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Théorème de Fermat

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Introduction

In this article, we approach Fermat's famous theorem in an original - and above all very simple - manner. To achieve this, I will use two applications, one of which is well-known, while the other seems to be unprecedented.

1 - The first is this one, which forms the basis of Pythagoras' theorem.

$$n = \frac{\left(\frac{n+d}{2}\right)^2 - \left(\frac{n-d}{2}\right)^2}{d}$$

2 - The second, to my knowledge, is novel, and it will serve us in a pivotal manner:

For the odd integers, we can express: $n = 2*m+1 \implies f(n) = \frac{5n^2-29}{4}$

And for even integers, it will be like this: : $n = 2*m \implies f(n) = \frac{5n^2}{4}$

This function will allow us to classify all odd integers into a single category

$$f(2n+1) = 10k + 4$$

With 4 subcategories that will operate in pairs :

10k + 04 ; 10k +24 ; 10k + 44 ; 10k +54 ; 10k +74 ; 10k + 94)

We will see that each subcategory will be expressed as a sequence composed of two identical sequences, differing only in the first term. These sequences will enable us to generate similar sequences that will classify the natural numbers forming the Pythagorean triplets.
PS: I do not need all these developments for this first article.

- And it will allow us to classify odd integers into only two categories:

$$f(2n) = 10k + 5$$
$$f(2n) = 10k$$

Let us nonetheless highlight this first necessary condition for Pythagorean triplets:

The difference between the images of the odd elements of the triplets can never be 10k + 5
It can only be expressed in the form of 10K
And thus the odd elements of the triplets must always be expressed as: 4*k

II - Pythagorean Triples

1 - General Formula

PS: One could reconstruct all the triples starting from this observation for the even element of the triples, but for now, let's stick to this well-known formula.

$$4 n^2 m^2 = (n^2 + m^2)^2 - (n^2 - m^2)^2$$

Or, at the most, to this one.

$$b^2 - a^2 == c^2$$

with :

$$a = 2 (1 + m) (2 k + m)$$
$$b = 1 + 4 k^2 + 2 m + 4 k m + 2 m^2$$
$$c = (-1 + 2 k) (1 + 2 k + 2 m)$$

Hence the following table provides an overview of the triplets, and will serve as an illustration for our demonstration.

```
{{{3, 4, 5}, {15, 8, 17}, {35, 12, 37}, {63, 16, 65}, {99, 20, 101}, {143, 24, 145}}, etc.
  {{5, 12, 13}, {21, 20, 29}, {45, 28, 53}, {77, 36, 85}, {117, 44, 125}, {165, 52, 173}}, etc.
  {{7, 24, 25}, {27, 36, 45}, {55, 48, 73}, {91, 60, 109}, {135, 72, 153}, {187, 84, 205}}, etc.
  {{9, 40, 41}, {33, 56, 65}, {65, 72, 97}, {105, 88, 137}, {153, 104, 185}, {209, 120, 241}}, etc.
  {{11, 60, 61}, {39, 80, 89}, {75, 100, 125}, {119, 120, 169}, {171, 140, 221}, {231, 160, 281}}}, etc.
 etc.
```

_Important Note To find all possible triplets, it will suffice to multiply each term by the same integer, whether even or odd.
However, they can also be generated in the following manner:

$$n = x * y$$

We would then have:

$$y * x^2 = \frac{a^2 - b^2}{y}$$

Which gives us this formula for multiple triplets:

$$y x^2 = \frac{\left(\frac{y*x^2+y}{2}\right)^2 - \left(\frac{y*x^2-y}{2}\right)^2}{y}$$

NB: Let us recall that by simplifying by y, we obtain:

$$x^2 = \left(\frac{x^2+1}{2}\right)^2 - \left(\frac{x^2-1}{2}\right)^2$$

In other words:

$$x^2 = x^2 * 1^2$$

1 - the first transformation

Let the Pythagorean triplet be:

(c , a , b)

$$f\,(c) = \frac{5\,c^2-29}{4}$$
$$f\,(b) = \frac{5\,b^2-29}{4}$$
$$f\,(a) = \frac{5\,a^2}{4}$$

Note, the following tables will be central to our demonstration.

```
Table[{c, a, b}, {m, 0, 5}, {k, 1, 5}]
table
Table[{F, A, B}, {m, 0, 5}, {k, 1, 5}]
table
```

```
{{{3, 4, 5}, {15, 8, 17}, {35, 12, 37}, {63, 16, 65}, {99, 20, 101}},
 {{5, 12, 13}, {21, 20, 29}, {45, 28, 53}, {77, 36, 85}, {117, 44, 125}},
 {{7, 24, 25}, {27, 36, 45}, {55, 48, 73}, {91, 60, 109}, {135, 72, 153}},
 {{9, 40, 41}, {33, 56, 65}, {65, 72, 97}, {105, 88, 137}, {153, 104, 185}},
 {{11, 60, 61}, {39, 80, 89}, {75, 100, 125}, {119, 120, 169}, {171, 140, 221}},
 {{13, 84, 85}, {45, 108, 117}, {85, 132, 157}, {133, 156, 205}, {189, 180, 261}}}
```

```
{{{4, 20, 24}, {274, 80, 354}, {1524, 180, 1704}, {4954, 320, 5274}, {12 244, 500, 12 744}},
 {{24, 180, 204}, {544, 500, 1044}, {2524, 980, 3504}, {7404, 1620, 9024}, {17 104, 2420, 19 524}},
 {{54, 720, 774}, {904, 1620, 2524}, {3774, 2880, 6654}, {10 344, 4500, 14 844},
  {22 774, 6480, 29 254}}, {{94, 2000, 2094}, {1354, 3920, 5274},
  {5274, 6480, 11 754}, {13 774, 9680, 23 454}, {29 254, 13 520, 42 774}},
 {{144, 4500, 4644}, {1894, 8000, 9894}, {7024, 12 500, 19 524}, {17 694, 18 000, 35 694},
  {36 544, 24 500, 61 044}}, {{204, 8820, 9024}, {2524, 14 580, 17 104},
  {9024, 21 780, 30 804}, {22 104, 30 420, 52 524}, {44 644, 40 500, 85 144}}}
```

First, let's highlight this peculiarity of squares: their image by function f always yields this (in one way or another)

{2**74**, 80, 3**54**}
{15**24**, 180, 17**04**}
{122**44**, 500, 127**44**}
{1989**94**,2000,2009**94**}

This can be easily demonstrated. The odd integers raised to the power of 1:

(1, 3, 5, 7, 9)

which results in this for the squares:

(1 , 9, 5, 9, 1)

Or this for even powers that are multiples of 4:

(1 , 5)

II – Fermat's Theorem

1 - Study of the fourth power 4

$$a^4 - b^4 == c^4$$

Is it possible to have this equality with a, b, and c being natural numbers?

To answer this question, let us proceed as follows:

$$(c^2)^2 == (a^2)^2 - (b^2)^2$$

CWhich brings us back to Pythagorean triplets.
Let us first recall that there is no Pythagorean triplet that cannot be expressed using the formulas mentioned earlier. The proof is quite straightforward, and I will present it in the appendices if necessary.

PS: I know that the impossibility of quadruples can be demonstrated using the above formulas.
But for now, let us stick to the logistic function I mentioned earlier:

$$f(n) = \frac{5\,n^2-29}{4}$$

Let (c, a, b) be a Pythagorean triplet. Let's construct a quadruplet:

$$f\,(c) = \frac{5\,c^4-29}{4}$$
$$f\,(b) = \frac{5\,b^4-29}{4}$$
$$f\,(a) = \frac{5\,a^4}{4}$$

This will result in the following schematic:

```
Table[{F1, A1, B1}, {m, 0, 5}, {k, 1, 5}]
table

{{{94, 320, 774}, {63 274, 5120, 104 394}, {1 875 774, 25 920, 2 342 694},
  {19 691 194, 81 920, 22 313 274}, {120 074 494, 200 000, 130 075 494}},
 {{774, 25 920, 35 694}, {243 094, 200 000, 884 094}, {5 125 774, 768 320, 9 863 094},
  {43 941 294, 2 099 520, 65 250 774}, {234 235 894, 4 685 120, 305 175 774}},
 {{2994, 414 720, 488 274}, {664 294, 2 099 520, 5 125 774}, {11 438 274, 6 635 520, 35 497 794},
  {85 718 694, 16 200 000, 176 447 694}, {415 188 274, 33 592 320, 684 976 594}},
 {{8194, 3 200 000, 3 532 194}, {1 482 394, 12 293 120, 22 313 274}, {22 313 274, 33 592 320, 110 661 594},
  {151 938 274, 74 961 920, 440 344 194}, {684 976 594, 146 232 320, 1 464 188 274}},
 {{18 294, 16 200 000, 17 307 294}, {2 891 794, 51 200 000, 78 427 794},
  {39 550 774, 125 000 000, 305 175 774}, {250 667 394, 259 200 000, 1 019 663 394},
  {1 068 795 094, 480 200 000, 2 981 804 094}}, {{35 694, 62 233 920, 65 250 774},
  {5 125 774, 170 061 120, 234 235 894}, {65 250 774, 379 494 720, 759 466 494},
  {391 125 894, 740 301 120, 2 207 625 774}, {1 594 987 294, 1 312 200 000, 5 800 588 294}}}
```

First, let's note that in the vast majority of cases, the triplets exhibit this configuration (in one direction or the other)

{63274, 5120, 104394}

This configuration is not among the possibilities for triplets raised to the power of 2. Here too, it’s very easy to demonstrate.

However, occasionally, we encounter this problematic situation because it’s also found in pseudo-triplets of power 4.

{12007449 4, 200000, 13007549 4}

Note that this occurs in cases where the natural number a (even) is written in this way:

a = 10k

lTherefore, we should question whether there are, or are not, power 4 triplets among those we have identified at power 2.

Let’s first highlight that power 2 is found in triplets of the form.

$$(10 k_1 + 1 , 10k , 10 k_2 + 9)$$

ainsi que $(10 k_1 + 9 , 10k , 10 k_2 + 1)$

But it is primarily the following case that interests us for the power of 4

$$(10 k_1 + 1 , 10k , 10 k_2 + 1)$$

Let us recall that: : $a^4 - b^4 == c^4$, is also written as $(c^2)^2 == (a^2)^2 - (b^2)^2$

RLet us also recall the table of endings for the squares of all odd integers : (1, 9, 5, 9, 1)

We will Therefore need to eliminate the cases **3 * 7**

to retain only the following two cases:

9 * 9

1 * 1

That said, let’s move on to the demonstration:

We will show that in the Pythagorean triplet written as: : $(10 k_1 + 1 , 10k , 10 k_2 + 1)$, the first term can never be a perfect square. Referring back to the formula mentioned above, this term is expressed as:

$$(n^2 - m^2) = (n - m) (n + m)$$

OTo find an integer ending in 1, the product of (n - m) and (n + m) must necessarily occur between two integers both ending in either 9 or 1.

Thus, we will have the following form: (a and b being integers)

$$(n^2 - m^2) = (10 a + 1) (10 b + 1)$$

or

$$(n^2 - m^2) = (10 a + 9) (10 b + 9)$$

And this is where the crux of the proof lies:

In certain instances, the function f of one or the other of these numbers yields a number ending in **94**, whereas in other scenarios, it ends in **44**.

Thus, for example, consider the following examples:

$(11, 60, 61) \Longrightarrow 44$
$(171, 140, 221) \Longrightarrow 44$
$(231, 160, 281) \Longrightarrow 94$
$(551, 240, 601) \Longrightarrow 94$
$(651, 260, 701) \Longrightarrow 44$
etc.

Now, the explanation is quite straightforward, although it may be somewhat laborious as will be detailed in the appendix.

Here it is:

The scenarios in which the termination **44** occurs must satisfy this condition:

2n*m = 2* (2k +1) * 10

While the cases where we find the ending 94 must fulfill this condition:

2n*m = 2* (4k) * 10

and

n = (10 *b* + 1)

m = (10 *a* + 1)

which ultimately gives us:

$$n + m = 4*(5k + 4) + 5$$
$$n - m = 4*(5k + 4) - 5$$

This allows us to conclude as follows:

$$(n + m) - (n - m) = 10$$

This allows us to write:

$$n - m = (10k + 1)$$
$$n + m = (10k + 11)$$

And definitively conclude that:

(n + m) * (n - m) can never be a perfect square

Quick proof:

For this product to be a perfect square, it would be necessary that (10k + 11) = (10k+1) * x^2

We arrive at this:

$$k = 1 - \frac{10}{-1+x^2}$$

k can in no way be a natural number.

2 - Study of Even Powers Greater Than 2

We will see that through the function f(n), all even powers produce a situation identical to that of power 4. As in this example with power 8

```
F2 = (5 * c^8 - 29) / 4;
B2 = (5 * b^8 - 29) / 4;
A2 = (5 * a^8) / 4;

Table[{F2, A2, B2}, {m, 0, 5}, {k, 1, 5}]
table
```

```
Out[125]= {{{8194, 81 920, 488 274}, {3 203 613 274, 20 971 520, 8 719 696 794},
   {2 814 844 238 274, 537 477 120, 4 390 599 317 394},
   {310 194 725 334 394, 5 368 709 120, 398 306 016 113 274},
   {11 534 308 680 348 994, 32 000 000 000, 13 535 708 820 350 994}},
  {{488 274, 537 477 120, 1 019 663 394}, {47 278 574 194, 32 000 000 000, 625 308 016 194},
   {21 018 906 738 274, 472 252 497 920, 77 824 613 014 194},
   {1 544 670 364 434 594, 3 526 387 384 320, 3 406 131 562 988 274},
   {43 893 165 947 519 794, 17 560 279 531 520, 74 505 805 969 238 274}},
  {{7 205 994, 137 594 142 720, 190 734 863 274},
   {353 036 920 594, 3 526 387 384 320, 21 018 906 738 274},
```

The proof is trivial: powers greater than 2 are ultimately powers of 2.
Therefore, we will inevitably have the same result:

The impossibility of a Pythagorean triplet for even powers greater than 2.

3 - Étude de la puissance 3

Is it possible to have a triplet (b, a, c) such that:

$$a^3 = c^3 - b^3$$

Here again, the function f(n) will be of great utility to us. Hence the following table:

```
b = 2 * r + 1;
a = 2 * s;
Table[{b, (5 * b^3 - 29) / 2} , {r, 1, 10}]
table
Table[{a, (5 * a^3) / 2}, {s, 1, 20}]
table
```

```
Out[156]= {{3, 53}, {5, 298}, {7, 843}, {9, 1808}, {11, 3313}, {13, 5478},
  {15, 8423}, {17, 12268}, {19, 17133}, {21, 23138}}

Out[157]= {{2, 20}, {4, 160}, {6, 540}, {8, 1280},
  {10, 2500}, {12, 4320}, {14, 6860}, {16, 10240},
  {18, 14580}, {20, 20000}, {22, 26620}, {24, 34560},
  {26, 43940}, {28, 54880}, {30, 67500}, {32, 81920},
  {34, 98260}, {36, 116640}, {38, 137180}, {40, 160000}}
```

Let us note the following:

f (b) = 10*k

It would then be necessary for f(b) and f(c) to both be:

- either in the form of: **10*r + 3**
- or in the form of: **10*s + 8**

And therefore, all the odd integers will belong to one of the two following sequences.

Suite n°1 ⟹ f (n) = 3 + 4* k

Suite n°2 ⟹ f (n) = 5 + 4* k

Our triplets must necessarily be written as follows:

{ (3 + 4* k1) , 10*k , (3 + 4* k2)}

{ (3 + 4* k1) , 10*k , (3 + 4* k2)}

We will immediately demonstrate that this is impossible.

Consider two odd numbers a and c such that:

3 + 4* k2 = 3 + 4*(k1 + r)

And calculate:

$$(3 + 4\,k2)^2 - (3 + 4\,k1)^2 = 8\,r\,(3 + 4\,k + 2\,r)$$

For it to be a power of 2, it would need to:

$$(3+4\,k+2\,r) = 2\,r\,s^2$$

Which gives us:

$$k = \frac{1}{4}\left(-3 - 2\,r + 2\,r\,s^2\right)$$

$(-3 - 2\,r + 2\,r\,s^2)$ being odd, k cannot in any case be a natural integer

It is therefore impossible to have a Pythagorean triplet for the power of 3

2 - Study of Odd Powers Greater Than 3

LOnce again, the behavior of powers greater than 3 is identical to the power of 3
As demonstrated by this table:

```
Table[{n, (5*n^3 - 29) / 2}, {m, 1, 5}]
table
Table[{n, (5*n^5 - 29) / 2}, {m, 1, 5}]
table
Table[{n, (5*n^7 - 29) / 2}, {m, 1, 5}]
table
Table[{n, (5*n^9 - 29) / 2}, {m, 1, 10}]
table
Table[{n, (5*n^11 - 29) / 2}, {m, 1, 5}]
table
```

```
Out[12]= {{3, 53}, {5, 298}, {7, 843}, {9, 1808}, {11, 3313}}

Out[13]= {{3, 593}, {5, 7798}, {7, 42 003}, {9, 147 608}, {11, 402 613}}

Out[14]= {{3, 5453}, {5, 195 298}, {7, 2 058 843}, {9, 11 957 408}, {11, 48 717 913}}

Out[15]= {{3, 49 193}, {5, 4 882 798}, {7, 100 884 003}, {9, 968 551 208},
          {11, 5 894 869 213}, {13, 26 511 248 418}, {15, 96 108 398 423},
          {17, 296 469 691 228}, {19, 806 719 244 433}, {21, 1 985 700 116 438}}

Out[16]= {{3, 442 853}, {5, 122 070 298},
          {7, 4 943 316 843}, {9, 78 452 649 008}, {11, 713 279 176 513}}
```

The proof is straightforward: the endings of odd powers are nearly identical, and notably, they reproduce in alternating sequences:

Suite n°1 $\Longrightarrow$ (1, 5, 9, etc.)

Suite n°2 $\Longrightarrow$ (3, 7, 11, etc.)

General Conclusion:

Therefore, there is no possibility of having a Pythagorean triplet,

- both for even powers greater than 2
- and for odd powers.

Mustapha Kharmoudi, Besançon le 28 févier 2025