A new Natario warp drive vector nY in $2D$ polar coordinates with the Hodge Star over the y-axis and a constant speed vs

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Abstract

The Natario warp drive appeared for the first time in 2001.Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.Natario defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates.In this work we present a new warp drive vector $nY = vs * (dy)$ where vs is the constant speed of the warp bubble and $*(dy)$ is the Hodge Star taken over the y-axis of motion in Polar Coordinates.Our new proposed warp drive vector also satisfies the Natario requirements for a warp drive spacetime.

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1 Introduction:

The Natario warp drive appeared for the first time in 2001.([1]).Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.

This propulsion vector nX uses the form $nX = X^i e_i$ where X^i are the shift vectors responsible for the spaceship propulsion or speed and e_i are the Canonical Basis of the Coordinates System where the shift vectors are based or placed.

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).(see Appendix A for the complete mathematical demonstration of the Natario calculations for the Hodge Star).The final form of the original Natario warp drive vector nX is given by $nX = vs * d(r\cos\theta)$ or better:

$$nX = -2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{1}$$

or

$$nX = 2v_s f \cos\theta \mathbf{e}_r - v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{2}$$

We prefer the latter expression above:

In this work we present a new warp drive vector $nY = vs * (dy)$ where vs is the constant speed of the warp bubble and $*(dy)$ is the Hodge Star taken over the y-axis of motion in Polar Coordinates.(see Appendix B for the complete mathematical demonstration of the calculations for the Hodge Star).The final form of the new Natario warp drive vector nY is given by $nY = vs * d(r\sin\theta)$ or better:

$$nY = 2v_s f \sin\theta \mathbf{e}_r + v_s(2f + rf')\cos\theta \mathbf{e}_\theta \tag{3}$$

Compare the above expression with the original Natario warp drive vector.

We adopted in this work a pedagogical language and a presentation style that perhaps will be considered as tedious,monotonous, exhaustive or extensive by experienced or seasoned readers and we designated this work for novices,newcomers,beginners or intermediate students providing in our work all the mathematical background needed to understand the process used to generate these Natario warp drive vectors nX and nY from the Hodge Star and retaining all the Natario physical features and properties.

We hope our paper is suitable to fill this proposed task.

This work was designed as a companion work to our work in [3].

2 The equation of the original Natario warp drive vector nX in $2D$ polar coordinates over the x-axis with a constant speed vs

The equation of the original Natario warp drive vector nX is given by:

$$nX = X^r e_r + X^\theta e_\theta \tag{4}$$

With the contravariant shift vector components X^{rs} and X^θ given by:(see pg 5 in [1])(see also Appendix A for details)

$$X^{rs} = 2v_s f(rs) \cos\theta \tag{5}$$

$$X^\theta = -v_s(2f(rs) + (rs)f'(rs)) \sin\theta \tag{6}$$

Considering a valid $f(rs)$ as a shape function being $f(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $f(rs) = 0$ for small rs(inside the warp bubble) while being $0 < f(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(see pg 5 in [1]):

We must demonstrate that the original Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a Natario warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(see pg 4 in [1])(see also Appendices C and D).

Natario in its warp drive uses the polar coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ only or the horizontal plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).

In a $1 + 1$ spacetime the horizontal plane we get¿:

$$nX = X^r e_r \tag{7}$$

The contravariant shift vector component X^{rs} is then:

$$X^{rs} = 2v_s f(rs) \tag{8}$$

Remember that Natario(see pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f(rs) = 0$ resulting in a $X^{rs} = 0$ and outside the bubble $f(rs) = \frac{1}{2}$ resulting in a $X^{rs} = vs$ and this illustrates the Natario definition for a warp drive spacetime.(see pg 4 in [1])(see also Appendices C and D).(see Appendix E about Polar Coordinates).

3 The equation of the new Natario warp drive vector nY in $2D$ polar coordinates over the y-axis with a constant speed vs

The equation of the new Natario warp drive vector nY is given by:

$$nY = Y^r e_r + Y^\theta e_\theta \tag{9}$$

With the contravariant shift vector components Y^{rs} and Y^θ given by:(see Appendix B for details)

$$Y^{rs} = 2v_s f(rs)\sin\theta \tag{10}$$

$$Y^\theta = v_s(2f(rs) + (rs)f'(rs))\cos\theta \tag{11}$$

Considering a valid $f(rs)$ as a shape function being $f(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $f(rs) = 0$ for small rs(inside the warp bubble) while being $0 < f(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(see pg 5 in [1]):

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nY generates a Natario warp drive spacetime if $nY = 0$ and $Y = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nY = vs(t)$ with $Y = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(see pg 4 in [1])(see also Appendices C and D).

Natario in its warp drive uses the polar coordinates rs and θ.In order to simplify our analysis we consider motion in the $y - axis$ only or the vertical plane rs where $\theta = 90$ $\sin(\theta) = 1$ and $\cos(\theta) = 0$.(see pgs 4,5 and 6 in [1]).

In a $1 + 1$ spacetime the vertical plane we get¿:

$$nY = Y^r e_r \tag{12}$$

The contravariant shift vector component Y^{rs} is then:

$$Y^{rs} = 2v_s f(rs) \tag{13}$$

Remember that we now defines the y axis as the axis of motion.Inside the bubble $f(rs) = 0$ resulting in a $Y^{rs} = 0$ and outside the bubble $f(rs) = \frac{1}{2}$ resulting in a $Y^{rs} = vs$ and this illustrates the Natario definition for a warp drive spacetime.(see pg 4 in [1])(see also Appendices C and D).(see Appendix E about Polar Coordinates).

4 Composed Natario warp drive vectors:multi-layered warp fields or perhaps a possible future transwarp drive??

In the science fiction movies of Star Trek[1] namely the Star Trek III The Search for Spock or the Star Trek VI The Undiscovered Country the starship $NCC-1701$ USS Enterprise is often described as a single warp drive starship possessing a single warp drive bubble while the starship $NX-2000$ USS Excelsior is described as a transwarp drive starship possessing multi-layered warp bubbles.According with the movies a transwarp drive with multi-layered warp bubbles allows faster speeds when compared to the speeds achieved by a single warp drive.

Now leaving the science fiction behind and concentrating ourselves in the mathematics of the warp drive vectors lets us conjecture the following possibility:multi-layered warp fields or perhaps a possible future transwarp drive:

Consider a vector A defined as $A = (A^1)_i + (A^2)_j + (A^3)_k$ with A^1,A^2 and A^3 being the contravariant components of the vector and i, j, k being the Canonical Basis of the R^3.Consider now another vector B defined as $B = (B^1)_i + (B^2)_j + (B^3)_k$ with B^1,B^2 and B^3 being the contravariant components of the vector and i, j, k being the same Canonical Basis of the R^3.

If we add the vectors A and B giving a new vector $C = A + B$ then C is defined as being $C = (C^1)_i + (C^2)_j + (C^3)_k$ with C^1,C^2 and C^3 being the contravariant components of the vector and i, j, k being the same Canonical Basis of the R^3.

According with the laws of vectorial addition the vector C is equal to $C = (A^1+B^1)_i+(A^2+B^2)_j+(A^3+B^3)_k$

Applying the same procedures of vectorial addition to the Natario warp drive vectors nX and nY defining a new composed Natario warp drive vector $nT = nX + nY$ as being:[2]

$$\mathbf{nX} \sim 2v_s f \cos\theta \mathbf{e}_r - v_s(2f + rf') \sin\theta \mathbf{e}_\theta \tag{14}$$

$$\mathbf{nY} \sim 2v_s f \sin\theta \mathbf{e}_r + v_s(2f + rf') \cos\theta \mathbf{e}_\theta \tag{15}$$

$$\mathbf{nT} \sim 2v_s f \sin\theta \mathbf{e}_r + 2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf') \cos\theta \mathbf{e}_\theta - v_s(2f + rf') \sin\theta \mathbf{e}_\theta \tag{16}$$

$$\mathbf{nT} \sim 2v_s f(\sin\theta \mathbf{e}_r + \cos\theta \mathbf{e}_r) + v_s(2f + rf')(\cos\theta \mathbf{e}_\theta - \sin\theta \mathbf{e}_\theta) \tag{17}$$

$$\mathbf{nT} \sim 2v_s f(\sin\theta + \cos\theta)\mathbf{e}_r + v_s(2f + rf')(\cos\theta - \sin\theta)\mathbf{e}_\theta \tag{18}$$

Above is depicted the Natario transwarp drive vector as an addition $nT = nX + nY$ or superposition of the Natario warp drive vectors nX and nY.A complete study of this new vector will appear in a future work.At first sight interesting results are obtained when $\theta = 45$ degrees.

[1] Paramount Pictures
[2] we chooses T for the Transwarp

5 Conclusion

In this work we introduced the new Natario warp drive vector nY using the y-axis as the main axis of motion or better:the $*(dy)$ as being the Hodge Star taken over the y-axis in Polar Coordinates.We focused ourselves in the $2D$ polar coordinates for constant speeds.

But remember that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must uses all the tridimensional $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ so there is still a work to be done:a real tridimensional $3D$ warp drive vector nY also based over the y-axis but resembling the work in [3].

The Natario warp drive is probably the best candidate(known until now) for an interstellar space travel considering the fact that a spaceship in a real superluminal spaceflight will encounter(or collide against) hazardous objects(asteroids,comets,interstellar dust and debris etc) and the Natario spacetime offers an excellent protection to the crew members as depicted in the works [8],[9],[10] and [11].

Remember also that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model so our Natario warp drive vector nY with constant speeds will have a future version encompassing variable speeds resembling the work in [3].

The application of the new Natario warp drive vector nY wether in constant or variable speeds to the ADM(Arnowitt-Dresner-Misner) formalism equations in General Relativity using the approach of MTW(Misner-Thorne-Wheeler)resembling the works [4],[5],[6] and [7] will appear in a future work.

6 Appendix A:differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vs * dx$ and $nX = vs * dx$ for a constant speed vs over the x-axis in Polar Coordinates in a R^3 space basis

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{19}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim (r \sin\theta d\varphi) \wedge dr \sim r \sin\theta (d\varphi \wedge dr) \tag{20}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r \sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{21}$$

From above we get the following results

$$dr \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{22}$$

$$rd\theta \sim r \sin\theta (d\varphi \wedge dr) \tag{23}$$

$$r \sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{24}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see eq 3.72 pg 69$(a)(b)$ in [2]):

$$*dr = r^2 \sin\theta (d\theta \wedge d\varphi) \tag{25}$$

$$*rd\theta = r \sin\theta (d\varphi \wedge dr) \tag{26}$$

$$*r \sin\theta d\varphi = r(dr \wedge d\theta) \tag{27}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [1]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta \cos\theta d\theta \wedge d\varphi + r \sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2} r^2 \sin^2\theta d\varphi\right) \tag{28}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{29}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{30}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{31}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi\wedge dr)] \tag{32}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] - [r\sin^2\theta(d\varphi\wedge dr)] \tag{33}$$

We know that the following expression holds true(see eq 3.79 pg 70$(a)(b)$ in [2]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{34}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] + [r\sin^2\theta(dr\wedge d\varphi)] \tag{35}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [1]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{36}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{37}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{38}$$

According to eq 3.90 pg 74$(a)(b)$ in [2] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{39}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{40}$$

Because and according to eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.2 pg 68$(a)(b)$ in [2]:

$$*d(\alpha+\beta) = d\alpha + d\beta \tag{41}$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{42}$$

$$*d(dx) = d(dy) = d(dz) = 0 \tag{43}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{44}$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \tag{45}$$

And then we derived again the Natario result of pg 5 in [1]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \tag{46}$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [1] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \tag{47}$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \tag{48}$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{49}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{50}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi)+2f(r)r\sin^2\theta(dr\wedge d\varphi)+r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \quad (51)$$

Comparing the above expressions with the Natario definitions of pg 4 in [1]):

$$e_r\equiv\frac{\partial}{\partial r}\sim dr\sim(rd\theta)\wedge(r\sin\theta d\varphi)\sim r^2\sin\theta(d\theta\wedge d\varphi) \quad (52)$$

$$e_\theta\equiv\frac{1}{r}\frac{\partial}{\partial\theta}\sim rd\theta\sim(r\sin\theta d\varphi)\wedge dr\sim r\sin\theta(d\varphi\wedge dr)\sim -r\sin\theta(dr\wedge d\varphi) \quad (53)$$

$$e_\varphi\equiv\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\sim r\sin\theta d\varphi\sim dr\wedge(rd\theta)\sim r(dr\wedge d\theta) \quad (54)$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta\wedge d\varphi)]+2f(r)\ sin\theta[r\sin\theta(dr\wedge d\varphi)]+f'(r)r\sin\theta[r\sin\theta(dr\wedge d\varphi)] \quad (55)$$

$$2f(r)\ cos\theta e_r-2f(r)\ sin\theta e_\theta-rf'(r)\sin\theta e_\theta \quad (56)$$

$$*d[f(r)r^2\sin^2\theta d\varphi]=2f(r)\ cos\theta e_r-[2f(r)+rf'(r)]\sin\theta e_\theta \quad (57)$$

Defining the original Natario Vector as in pg 5 in [1] with the Hodge Star operator * explicitly written :

$$nX=vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \quad (58)$$

$$nX=-vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \quad (59)$$

We can get finally the latest expressions for the original Natario Vector nX also shown in pg 5 in [1]

$$nX=2vs(t)f(r)\ cos\theta e_r-vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \quad (60)$$

$$nX=-2vs(t)f(r)\ cos\theta e_r+vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \quad (61)$$

$$nX=-2v_sf\cos\theta\mathbf{e}_r+v_s(2f+rf')\sin\theta\mathbf{e}_\theta \quad (62)$$

or

$$nX=2v_sf\cos\theta\mathbf{e}_r-v_s(2f+rf')\sin\theta\mathbf{e}_\theta \quad (63)$$

$$nX = X^r e_r + X^\theta e_\theta \tag{64}$$

$$X^{rs} = -2v_s f(rs)\cos\theta \tag{65}$$

$$X^\theta = +v_s(2f(rs) + (rs)f'(rs))\sin\theta \tag{66}$$

or

$$X^{rs} = 2v_s f(rs)\cos\theta \tag{67}$$

$$X^\theta = -v_s(2f(rs) + (rs)f'(rs))\sin\theta \tag{68}$$

7 Appendix B:differential forms,Hodge star and the mathematical demonstration of the Natario vector $nY = vs * dy$ for a constant speed vs over the y-axis in Polar Coordinates in a R^3 space basis

This appendix is also being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods we used to arrive at the final expression of the Natario Vector nY

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{69}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{70}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{71}$$

From above we get the following results

$$dr \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{72}$$

$$rd\theta \sim r\sin\theta(d\varphi \wedge dr) \tag{73}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{74}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see eq 3.72 pg 69$(a)(b)$ in [2]):

$$*dr = r^2 \sin\theta(d\theta \wedge d\varphi) \tag{75}$$

$$*rd\theta = r\sin\theta(d\varphi \wedge dr) \tag{76}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{77}$$

Look that

$$dy = d(r\sin\theta) = \sin\theta dr + r\cos\theta d\theta \tag{78}$$

Or

$$dy = d(r\sin\theta) = \sin\theta dr + \cos\theta r d\theta \tag{79}$$

Applying the Hodge Star operator * to the above expression:

$$*dy = *d(r\sin\theta) = \sin\theta(*dr) + \cos\theta(*rd\theta) \tag{80}$$

From

$$*dr = r^2\sin\theta(d\theta\wedge d\varphi) \tag{81}$$

$$*rd\theta = r\sin\theta(d\varphi\wedge dr) \tag{82}$$

We have:

$$*dy = *d(r\sin\theta) = \sin\theta[r^2\sin\theta(d\theta\wedge d\varphi)] + \cos\theta[r\sin\theta(d\varphi\wedge dr)] \tag{83}$$

$$*dy = *d(r\sin\theta) = [r^2\sin^2\theta(d\theta\wedge d\varphi)] + [r\sin\theta\cos\theta(d\varphi\wedge dr)] \tag{84}$$

$$*dy = *d(r\sin\theta) = \sin\theta e_r + \cos\theta e_\theta \tag{85}$$

Now examining the expression:

$$d\left(\frac{1}{2}\sin^2\theta d\varphi\right) \tag{86}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}\sin^2\theta d\varphi\right) \tag{87}$$

$$*d\left(\frac{1}{2}\sin^2\theta d\varphi\right) \sim \frac{1}{2}*d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta*[dd\varphi] \tag{88}$$

According to eq 3.90 pg 74$(a)(b)$ in [2] the term $\frac{1}{2}\sin^2\theta*d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}*d[(\sin^2\theta)d\varphi] \sim \frac{1}{2}2\sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{89}$$

$$\frac{1}{2} * d[(\sin^2\theta)d\varphi] \sim \frac{1}{2} 2\sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{90}$$

$$\frac{1}{2} * d[(\sin^2\theta)d\varphi] \sim \sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{91}$$

Because and according to eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.2 pg 68$(a)(b)$ in [2]:

$$*d(\alpha+\beta) = d\alpha + d\beta \tag{92}$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{93}$$

$$*d(dx) = d(dy) = d(dz) = 0 \tag{94}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{95}$$

Now examining the expression:

$$[(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] \tag{96}$$

$$[(r^2)(\tan\theta)][\sin\theta\cos\theta(d\theta\wedge d\varphi)] = [(r^2)(\frac{\sin\theta}{\cos\theta})][\sin\theta\cos\theta(d\theta\wedge d\varphi)] \tag{97}$$

$$[(r^2)(\frac{\sin\theta}{\cos\theta})][\sin\theta\cos\theta(d\theta\wedge d\varphi)] = [(r^2)][\sin^2\theta(d\theta\wedge d\varphi)] = \sin\theta e_r \tag{98}$$

Now examining the expression:

$$d\left(\frac{1}{2}r^2 d\varphi\right) \tag{99}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2 d\varphi\right) \tag{100}$$

$$*d\left(\frac{1}{2}r^2 d\varphi\right) \sim \frac{1}{2} * [d(r^2)d\varphi] + \frac{1}{2}r^2 * d[(d\varphi)] \tag{101}$$

According to eq 3.90 pg 74$(a)(b)$ in [2] the term $\frac{1}{2}r^2 * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2} * [d(r^2)d\varphi] \sim \frac{1}{2} 2r(dr\wedge d\varphi) \tag{102}$$

Because and according to eqs 3.90 and 3.91 pg $74(a)(b)$ in [2],tb 3.2 pg $68(a)(b)$ in [2]:

$$*d(\alpha+\beta) = d\alpha + d\beta \tag{103}$$

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{104}$$

$$*d(dx) = d(dy) = d(dz) = 0 \tag{105}$$

From above we can see for example that

$$*[d(r^2)d\varphi] = 2rdr \wedge d\varphi + r^2 \wedge dd\varphi = 2r(dr \wedge d\varphi) \tag{106}$$

$$\frac{1}{2} * [d(r^2)d\varphi] \sim \frac{1}{2}2r(dr \wedge d\varphi) \sim r(dr \wedge d\varphi) = r(dr \wedge d\varphi) = -r(d\varphi \wedge dr) \tag{107}$$

We know that the following expression holds true(see eq 3.79 pg $70(a)(b)$ in [2]):

$$d\varphi \wedge dr = -dr \wedge d\varphi \tag{108}$$

$$\frac{1}{2} * [d(r^2)d\varphi] \sim -r(d\varphi \wedge dr) \tag{109}$$

Now examining the expression:

$$(-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] = (-1)(\sin\theta)(\cos\theta)[-r(d\varphi \wedge dr)] \tag{110}$$

$$(-1)(\sin\theta)(\cos\theta)[-r(d\varphi \wedge dr)] = [r\sin\theta\cos\theta(d\varphi \wedge dr)] = \cos\theta e_\theta \tag{111}$$

Combining the expressions:

$$[(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] = \sin\theta e_r \tag{112}$$

and

$$(-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] = \cos\theta e_\theta \tag{113}$$

As being

$$[(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + (-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{114}$$

We obtain the same result of the Hodge Star for the y-axis

$$*dy = *d(r\sin\theta) = [r^2\sin^2\theta(d\theta \wedge d\varphi)] + [r\sin\theta\cos\theta(d\varphi \wedge dr)] \tag{115}$$

$$*dy = *d(r\sin\theta) = \sin\theta e_r + \cos\theta e_\theta \tag{116}$$

Then we have:

$$*dy = *d(r\sin\theta) = \sin\theta e_r + \cos\theta e_\theta \tag{117}$$

$$*dy = [(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + (-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{118}$$

Now using the following expression:

$$[2f(r)][(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + [2f(r) + rf'(r)](-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{119}$$

With these ones:

$$[(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] = \sin\theta e_r \tag{120}$$

$$(-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] = \cos\theta e_\theta \tag{121}$$

We have finally

$$[2f(r)][(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + [2f(r) + rf'(r)](-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{122}$$

$$[2f(r)]\sin\theta e_r + [2f(r) + rf'(r)]\cos\theta e_\theta \tag{123}$$

Defining the new Natario Vector nY with the Hodge Star operator * explicitly resolved :

$$nY = vs(t)[2f(r)]\sin\theta e_r + vs[2f(r) + rf'(r)]\cos\theta e_\theta \tag{124}$$

$$nY = 2vs(t)f(r)\ sin\theta e_r + vs(t)[2f(r) + rf'(r)]\cos\theta e_\theta \tag{125}$$

compare the new Natario Vector nY with the original Natario Vector nX

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{126}$$

$$nY = 2vs(t)f(r)\ sin\theta e_r + vs(t)[2f(r) + rf'(r)]\cos\theta e_\theta \tag{127}$$

Do they look familiar ?

$$nY = Y^r e_r + Y^\theta e_\theta \tag{128}$$

$$Y^{rs} = 2v_s f(rs)\sin\theta \tag{129}$$

$$Y^\theta = +v_s(2f(rs) + (rs)f'(rs))\cos\theta \tag{130}$$

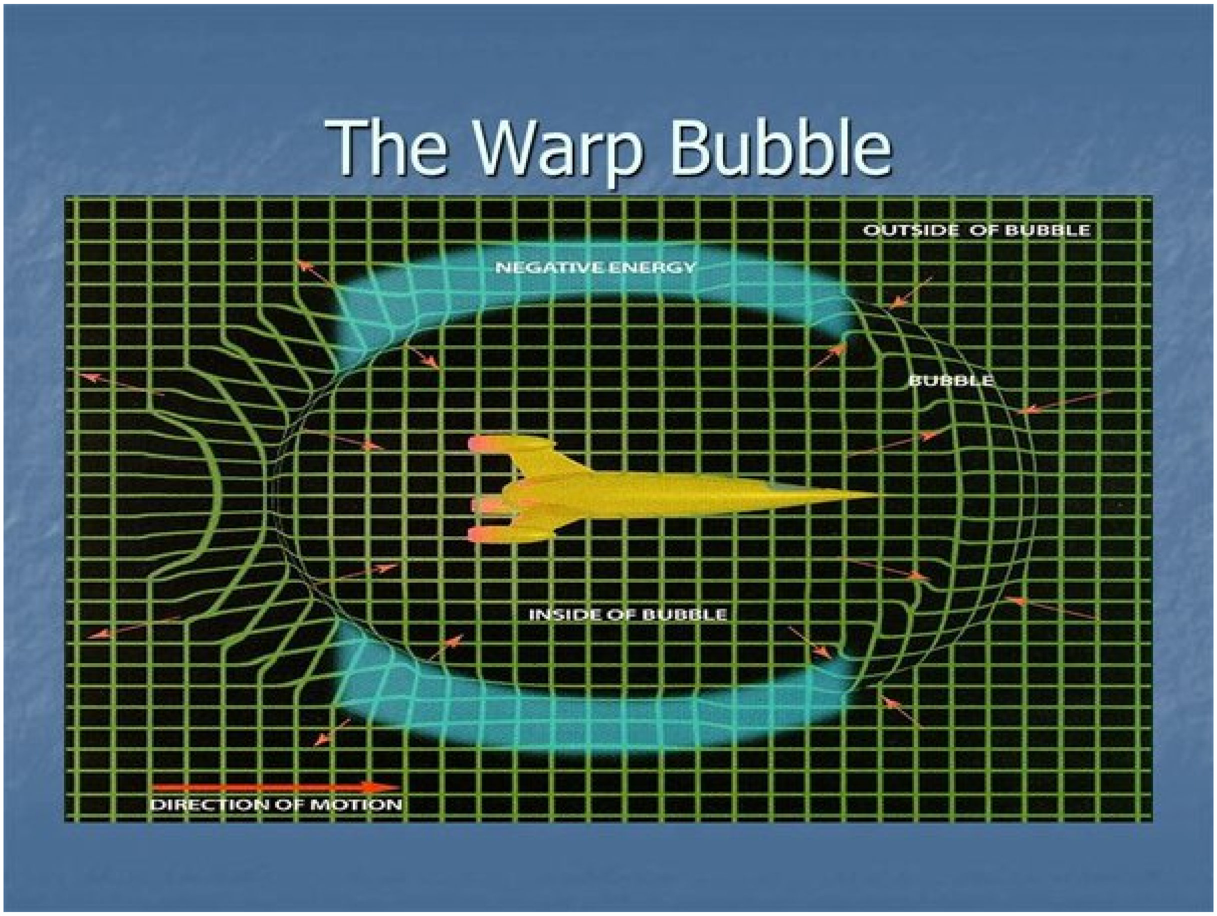

Figure 1: Artistic Presentation of a Warp Bubble.(Source:Internet)

8 Appendix C:Artistic Presentation of a Warp Bubble

In 2001 the Natario warp drive appeared.([1]).This warp drive deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [1]). Imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium.An observer at the rest in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Since the fish is at the rest inside the aquarium the fish would see the observer in the margin passing by him with a large relative speed since for the fish is the margin that moves with a large relative velocity

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])

Lets explain better this statement:Natario considered in this case a coordinates reference frame placed inside the bubble where the fish inside the aquarium or the astronaut in a spaceship inside the bubble depicted above are at the rest with respect to their local neighborhoods.Then any Natario vector must be zero inside the bubble or the aquarium or the spaceship.

On the other hand since the fish sees the margin passing by him with a large relative velocity or the astronaut would see a stationary observer in outer space outside the bubble passing by him with a large relative velocity then any Natario vector outside the bubble must have a value equal to the relative velocity seen by both the fish and the astronaut.

Considering a valid f as a Natario shape function being $f = \frac{1}{2}$ for large r(outside the warp bubble) and $f = 0$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):The walls of the bubble the Natario warped region corresponds to the distorted region in the picture depicted in this Appendix.

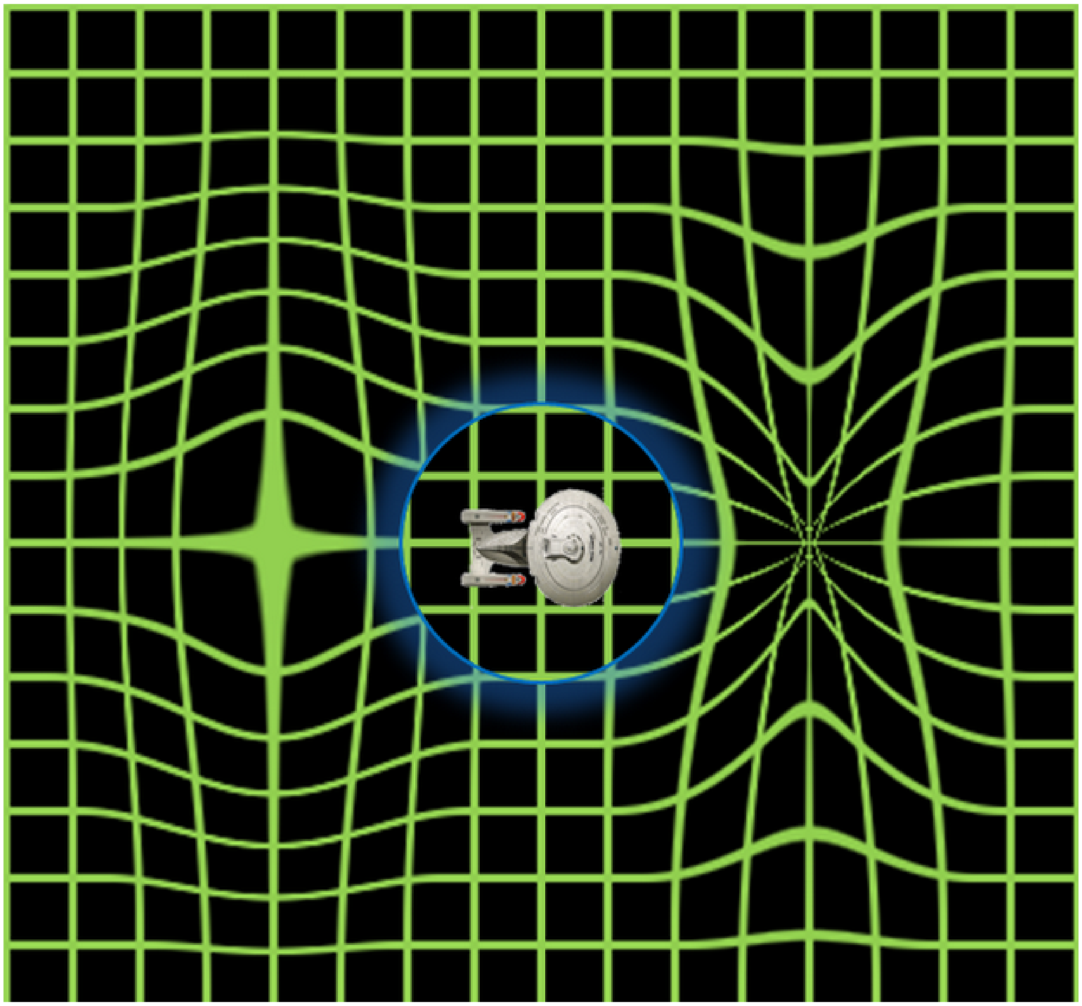

Figure 2: Another Artistic Presentation of a Warp Bubble.(Source:Internet)

9 Appendix D:Another Artistic Presentation of a Warp Bubble

Natario considered a coordinates reference frame placed inside the bubble.Now we must consider a coordinates reference frame placed outside the bubble:In this case the observer at the rest in the margin of the river would see the aquarium passing by him with a large velocity with the fish inside.Also a stationary observer at the rest in outer space would see the spaceship depicted in the picture above passing by him with a large velocity with the astronaut inside.

Now the rules originally defined by Natario are interchanged:

Since the observer in the margin and the observer in outer space are at the rest any Natario vector in this case must be zero outside the bubble.

But since the fish and the spaceship are being seen by the observer at the rest in the margin and the observer at the rest in outer space both fish and spaceship with a large velocity then the Natario vector inside the bubble must have a value equal to the velocity seen by both observers.

Considering a valid f as a Natario shape function being $f = 0$ for large r(outside the warp bubble) and $f = \frac{1}{2}$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region:The walls of the bubble the Natario warped region corresponds to the distorted region the "blue circle" in the picture depicted in this Appendix.

For an introductory explanation about remote frames outside the bubble or ship frames inside the bubble or comoving coordinates frames see pg 8 in [12].

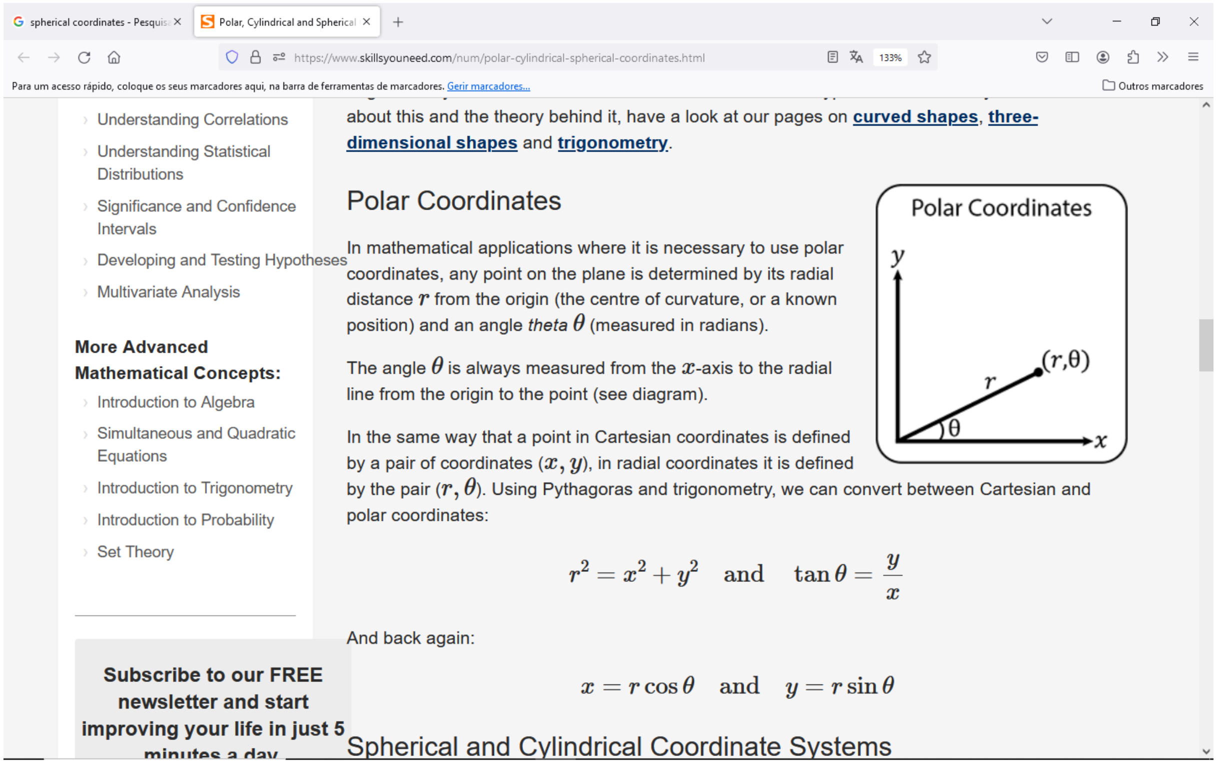

Figure 3: Polar Coordinates.(Source:Internet)

10 Appendix E:Polar Coordinates

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the **constant** speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in **Polar Coordinates** giving $nX = vs * d(r\cos\theta)$(See pg 4 in [1].(See also Appendix A for the detailed calculations).

Consequently if we set exactly what Natario did in pg 5 in [1]:

$$\mathbf{X} \sim -2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{131}$$

$$\mathbf{X} \sim 2v_s f \cos\theta \mathbf{e}_r - v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{132}$$

We prefer the latter expression:

$$nX = X^r e_r + X^\theta e_\theta \tag{133}$$

$$X^{rs} = 2v_s f \cos\theta \tag{134}$$

$$X^\theta = -v_s(2f + rf')\sin\theta \tag{135}$$

Considering a valid f as a Natario shape function being $f = \frac{1}{2}$ for large r(outside the warp bubble) and $f = 0$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(see pg 4 in [1])(see also Appendices C and D)..

Inside the bubble $f = 0$ and the Natario vector components are zero too.Outside the bubble $f = \frac{1}{2}$,$X^{rs} = v_s \cos\theta$ and $X^{\theta} = -v_s \sin\theta$.In motion over the x-axis only in the horizontal plane $X^{rs} = v_s$ because $\cos\theta = 1$ and $X^{\theta} = 0$ because $\sin\theta = 0$.

We defined a new warp drive vector $nY = vs * (dy)$ where vs is the **constant** speed of the warp bubble and $*(dy)$ is the Hodge Star taken over the y-axis of motion in **Polar Coordinates** giving $nY = vs * d(r\sin\theta)$.(See Appendix B for the detailed calculations).

Consequently if we set:

$$\mathbf{Y} \sim 2v_s f \sin\theta \mathbf{e}_r + v_s(2f + rf')\cos\theta \mathbf{e}_\theta \tag{136}$$

$$nY = Y^r e_r + Y^\theta e_\theta \tag{137}$$

$$Y^{rs} = 2v_s f \sin\theta \tag{138}$$

$$Y^\theta = v_s(2f + rf')\cos\theta \tag{139}$$

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nY generates a warp drive spacetime if $nY = 0$ and $Y = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nY = vs(t)dy$ with $Y = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(see pg 4 in [1])(see also Appendices C and D)..

Inside the bubble $f = 0$ and the Natario vector components are zero too.Outside the bubble $f = \frac{1}{2}$,$Y^{rs} = v_s \sin\theta$ and $Y^{\theta} = v_s \cos\theta$.In motion over the y-axis only in the vertical plane $Y^{rs} = v_s$ because $\sin\theta = 1$ and $Y^{\theta} = 0$ because $\cos\theta = 0$.Compare both Natario warp drive vectors given below:

$$\mathbf{X} \sim 2v_s f \cos\theta \mathbf{e}_r - v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{140}$$

$$\mathbf{Y} \sim 2v_s f \sin\theta \mathbf{e}_r + v_s(2f + rf')\cos\theta \mathbf{e}_\theta \tag{141}$$

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