On the osculating circle using complex numbers

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Abstract

In this letter a theorem is stated on the complex-valued represention of the radius vector of an osculating circle. This theorem can be used in education in mathematics and physics. To develop exercises for education a construction is presented for a class of curves characterized by a parameter. A numerical example is provided to illustrate the educational potential of the theorem and the constuction of corresponding exercises.

A curve is given by the function $Y(x) = y$. Let Z a point on the curve with coordinates (x, y). Then the coordinates (C_x, C_y) of the center C and the radius R of the osculating circle at the point Z are

$$\begin{cases} C_x := x + R_x \\ C_y := y + R_y \end{cases}, \quad R := \sqrt{R_x^2 + R_y^2}$$

where the coordinates (R_x, R_y) of the radius vector $\underline{R}$ are obtained as

$$\begin{cases} R_x := -\dfrac{1 + (Y')^2}{Y''} \cdot Y' \\ R_y := +\dfrac{1 + (Y')^2}{Y''} \end{cases}$$

1 Theorem

Let $\underline{Z} := x + Y(x) \cdot i$, $\underline{C} = C_x + C_y \cdot i$ and $\underline{R} = R_x + R_y \cdot i$ be the complex-valued represention of the point Z, the center C and the radius vector $\underline{R}$ of the osculating respectively. Then the coordinates (C_x, C_y) of the center C and the radius R of the osculating circle at the point Z obtained as

$$\underline{C} := \underline{Z} + \underline{R}, \quad R := |\underline{R}|$$

where the coordinates (R_x, R_y) of the radius vector $\underline{R}$ can be obtained as

$$\underline{R} := -\frac{|\underline{Z}'|^2 \cdot \underline{Z}'}{\underline{Z}''}$$

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Proof

$\underline{Z}(x) \coloneqq x + y \cdot i = x + Y(x) \cdot i$

$$\underline{Z}'(x) \coloneqq \frac{d}{dx}\underline{Z}(x) = \frac{d}{dx}(x + Y(x) \cdot i) = 1 + Y'(x) \cdot i$$

$$\underline{Z}''(x) \coloneqq \frac{d}{dx}\underline{Z}'(x) = \frac{d}{dx}(1 + Y'(x) \cdot i) = Y''(x) \cdot i$$

$$\underline{R} \coloneqq R_x + R_y \cdot i = -\frac{1 + (Y')^2}{Y''} \cdot Y' + \frac{1 + (Y')^2}{Y''} \cdot i = \frac{1 + (Y')^2}{Y''} \cdot (-Y' + i)$$

$$= \frac{1 + (Y')^2}{Y''} \cdot (1 + Y' \cdot i) \cdot i = -\frac{|1 + Y' \cdot i|^2}{i \cdot Y''} \cdot (1 + Y' \cdot i) = -\frac{\left|\underline{Z}'\right|^2 \cdot \underline{Z}'}{\underline{Z}''}$$

□

2 Example

Given the function of the curve $Y(x) \coloneqq \sqrt{2 \cdot p \cdot x}$ with parameter p, the coordinates $R_x = 10$ and $R_y = -5$ of the radius vector $\underline{R}$, shown in Figure 1

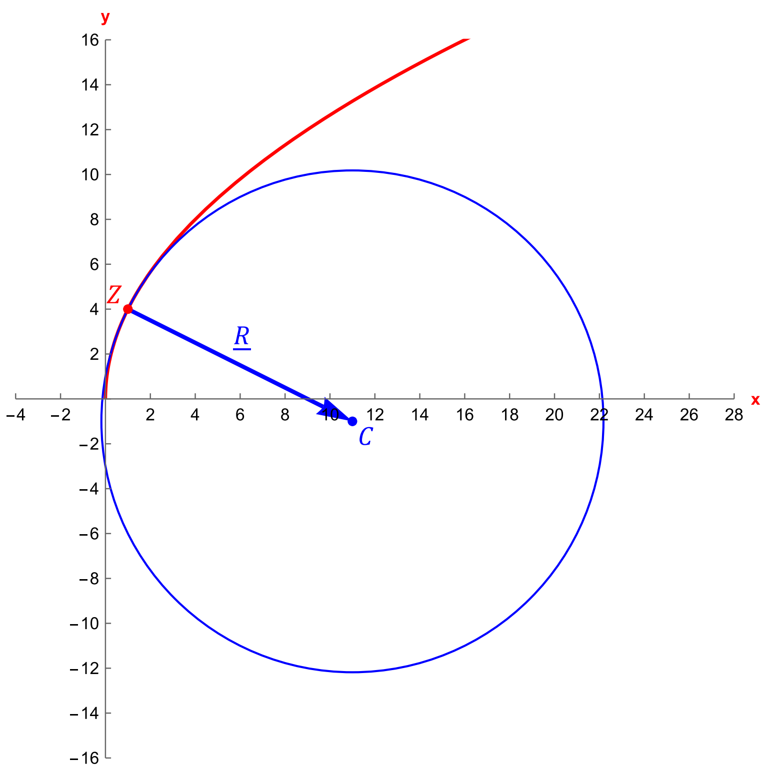

Figure 1: The osculating circle at the point Z, its center C and radius vector $\underline{R}$.

From $\tilde{y} = Y(\tilde{x}) = \sqrt{2 \cdot p \cdot \tilde{x}}$ and

$$\begin{cases} Y'(\tilde{x}) = -\dfrac{R_x}{R_y} \\ Y''(\tilde{x}) = \dfrac{R^2}{R_y^3} \end{cases}$$

the coordinates $(\tilde{x}, \tilde{y})$ of the point Z and the parameter p can be constructed as

$$\begin{cases} \tilde{x} = +\dfrac{R_x \cdot R_y^2}{2 \cdot R^2} \\ \tilde{y} = -\dfrac{R_x^2 \cdot R_y}{R^2} \\ p = \dfrac{R_x^3}{R^2} \end{cases}$$

The function of the curve then can be expressed as

$$Y(x) = \frac{R_x}{R} \cdot \sqrt{2 \cdot R_x \cdot x}$$

For the given coordinates $R_x = 10$ and $R_y = -5$ of the radius vector $\underline{R}$ we then have

the radius R, see Figure 1:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(10)^2 + (-5)^2} = 5\sqrt{5}$$

the coordinates $(\tilde{x}, \tilde{y})$ of the point Z, see Figure 1:

$$\tilde{x} = +\frac{R_x \cdot R_y^2}{2 \cdot R^2} = \frac{10 \cdot (-5)^2}{2 \cdot \left(5\sqrt{5}\right)^2} = 1$$

$$\tilde{y} = -\frac{R_x^2 \cdot R_y}{R^2} = \frac{(10)^2 \cdot (-5)}{\left(5\sqrt{5}\right)^2} = 4$$

the parameter p of the function $Y(x) = \sqrt{2 \cdot p \cdot x}$, see Figure 1:

$$p = \frac{R_x^3}{R^2} = \frac{(10)^3}{\left(5\sqrt{5}\right)^2} = 8$$

the function $Y(x)$ of the curve, see Figure 1:

$$Y(x) \coloneqq \sqrt{2 \cdot p \cdot x} = \sqrt{2 \cdot 8 \cdot x} = 4 \cdot \sqrt{x}$$

the function $Y(x)$ in the point Z:

$$Y(\tilde{x}) = Y(1) = 4 \cdot \sqrt{1} = 4 = \tilde{y}$$

the first and second derivative of the function $Y(x)$:

$$Y'(x) = \frac{2}{\sqrt{x}}$$

$$Y''(x) = \frac{-1}{x \cdot \sqrt{x}}$$

the first and second derivative of the function $Y(x)$ in the point Z:

$$Y'(\tilde{x}) = Y'(1) = \frac{2}{\sqrt{1}} = 2$$

$$Y''(\tilde{x}) = Y''(1) = \frac{-1}{1 \cdot \sqrt{1}} = -1$$

the complex-valued represention of the point Z, see Figure 1:

$$\underline{Z}(\tilde{x}) = \underline{Z}(1) = 1 + Y(1) \cdot i = 1 + 4 \cdot i = \tilde{x} + \tilde{y} \cdot i$$

and the first and second derivative of the function $\underline{Z}(x)$ in the point Z:

$$\underline{Z}'(\tilde{x}) = \underline{Z}'(1) = 1 + Y'(1) \cdot i = 1 + 2 \cdot i$$

$$\underline{Z}''(\tilde{x}) = Y''(1) \cdot i = -1 \cdot i = -i$$

Applying the theorem

The complex-valued represention of the radius vector $\underline{R}$, see Figure 1:

$$\underline{R}(\tilde{x}) = \underline{R}(1) = -\frac{\left|\underline{Z}'(1)\right|^2 \cdot \underline{Z}'(1)}{\underline{Z}''(1)} = -\frac{|1+2\cdot i|^2 \cdot (1+2\cdot i)}{-i} = 10 - 5\cdot i = R_x(1) + R_y(1)\cdot i$$

$$R_x(\tilde{x}) = R_x(1) = 10$$

$$R_y(\tilde{x}) = R_y(1) = -5$$

The radius R, see Figure 1:

$$R(\tilde{x}) = R(1) \coloneqq \left|\underline{R}(1)\right| = |10 - 5\cdot i| = 5\sqrt{5}$$

The complex-valued represention of the center C, see Figure 1:

$$\underline{C}(\tilde{x}) = \underline{C}(1) \coloneqq \underline{Z}(1) + \underline{R}(1) = (1 + 4\cdot i) + (10 - 5\cdot i) = 11 - i = C_x(1) + C_y(1)\cdot i$$

$$C_x(\tilde{x}) = C_x(1) = 11$$

$$C_y(\tilde{x}) = C_y(1) = -1$$

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