutative and non-linear sum meaning that

$$\lambda((x, y) \oplus (x, z)) \neq (\lambda(x, y)) \oplus (\lambda(x, z))$$

the very concept of a linearly independent and generating set of directions at x is still a well defined concept albeit I believe this does not imply that each vector can be written in a unique way by means of $\oplus$ and scalar multiplication. So, the concept of a basis is somewhat less restrictive but it is still well defined as a minimal set of independent and generating directions. The dimension is then an ordinary integer defined by the number of basis directions; these observations allow one to transport the entire cnstruction of tangent and cotangent spaces to our setting. But beware, we work very differently here as in the case of the ordinary theory; here it are the connections which determine the tangent bundle as well as its dimension, a much more intrinsic approach as the usual one where the backbone differential structure defines the connections. So, a linear functional, or covector, is defined by means of a continuous functional ω_X on the displacements (x, y) satisfying

$$\frac{1}{\epsilon}\left(\omega_X((x, z) \oplus (x, y)) - \omega_X((x, y)) - \omega((x, z))\right) = 0$$

and

$$\frac{1}{\epsilon}\left(\omega_X(\lambda(x, y)) - \lambda\omega_X((x, y))\right) = 0 \in \mathbb{R}$$

in the limit for $d(x, z) = \epsilon a,\ d(x, y) = \epsilon b$ for $a, b > 0$ constant and $\epsilon \to 0$. Note that we cannot request $\omega_X((y, z) \oplus (x, y)) = \omega_X((x, y)) + \omega((y, z))$ for finite displacements given that the sum operation allows for ambiguities non-locally. Furthermore, if ω_X were a field, then we could define it to constant meaning that

$$\omega_X(\nabla_{(x,y)}(x, z)) = \omega_X((x, z)).$$

Just as in ordinary functional analysis, we can define the weaker notions of continuity and differentiability of functions regarding convergence properties with respect to linear functions which all define semi-norms when suitably rescaled in the infinitesimal limit given by

$$||(x, y)|| := |\omega_X((x, y))|.$$

All proceeds now in a fairly trivial way: given our geodesics (with or without torsion), we have, as mentioned before directions which are endowed with a natural notion of length and angles between them. You can consider generalizations of tensors in those directions which upon suitable rescaling in the infinitesimal limit might become ordinary linear objects. We leave such developments to the reader.

3 Riemannian geometry.

In this section, we shall take a very different point of view as in the previous one; the latter was delicate and subtle and very much in line with the standard manifold treatment. Note that we have sidestepped the issue of existence of connections something which seems not totally obvious to prove and might be too delicate for practical purposes. For example, regarding hyperbolic spaces with conical singularities, it is rather obvious that no connection exists at the singular points. To give away the detail, take a couple of equilateral flat triangles (all angles having 60 degrees) and glue them together along their edges such that one has the situation where an interior vertex meets $n > 6$ triangles; in either the internal angle meaure exceeds 360 degrees. Take now any half line starting from the vertex, then it will have an angle of π with all other half lines in a range of $(n-6)2\pi$. Obviously, it is impossible for any mapping to preserve angles when it returns to a normal region where the measure of the circle equals 2π. The situation is the reverse for conical spherical spaces where no mapping towards such points exist. Nevertheless, our coarse grained notion of curvature is still able to capture the curvature around such vertex whereas local curvature fails. I invite the reader to think about this; after all, the integrability condition was together with preservation of distances by far the most important criterion. But it is not sufficient either, so maybe we should be clever enough to find a weaker condition as the preservation of angles which amounts in the manifold case to precisely that. For example, a weaker criterion would be that the angles with the direction of propagation need to be preserved as well as the angles amongst themselves as long as both angles with respect to the direction of propagation are less than π. This definition would certainly fit all path metrical spaces and coincide with the usual lore of differential geometry. This does not change anything to what we have said in the previous section, but merely generalizes the setting to which it be applied. Nevertheless, the downside of the connection theory is that in general it is impossible to give a concrete prescription something which made the Christoffel connection so powerful. There are people who think you should give an easy prescription to calculate curvature even without constructing geodesics which might be a very daunting if not impossible task for a general path metric. Now, I am someone who is very found of geodesics, which are barely manageable in a general curved Riemannian space but I also sympathize with such an idea. The least you should know, I believe are distances and the work done in this section does precisely that. The price to pay is that we cannot speak any longer of vectors, but we have to directly calculate the scalar invariants.

With this in mind, we work now on general path metric spaces (X, d). We have the following definitions:

- Alexandrov curvature: in flat Euclidean geometry, the midpoint r of a line segment $[ab]$ satisfies

$$\vec{xr} = \frac{1}{2}(\vec{xa} + \vec{xb})$$

for any x. Hence, one arrives at

$$d(x,r)^2 = \frac{1}{4}(d(x,a)^2 + d(x,b)^2 + 2d(x,a)d(x,b)\cos(\theta_x(a,b))).$$

We define the nonlocal Alexandrov curvature as

$$R(x,y,z) = \frac{-2d(x,y)^2 - 2d(x,z)^2 + d(y,z)^2 + 4d(x,r)^2}{d(x,y)^2 d(x,z)^2 \sin^2(\theta_x(y,z))}.$$

Taking again geodesic segments between (x,y) and (x,z) parametrized by ϵ and corresponding to the vectors V, W respectively then, as before

$$y = x + \epsilon V - \frac{\epsilon^2}{2}\Gamma(V,V) - \frac{\epsilon^3}{6}\left((V\Gamma)(V,V) - \Gamma(\Gamma(V,V),V) - \Gamma(V,\Gamma(V,V))\right)$$

and

$$d(x,y)^2 = \epsilon^2 h(V,V)$$

by the very property of the exponential map. To find the midpoint between y and z we solve for

$$x + \epsilon V - \frac{\epsilon^2}{2}\Gamma(V,V) - \frac{\epsilon^3}{6}\left((V\Gamma)(V,V) - \Gamma(\Gamma(V,V),V) - \Gamma(V,\Gamma(V,V))\right)$$

$$+\epsilon Z - \frac{\epsilon^2}{2}\Gamma_y(Z,Z) - \frac{\epsilon^3}{6}\left((Z\Gamma_y)(Z,Z) - \Gamma_y(\Gamma_y(Z,Z),Z) + \Gamma_y(Z,\Gamma_y(Z,Z))\right) =$$

$$x + \epsilon W - \frac{\epsilon^2}{2}\Gamma(W,W) - \frac{\epsilon^3}{6}\left((W\Gamma)(W,W) - \Gamma(\Gamma(W,W),W) - \Gamma(W,\Gamma(W,W))\right)$$

leading to

$$Z := W - V + \epsilon\left(\Gamma(V,V) - \Gamma(W,V)\right) +$$

$$\epsilon^2\left(\frac{1}{2}(V\Gamma)(V,V) - \frac{2}{3}(V\Gamma)(W,V) + \frac{1}{3}(V\Gamma)(W,W) + \frac{1}{6}(W\Gamma)(V,V) - \frac{1}{3}(W\Gamma)(W,V)\right)$$

$$+\epsilon^2\left(\frac{2}{3}\Gamma(W,\Gamma(V,V)) - \frac{1}{3}\Gamma(W,\Gamma(W,V)) - \Gamma(V,\Gamma(V,V)) + \frac{1}{3}\Gamma(V,\Gamma(W,W)) + \frac{1}{3}\Gamma(V,\Gamma(V,W))\right).$$

This implies that the midpoint has coordinates, up to third order in ϵ given by

$$r = x + \epsilon\left(\frac{V+W}{2}\right) - \frac{\epsilon^2}{2}\Gamma\left(\frac{V+W}{2},\frac{V+W}{2}\right) - \frac{\epsilon^3}{6}\left(\frac{V+W}{2}\Gamma\right)\left(\frac{V+W}{2},\frac{V+W}{2}\right)$$

$$-\frac{\epsilon^3}{6}\left(\frac{1}{2}R(V,W)V + \frac{1}{2}R(W,V)W\right)$$

$$-\frac{\epsilon^3}{6}\left(-2\Gamma\left(\Gamma\left(\frac{V+W}{2},\frac{V+W}{2}\right),\frac{V+W}{2}\right)\right)$$

This shows that

$$d(x,r)^2 = \frac{\epsilon^2}{4}\left(h(V,V) + h(W,W) + 2h(V,W)\right) - \frac{\epsilon^4}{6}h(R(V,W)V,W) + O(\epsilon)^6$$

and because

$$d(y,z)^2 = \epsilon^2\left(h(V,V) + h(W,W) - 2h(V,W)\right) + \frac{\epsilon^4}{3}h(R(V,W)V,W)$$

the Alexandrov curvature equals

$$-\frac{h(R(V,W)V,W)\epsilon^4 + \dots}{3\epsilon^4(h(V,V)h(W,W) - h(V,W)^2) + \dots}$$

which in the limit for ϵ to zero provides for $\frac{1}{3}$ times the sectional curvature. The reader might have guessed this result apart from the front factor based upon the symmetries of the Alexandrov curvature and the Riemann tensor.

- We now arrive to the notion of Riemann curvature; here, we shall ahve to take midpoints of midpoints. To understand why this is the case, consider the following expression

$$h\left(R\left(\frac{V+X}{2},\frac{W+Y}{2}\right)\frac{V+X}{2},\frac{W+Y}{2}\right)=$$

$$-\frac{1}{16}\left(h(R(V,W)V,W)+h(R(V,Y)V,Y)+h(R(X,W)X,W)+h(R(X,Y)X,Y)\right)+$$
$$\frac{1}{4}\left(h\left(R\left(\frac{V+X}{2},W\right)\frac{V+X}{2},W\right)+h\left(R\left(V,\frac{W+Y}{2}\right)V,\frac{W+Y}{2}\right)\right)$$
$$+\frac{1}{4}\left(h\left(R\left(X,\frac{W+Y}{2}\right)X,\frac{W+Y}{2}\right)+h\left(R\left(\frac{V+X}{2},Y\right)\frac{V+X}{2},Y\right)\right)$$
$$+\frac{1}{8}h(R(V,Y)X,W)+\frac{1}{8}h(R(X,Y)V,W)$$

Now, to undo the symmetrization in the curvature terms

$$\frac{1}{8}h(R(V,Y)X,W)+\frac{1}{8}h(R(X,Y)V,W)$$

note that by means of the Bianchi identity, this can be rewritten as

$$-\frac{1}{4}h(R(Y,X)V,W)+\frac{1}{8}h(R(V,X)Y,W)$$

so that we have broken the coefficient symmetry. Considering therefore the expression

$$h\left(R\left(\frac{V+X}{2},\frac{W+Y}{2}\right)\frac{V+X}{2},\frac{W+Y}{2}\right)-h\left(R\left(\frac{V+Y}{2},\frac{W+X}{2}\right)\frac{V+Y}{2},\frac{W+X}{2}\right)=$$

$$-\frac{1}{16}\left(h(R(V,Y)V,Y)+h(R(X,W)X,W)-h(R(V,X)V,X)-h(R(Y,W)Y,W)\right)+$$
$$\frac{1}{4}\left(h\left(R\left(\frac{V+X}{2},W\right)\frac{V+X}{2},W\right)+h\left(R\left(V,\frac{W+Y}{2}\right)V,\frac{W+Y}{2}\right)\right)$$
$$-\frac{1}{4}\left(h\left(R\left(\frac{V+Y}{2},W\right)\frac{V+Y}{2},W\right)+h\left(R\left(V,\frac{W+X}{2}\right)V,\frac{W+X}{2}\right)\right)+$$
$$\frac{1}{4}\left(h\left(R\left(X,\frac{W+Y}{2}\right)X,\frac{W+Y}{2}\right)+h\left(R\left(\frac{V+X}{2},Y\right)\frac{V+X}{2},Y\right)\right)$$
$$-\frac{1}{4}\left(h\left(R\left(Y,\frac{W+X}{2}\right)Y,\frac{W+X}{2}\right)+h\left(R\left(\frac{V+Y}{2},X\right)\frac{V+Y}{2},X\right)\right)$$
$$+\frac{3}{8}h(R(X,Y)V,W)$$

which is the result we needed. Denoting by $\widehat{(y,z)}$ the midpoint between y,z, we arrive at the following prescription for the curvature

$$S(x,y,z,p,q)=-8\left(S(x,\widehat{(y,p)},\widehat{(z,q)})-S(x,\widehat{(p,z)},\widehat{(y,q)})\right)$$
$$-\frac{1}{2}\left(S(x,p,z)+S(x,y,q)-S(x,p,y)-S(x,z,q)\right)$$
$$+2\left(S(x,\widehat{(p,y)},q)+S(x,\widehat{(q,z)},p)-S(x,\widehat{(p,z)},q)-S(x,\widehat{(y,q)},p)\right)$$
$$+2\left(S(x,\widehat{(z,q)},y)+S(x,\widehat{(p,y)},z)-S(x,\widehat{(q,y)},z)-S(x,\widehat{(p,z)},y)\right)$$

where

$$S(x,y,z) = -2d(x,y)^2 - 2d(x,z)^2 + d(y,z)^2 + 4d(x,\widehat{(y,z)})^2.$$

The reader verifies that all symmetries of the Riemann tensor hold, meaning

$$S(x,y,z,p,q) = -S(x,z,y,p,q) = -S(x,y,z,q,p) = S(x,p,q,y,z)$$

and

$$S(x,y,z,p,q) + S(x,p,y,z,q) + S(x,z,p,y,q) = 0.$$

This concludes our definition of the Riemann tensor.

- We shall now first define a notion of measure attached to any metric very much like the canonical volume element attached to a Riemannian metric tensor; there are several ways to proceed here. Define for any subset $S \subset X$, the outer measure of scale $\delta > 0$ and dimension d as

$$\mu^d_\delta(S) = \inf\{\sum_i r_i^d | B(x_i, r_i) \text{ is a countable cover of open balls of radius } r_i < \delta \text{ around } x_i \text{ of } S\}.$$

  Obviously, the $\mu^d_\delta(S)$ increase as δ decreases so we define

$$\mu^d(S) = \lim_{\delta \to 0} \mu^d_\delta(S).$$

  The reader verifies that this defines a measure on the Borel sets of X and morover $\mu^d(S)$ is a decreasing function of d which is infinity for $d = 0$, in case X does not consist out of a finite number of points, and 0 for $d = \infty$. Upon defining α as

$$\alpha = \inf\{d|\mu^d(X) = 0\} = \sup\{d|\mu^d(X) = \infty\}$$

  an equality which holds as the reader should prove and it is $\mu^\alpha(S)$ which is of interest. α is called the Hausdorff dimension of X. I invite the reader to "localize" this concept such that one can speak of the local dimension of a space at a point and not just a global one.

- We define now a one parameter family of "scalar products" by means of

$$g^\epsilon(x,a,b) = \frac{d(x,a)d(x,b)\cos(\theta_x(a,b))}{\epsilon^2}.$$

  The reader notices the scaling here as we shall be interested in taking the limit for ϵ to zero in a well defined way. Note that we could replace the metric compatibility of our connections in the previous section by the single demand that $g^\epsilon(x,a,b)$ is preseved under transport meaning that

$$g^\epsilon(y, \pi_2(\nabla_{(x,y)}(x,a)), \pi_2(\nabla_{(x,y)}(x,a))) = g^\epsilon(x,a,b).$$

  We want now, in full analogy with the standard treatment in differential geometry define contractions of the Riemann "tensor" in order to construct the Ricci and Einstein tensor. Note that we do not necessarily dispose of a connection here and therefore we have no addition of vectors, seen as defining a direction. Therefore, we cannot rely upon the notion of a dual tensor associated to our functionals defind in the previous section. Nevertheless, we want to construct a notion of inverse which coincides in the latter cases with the more advanced linear concept. To set the ground for this discussion, note that there exists a natural generalization of the Dirac delta function regarding the Hausdorff measure. That is, there exists a symmetric $\delta(a,b)$ such that for all continuous functions f on X, it holds that

$$\int_X d\mu^{\alpha}(a)\delta(a,b)f(a) = f(b).$$

Defining now the nonlinear dual $\widehat{a}$ of a as

$$\widehat{b}(a) = \delta(a,b)$$

we define inverses $g^{\epsilon}(x,\widehat{a},\widehat{b})$ as

$$\frac{\int_{B(x,\epsilon)} d\mu^{\alpha}(b) g^{\epsilon}(x,\widehat{a},\widehat{b}) g^{\epsilon}(x,b,c)}{\mu^{\alpha}(B(x,\epsilon))} = \delta(a,c).$$

The existence of a uniqueness of the inverse follows from the fact that the former defines a Toeplitz operator with trivial kernel. It is to say, $g^{\epsilon}(x,\widehat{a},\widehat{b})$ is the standard Green's function of the metric regarding the Hausdorff measure. This holds of course only if the measure is well behaved and we leave such details to the reader.

Prior to defining contractions with the metric tensor, remark that

$$\int_{B(x,\epsilon)} \int_{B(x,\epsilon)} d\mu^{\alpha}(b) d\mu^{\alpha}(a) g^{\epsilon}(x,\widehat{a},\widehat{b}) g^{\epsilon}(x,b,a)$$

is ill defined and requires "a point splitting" procedure to obtain a well defined answer. Concretely, we consider

$$\alpha \frac{\int_{B(x,\epsilon)} \int_{B(x,\epsilon)} d\mu^{\alpha}(b) d\mu^{\alpha}(a) \int_{B(a,\delta)} d\mu^{\alpha}(c) g^{\epsilon}(x,\widehat{c},\widehat{b}) g^{\epsilon}(x,b,a)}{\mu^{\alpha}(B(x,\epsilon))^2}$$

an expression which is independent of $\delta > 0$. Note that the dimension α has been added here to restore for the correct trace.

- The reader may now define the rescaled Riemann curvature tensors $S(x,y,z,p,q,\epsilon) := \frac{S(x,y,z,p,q)}{\epsilon^4}$ and consider contractions with $g^{\epsilon}(\widehat{y},\widehat{q})$ to define the Ricci tensor $S(x,z,p,\epsilon)$ and from thereon the Ricci scalar. We leave this as an exercise to the reader.

4 Simplicial refinements.

Simplicial metric spaces are simple examples of generalizations of Riemannian manifolds and the metric structure is fully characterized by distances $d(v_0 v_1)$ on the edges $(v_0 v_1)$. Given that one has more structure than usual, it is possible to get more close to the manifold language which is what we shall develop partially in this concluding section. We first start by defining the operators $x_w(v_0 \ldots v_i) = (w v_0 \ldots v_i)$ en $\partial_w(w v_0 \ldots v_i) = (v_0 \ldots v_i)$ in case none of the v_j equals w. In case this would be true, $\partial_w(w) = \mathbf{1}$, $x_w \mathbf{1} = (w)$ where $\mathbf{1} = ()$ equals the empty simplex. From this follows that $(x_w)^2 = 0$ as well as $(\partial_w)^2 = 0$ giving rise to a natural point Grasmann algebraic structure. One notices that $\partial = \sum_{w \in S} \partial_w$ which shows that ∂_w is the correct partial differential operator associated to the Hodge boundary operator ∂ giving rise to a natural theory of k forms. The empty simplex constitutes the identity element regarding the cross product $*$ defined by

$$(v_0 \ldots v_i) * (w_0 \ldots w_j) = (v_0 \ldots v_i w_0 \ldots w_j).$$

One simply verifies indeed that $x_w x_v = -x_v x_w$ and likewise for the operators ∂_v, ∂_w as well as

$$\partial_v x_w + x_w \partial_v = \delta(v, w)$$

giving rise to the usual Heisenberg duality where ∂_v would be the associated momentum operator. One verifies that x_w, ∂_v satisfy the fermionic Leibniz rule with respect to the $*$ product and that $\mathbf{1}$ is bosonic with respect to the action of x_w regarding $*$. Bosonic operators are then formed by considering even simplicial structures; the line segment provides one with

$$\partial_{(vw)} = \partial_w \partial_v$$

which obeys

$$\partial_{(vw)}(yz) = \delta(v, y)\delta(w, z) - \delta(v, z)\delta(w, y)$$

providing one with an oriented derivative. The simplex algebra is generated by polynomials constituting of monomials which are free products of $(v_0 \ldots v_j)$ for all $j : 0 \ldots n$; mind, the formal product does *not* equal the cross product implying that $\mathbf{1}$ is no longer equal to unity. Since on general metric spaces, bi-relations are merely characterized by means of a metric d the function algebra is limited to monomials in $(v_0 v_1)$ given that higher simplices do not provide for independent higher invariants. Assuming, furthermore, that $\mathbf{1}$ is bosonic with respect to the action of x_w, also for the free product, taking into account that ∂_v, x_w are both fermionic operators, one arrives at

$$\partial_v((w)Q) = \partial_v((x_w \mathbf{1})Q) = \partial_v x_w(\mathbf{1}Q) - \partial_v(\mathbf{1} x_w Q) = (k+1)\delta(v, w)\mathbf{1}Q - x_w(\mathbf{1}\partial_v Q) - \partial_v(\mathbf{1} x_w Q)$$

which reduces to

$$(k+1)\delta(v, w)\mathbf{1}Q - (x_w)\partial_v Q - \mathbf{1} x_w \partial_v Q - \mathbf{1}\partial_v x_w Q = \delta(v, w)\mathbf{1}Q - (x_w)\partial_v Q$$

where k is the degree of the monomial Q, which means the number of factors. This follows immediately from the Leibniz rule for bosonic operators

$$x_w \partial_v + \partial_v x_w = \delta(v, w).$$

Henceforth, akin to the $*$ product, the even simplex variables behave bosonic whereas the odd ones behave fermionic. Indeed,

$$\partial_v((wz)Q) = \partial_v((x_w(z))Q) = \partial_v(x_w((z)Q) + ((z)x_w Q)) = -x_w \partial_v((z)Q) - (z)(\partial_v x_w Q)$$

what reduces to

$$= x_w((z)\partial_v Q) - (z)(\partial_v x_w Q) = (wz)\partial_v Q.$$

The reason for introducing the formal product as a supplementary structure over the $*$ product resides in the fact that the latter allows only for linear function in the edge variables and the standard operations on real numbers would have to be recuperated in a rather different fashion by means of infinite pulverisation (excluding diagonal terms) instead of direct comparison with the simplicial line segments.

Standard derivatives are defined by means of an infinitesimal line segment

$(x-|\epsilon|, x+|\epsilon|)$ where $f(v+\epsilon, v-\epsilon)$ has been defined by means of $f(x)$. This is logical because the $v \pm \epsilon$ are fermionic and independent whereas the segments $(v-\epsilon, v+\epsilon) \sim x$ are bosonic. Note that formal products of the kind $(v-\epsilon)(v+\epsilon)$ may be further derived and that

$$\partial_x f(x) = \mathbf{L}\left[\partial_{(v-\epsilon,v+\epsilon)} f(v-\epsilon, v+\epsilon)\right]$$

whereby $\mathbf{L}$ only retains monomials depending upon the line segments. To under stand this, consider $(vw)^2$ whose (vw) derivative equals

$$2(vw) - 2(v)(w).$$

In order to define the standard bosonic multiplication operator on line segments (vw), we define

$$\widehat{x}_{(vw)} Q := x_{(vw)} x_{\mathbf{1}} Q$$

where Q is a free polynomial in the line segments (r,s) and $x_{(vw)}$ is a bosonic Leibniz operator defined by means of

$$x_{(vw)}(v_0 \ldots v_j) = (vwv_0 \ldots v_j).$$

By definition, one has that

$$x_{(vw)}(rs) = 0$$

if and only if r or s equals v, w as well as

$$(x_{(vw)} + x_{(rs)})((vw) + (rs)) = 2(vwrs)$$

which vanishes identically unless (r,s) equals the opposite side of a four simplex which is never possible for geodesics amongst other curves. For geodesics

$$\gamma(v_0 v_i) := (v_0 v_1) + (v_1 v_2) + \ldots (v_{i-1} v_i)$$

we have that

$$x_{\gamma(v_0 v_i)} := \sum_{j=1}^{i} x_{(v_{j-1} v_j)}$$

and therefore

$$x_{\gamma(v_0 v_i)} \gamma(v_0, v_i) = 0.$$

Next, we define derivatives

$$\partial_{\gamma(v_0, v_i)} := \sum_{j=1}^{i} \partial_{(v_{j-1} v_j)}$$

and consider the operator

$$\widehat{\partial}_{\gamma(v_0,v_i)} = \mathbf{L} \circ \partial_{\gamma(v_0,v_i)}$$

which satisfies

$$\widehat{\partial}_{\gamma(v_0,v_i)} \widehat{x}_{\gamma(v_0,v_i)} - \widehat{x}_{\gamma(v_0,v_i)} \widehat{\partial}_{\gamma(v_0,v_i)} = 1$$

on the function space of monomials Q of the form $(\gamma(v_0, v_i))^k$ where $k > 0$.

5 The Lorentzian case.

The question now is how to generalize the above setting to spaces equipped with a Lorentz distance. That is, we consider spaces (X, d) with a compact topology such that $d : X \times X \to \mathbb{R}^+$ is continuous and satisfies

- $d(x, y) \geq 0$ and $d(x, x) = 0$
- $d(x, y) > 0$ implies that $d(y, x) = 0$
- $d(x, y) > 0$ and $d(y, z) > 0$ implies that $d(x, z) > 0$.

As is well known, this defines a chronology relation $y \in I^+(x)$ if and only if $d(x, y) > 0$ where $I^+(x)$ is the set of all events lying in the chronological future of x. Likewise, one has the chronological past $I^-(x)$ containing all y such that $d(y, x) > 0$. Now, we assume the following regarding the partial order $\prec$ defined by $x \prec y$ if and only if $d(x, y) > 0$. That is, for any open $\mathcal{O}$ around x one has points y, z such that $y \prec x \prec z$ and $I^-(z) \cap I^+(y) \equiv A(y, z) \subset \mathcal{O}$. The sets $A(x, y)$ called the Alexandrov sets clearly define the basis for a topology and what we are saying is that the Alexandrov topology must coincide with the space topology. Looking back at the construction of the Riemann tensor, taking into account that only the κ, λ terms do not vanish, one needs either that $a, b \in I^-(c) \cap I^-(d) \cap I^+(x)$ where timelike geodesics are defined by means of a maximization instead of minimization procedure. The other way around is $c, d \in I^-(a) \cap I^-(b) \cap I^+(x)$ or two similar options with a, b, c, d to the past of x. It is well known that a Lorentzian geometry does not provide for compact neighborhoods but that one can nevertheless try to define a Hausdorff measure identically than before for Alexandrov neighborhoods. However, such measure would be direction independent which is the case for manifolds where the metric exhibits Lorentz invariance but not true for piecewise linear manifolds with conal singularities where the result may be direction dependent. Henceforth, it is much better to choose an additional Riemannian metric d and define the Lorentzian metric tensor $g^{\pm}_{\epsilon}(a, b)$ on the pair of points $(a, b) \in I^{\pm}(x)$ for which holds that $d(a, b) > 0$ or $d(b, a) > 0$ so that hyperbolic angles, replacing sine and cosine by sinh and cosh, and so forth are well defined. Calling these regions $Z^{\pm}(x)$, we may define the inverse $g^{\pm,\epsilon}(\hat{a}, \hat{b})$ by means of integration over $(B(x, \epsilon) \times B(x, \epsilon)) \cap Z^{\pm}(x)$ respectively. Herefrom, it is obvious to define the remaining contractions and Ricci tensor and scalars. It is obviuous that for general spacetimes and Riemannian metrics, the limit of ϵ to zero is independent of the choice of the latter.

6 Conclusions.

We have made first steps with developing geometry for general metric spaces as well as a natural gravitational theory defined upon it. It would be interesting to generalize this to the setting of Lorentz spaces endowed with a supplementary Riemannian notion of locality provided by a particular class of metrics.

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