The Full Dedekind Cut and the Key to Leibnizian Mathematics

Adriaan van der Walt

Vanderwalttrust@Gmail.com

ABSTRACT

The aim of this document is to facilitate and motivate the reading of the document LEIBNIZIAN MATHEMATICS by investigating a compelling reason for introducing Leibnizian Mathematics. This document also motivates the extension of the *Dedekind Cut* to the *Full Dedekind Cut* and analyses some consequences.

First the relevant abstractions about Space shared by all are stated, which are then followed by stating the relevant basic assumptions of Abstract Mathematics. A tool is then developed that enables the identification and analysis of the consequences of these assumptions. This exposes the root motivations for, and the fundamental properties of, the tenets of Abstract Mathematics. The most consequential of these, in the present context, is the result that the total length of countable many points is zero. More than countable many points are therefore required to form a line of non-zero length. Also, that **countable many points can be added to or removed from a line without changing the length of the line** (this consequence is contrary to the current paradigm of Mathematics). The latter necessitated the introduction of the Full Dedekind Cut to preserve the real line and hence Euclidean Topology and Lebesgue theory.

The concepts "infinitesimal'' and "infinitesimal number" are introduced followed by a Riemann sum that results in a contradiction in Euclidean Mathematics by showing that there exists an example where countable many points form a line of length one.

Possible causes for this contradiction are discussed and it is concluded that the Riemann integral does not fit naturally into Abstract Mathematics, but that a second continuous model for space that leads to a different model for Mathematics, called Leibnizian Mathematics, must be developed to augment Abstract Mathematics. This model resolves the contradiction, accommodates the Riemann integral in a natural way and expands the paradigm of Mathematics.

A short list is appended describing the difference in meaning that some words have and the difference in the properties that they describe when used in different models.

0. ABSTRACT MATHEMATICS AND THE FULL DEDEKIND CUT

0.1.1 PARADIGMS

From the web[1]:

A paradigm is a framework containing the basic assumptions, ways of thinking, and methodology that are commonly accepted by members of a scientific community.

The aim of this treatise is to extend and to adapt the current paradigm of Mathematics in a non-trivial way by invoking the genius of Leibniz. To do this it is

[1] PARADIGM Definition & Meaning | Dictionary.com

necessary to make statements and study concepts that are unacceptable (and therefore wrong) in the current paradigm of Mathematics but may be valid in an extended paradigm.

This remark is best clarified by noting the possible models for Geometry that the paradigm of Geometry supports. Three different assumptions about the properties of Geometrical Space are made and statements that are true in one model are not necessarily true in the others. However, all three are accepted as valid Geometry.

Let a straight line and a point not on that line be given.

- **Euclidean Geometry** results if it is assumed that one, and only one, line parallel to the given line can be drawn through the given point. In Euclidean geometry the sum of the interior angles of a triangle is exactly 180°.
- **Riemannian Geometry** results if it is assumed that no line parallel to the given line can be drawn through the given point. In Riemannian Geometry the sum of the interior angles of a triangle is more than 180°.
- **Hyperbolic Geometry** (Lobachevsky Geometry) results if it is assumed that more than one line parallel to the given line can be drawn through the given point. In Hyperbolic Geometry the sum of the interior angles of a triangle is less than 180°.

0.1.2 **UNIVERSAL ASSUMPTIONS**

While young we are all exposed to the same physical environment. The relevant conclusions we form are therefore generally the same for all people. Some of these can be formally stated as:

Universal assumptions about Space, held as true for all Mathematics:

- **U1**: Solids exist and extend in three dimensions.
- **U2**: A surface is the interface between two abutting solids and extends in two dimensions.
- **U3**: A line is the interface between two intersecting surfaces and extends in one dimension.
- **U4**: A point is the interface between two intersecting lines. It is a place in space and do not extend in any direction.

0.1.3 **EUCLIDEAN ASSUMPTIONS**

About two and a half millennia ago the basic assumptions of Mathematics were decided on in Greece. These became part of the paradigm of Abstract Mathematics (Here called Euclidean Mathematics) and spell out the properties assigned to solids, surfaces and lines.

Assumptions about Space, particular to Euclidean Mathematics:

- **E1**: Axiom of Euclid: Points exist and are pieces of space with zero[2] extent.
- **E2**: A solid is a lump of points.
- **E3**: A surface is a single layer of points.
- **E4**: A line is a string of points.
- **E5**: The Real line: There is an order-preserving one to one mapping of the real numbers onto the points of a line.
- **E6**: Space is complete: A Cauchy sequence of points will always have a point as limit and a nested set of lines of which the lengths converge to zero will also always have a point as limit.

0.1.4 **ESSENTIAL ANATOMY[3] OF EUCLIDEAN MATHEMATICS**

A tool is needed to cut – and to lay bare - the heart of Euclidean Mathematics:

Let Z be an index set and let

$$\{A_\alpha | \alpha \epsilon Z\}$$

be that subset of the set of all points onto which Z is mapped one to one. Let

$$D = \sum_{\propto \epsilon Z} d(A_\alpha)$$

Where $d(A_\alpha)$ is the maximum diameter of the point A_α and therefore $d(A_\alpha) = 0$ for all α according to assumption **E1**.

Note that the cardinality of the set Z determines the number of points for which the sum is formed.

EA1) Z is a **finite** set, say Z = {1; 2; 3 …. n}:

then

[2] The Greeks did not have the number '0', but the paradoxes of Zeno revealed this as the intended meaning.

[3] The Greek word "Anatomy" translates to "Dissection"

$$D = D_n = \sum_{i=1}^{i=n} d(A_i) = n0 = 0$$

In this case D is a finite sum of zeros so that D=0.

EA2) Z is a **countable** set, say Z = {1; 2; 3; …… }:

In this case **continuity** requires D to be the limit, as n tends to infinity, of the partial sums to n terms; and these are all zero. Hence

$$D = \lim_{n\to\infty} D_n = 0.$$

Note that this implies that the total length of countable many points must always be zero.

This has a few additional compelling consequences for Euclidean Mathematics:

EA3) Firstly: If the points form a line, then D must be the length of the line and hence it must be larger than zero. In this case the cardinality of the set Z must therefore necessarily be **more than countable** otherwise the length would perforce be zero. This necessitates the introduction of the concept ‘more than countable’ into Euclidean Mathematics. In this model there must therefore also exist more than countable many points and thus there must exist more than countable many real numbers to form the real line.

EA4) As in EA2) above, D would **always** be zero whenever the sum is obtained by taking the limit of a finite or countable number of zeros. Therefore, to get D to be larger than zero no limits may be taken: thus, in this model more than countable many additions (or “actions”) must be performed one by one until the non-zero sum is complete. This introduces the essence of the axiom of choice into Euclidean Mathematics[i]. It also validates the assumption that in this model an irrational number is an infinite string of digits[ii] - with “infinite” as defined in EA5 - because it is possible to determine all the required digits down to the very last using this assumption. All this is obviously not possible in perceived reality, hence the name “Abstract Mathematics”.

EA5) In this model the words ‘infinite’ and ‘infinity’ both mean ‘an integer larger than all Natural Numbers’.

EA6) Completeness is a multifaceted concept.

- In the difference-topology a Cauchy sequence of numbers always has one and only one real number as limit.

- A set of nested intervals of which the lengths converge to zero has a point as limit - but this needs further clarification below.

0.2. EXTENSION OF THE NOMENCLATURE

According to U4 a place in space is indicated by where two lines cross. But a cross shows the place where four half-lines end. Henceforth a **place in space** (the 'point' named in U4) will be referred to as an **Endpoint of a line** or simply an **Endpoint** and the terms endpoint and place in space will be equivalent.

Hence the endpoint of a line is not a point in the Euclidean sense but is a property of a line and indicates the place in space where the line ends.

HOWEVER: Because the total length of countable many points is zero, countable many points in the Euclidean sense can be added to or removed at the endpoints of a line without changing the length of the line or the place in space where the line ends.

This implies that countable many points coincide at an endpoint of a line.

Note that this is incompatible with the existence of open and closed intervals. This complication is known to Mathematics (albeit somewhat disguised) where, to be able to define open and closed intervals, the Dedekind Cut is used to separate a single point from the end of a line. (This is required for the validity of the Lebesgue theory and Euclidean Topology.)

The fact that countable many points coincide at the endpoint of a line on the axis means that they all represent the same real number because a Cauchy sequence of rational numbers that converges to any single one converges to all. Therefore, all these points share the same equivalence class of Cauchy sequences of rational numbers converging to them and thus all these points represent the same infinite decimal fraction (or real number).

This invalidates the real line because a single number now maps onto countable many points: therefore, the Dedekind Cut must be extended to cover this situation as well:

The full Dedekind Cut[4]

Any countable set of points occurring at the same endpoint (place in space) can be linearly ordered by one-to-one association with linearly ordered identifiable real numbers.

This implies that post-infinite extensions must be appended to the infinite decimal fraction common to all these points. This is needed to generate additional real

[4] Whereas the Dedekind cut ensures that a Cauchy sequence of points will always have ONE limit point, the full Dedekind cut implies that the limit of a Cauchy sequence of points is formed by COUNTABLE many points. But, since the Cauchy sequence *itself* is focussed at the endpoint, the countable set of points forming the Cauchy sequence qualifies as a set of points that can be used as limit.

numbers[5] that are individualised, identified and linearly ordered. (The present Dedekind cut identifies the first or the last of these points.)

In this way the full Dedekind Cut re-establishes a one-to-one mapping between the real numbers and the points on a line. Therefore, the real line is also re-established (and the Lebesgue theory is re-validated).

To distinguish between whether the Dedekind cut or the full Dedekind cut is appended to the assumptions of Euclidean Mathematics, reference will be made to either 'Standard Euclidean Mathematics' (or equivalently to 'Abstract Mathematics') when the Dedekind cut is used or to 'non-standard Euclidean Mathematics' when the full Dedekind cut is used. Note that in Standard Euclidean Mathematics there is a single point located at every endpoint while in non-standard Euclidean Mathematics there are countable many points located at every endpoint.

Remark

For non-standard Euclidean Mathematics the cardinality of the set of lines along an axis from the origin is therefore still equal to the cardinality of the set of infinite sequences of digits as was proved by Cantor, but less than the cardinality of the real numbers.

Note that this is possible because referring to 'infinite decimal fractions' means that the post-infinite extensions are ignored.

An extension to the nomenclature is required to investigate the limit of a set of nested intervals:

Definition

A Cauchy sequence $\{s_n; n=1, 2, 3, \ldots\}$ which belongs to the equivalence class of Cauchy sequences converging to zero, is called an **infinitesimal number**.

The set of nested intervals $\{(A_n,A_0): n=1,2,3, \ldots\}$ is called an **infinitesimal focussed on A_0** if $\{s_n\}$ is an infinitesimal number and $s_n = L(A_n, A_0)$ is the length of the line (A_n,A_0) for all n.

According to EA6, in standard Euclidean Mathematics, the point A_0 is the limit of the sequence of nested intervals $\{(A_n, A_0)\}$. The effect of using the full Dedekind cut is studied later.

The extension of these definitions to the case where the point A_0 is internal to the intervals forming the set of nested intervals is trivial. This definition can be extended to areas and solids.

[5] A family forms - the first infinite number of digits act like a surname and the rest like a hidden name for each point.

0.3.1 DICHOTOMY

The first way of forming a line of non-zero length from points was by stringing together more than countable many points of zero length and then adding up their lengths - as was done in section 0.1.4 above for Euclidean Mathematics.

But an alternative way of forming a line of non-zero length from points would be (as is done in Calculus for the Riemann integral) to begin with a line of non-zero length and then divide it into ever shorter pieces. After doing this an infinite number of times (as can, according to EA4 above, implicitly be done in Euclidean Mathematics) the limits would all be single points, and the line would have been transformed into a string of points. (This will be investigated in more detail later)

Are these two ways of forming a line from points equivalent: Would this way yield more than countable many points to form a line; the same as in Euclidean Mathematics?

0.3.2 DECIDER

In Mathematics an example can prove nothing, a counterexample can disprove anything.

Consider the Riemann Integral in standard Euclidean Mathematics:

$$1 = \int_0^1 1 \cdot dx$$

Riemann sums can be formed by starting with an interval (line) of length one on the X-axis as partition zero, then form successive partitions by dividing each interval of the previous partition into three equal parts. In this way the n^{th} partition will consist of 3^n intervals, each of length 3^{-n}. If $x = a_i^n$ is at the centre of the i^{th} interval of the n^{th} partition, then

$$a_i^n = \frac{2i-1}{2} 3^{-n} \quad \text{For i = 1, 2, 3, …, } 3^n \text{ and n = 0, 1, 2, ….} \qquad [6A]$$

But the length of the whole interval is the sum of the lengths of the parts so that:

$$1 = \sum_{i=1}^{3^n} L(a_i^n - \tfrac{1}{2}3^{-n} \, , \, a_i^n + \tfrac{1}{2}3^{-n}) \qquad \text{for n=0, 1, 2, ….}$$

Where:

$$L\left(a_i^n - \frac{1}{2}3^{-n} \, , \, a_i^n + \frac{1}{2}3^{-n} \right) = 3^{-n}$$

Is the length of the line (with a_i^n at its centre) between the points $a_i^n - \frac{1}{2}3^{-n}$ and $a_i^n + \frac{1}{2}3^{-n}$ on the real line.

Since this sum is the same for all values of n

$$1 = \lim_{n\to\infty} \sum_{i=1}^{3^n} L(a_i^n - \tfrac{1}{2}3^{-n} \ , \ a_i^n + \tfrac{1}{2}3^{-n}) \qquad [6B]$$

Thus, the right-hand side of 6B is in some vague way a kind of multiple (that goes to infinity) of interval lengths (that all go to zero), and it is therefore some kind of indefinite form $\infty \cdot 0$

But these partitions have two specific properties that can be rigorously shown to be true, but can easily be seen by drawing three lines of unit length below each other and marking the partitions on them:

Firstly, when a point is in the middle of one part of a partition, it will be in the middle of a part for all subsequent partitions. Secondly, a set of nested intervals can be formed by selecting from consecutive partitions intervals having the same midpoint. The lengths of these intervals converge to zero while they remain symmetric about their common midpoint. *In Standard Euclidean Mathematics the set of intervals will have this internal point as a limit.*

A nested set of intervals of which the lengths converge to zero was defined above as an infinitesimal, so that here every infinitesimal is focussed on the common midpoint of all the intervals forming the infinitesimal.

The set of all the infinitesimals formed as described above is a directed set where the pre-order is defined by: "A<B is true when the first interval (part) of the infinitesimal B is contained in an interval that is a part of A". The infinitesimal that has the whole unit interval as first partition is the first element in this directed set. The Riemann integral can then be defined as a net on the directed set of infinitesimals mapping onto a value in the real numbers (in this case the number one).

But this net also maps onto the set:

D = { a_i^n: for some n and some valid i}

of points that are the limits of the infinitesimals in Standard Euclidean Mathematics. Note that these points all represent rational numbers.

But even though the set D of points is dense in the interval [0;1], no point that does not belong to D can be a limit for any one of the infinitesimals. This is so because the elements of D are all interior points of the intervals, and any other given point will eventually fall outside all the tail intervals forming any given infinitesimal. This informal argument then implies that *in the limit, the points in D and only the points in D, contribute to the sum of the lengths of the points forming the unit interval*.

Therefore, the sum of the lengths of all the points in D is one. [A]

But the numbers a_i^n are all rational numbers and therefore the cardinality of D is countable.

Therefore, the sum of the lengths of all the points in D must be zero. [B]

The contradiction formed by the results [A] and [B] of this (informal) argument is a necessary consequence of the assumption in standard Euclidean Mathematics that the limit of an interval of which the length goes to zero is a point. The contradiction should therefore be resolved by first accepting that the above informal argument cannot be replaced by a valid formal argument in Euclidean Mathematics that does not lead to the contradiction. Then to follow the example of the three models for Geometry by formulating a second model for Mathematics as a companion to Euclidean Mathematics, but in which the limit of the intervals forming the infinitesimal cannot be a point or points. Embedding the Riemann integral in this alternative model should nullify the contradiction.

The genius of Leibniz is that he could reach back over the millennia and posit that when the length of a line goes down to zero, the limit is not a point as required by EA6, but an entity, different from discrete points, of which the length is equivalent to zero but is not zero itself. Such numbers did not exist at the time, and he named them *infinitesimals*. Here the *spatial entity is called an infinitesimal* and its size is called an *infinitesimal number*.

0.3.3 NOTES ON THE NATURE OF THIS CONTRADICTION

It is highly unlikely that the countable many discrete limit points in the example above can be combined in some way to form a more than countable “sum” because countable many combinations of countable many objects are countable. Therefore, the contradiction above is the direct consequence of the assumptions in Standard Euclidean Mathematics that the infinitesimals have discrete limits, and that the Dedekind cut requires that these limits must be single points.

The same argument is true for non-standard Euclidean Mathematics where each of the limits of the infinitesimals consists of countable many points. Therefore, it is still highly unlikely that the assumptions of either standard or of non-standard Euclidean Mathematics can allow resolution of the contradiction.

On the other hand, vectors are lines that in the limit when their lengths converge to zero, do not have a point as limit but converge to the Null Vector - a vector with zero length and an undetermined direction. This is a pointer that the contradiction above may be resolved through introducing null lines, and in so doing introduce the concepts of ‘continuum’, and consequently also ‘more than countable’, into the argument. This will define a Non-standard Analysis that has “Null Lines” in lieu of points as limits for lines of which the lengths converge to zero.

The genius of Leibniz reached back over the millennia to introduce “Infinitesimals” to be the null lines that can act as alternative limits for sets of nested intervals of which the lengths converge to zero.

But in each of the two models for Euclidean Mathematics logic requires that a nested set of lines of which the lengths converge to zero cannot have both points and a null line as limit. Because in Euclidean Mathematics the real line is complete the limit of a set of nested lines is a point (or points) by definition. Hence null lines cannot be defined in either standard or in non-standard Euclidean Mathematics.

Another different set of assumptions for Mathematics must therefore be formulated in such a way that the existence of points as the building blocks for space is not introduced and where null lines can consequently be defined to take over the role of points.

Therefore, referring to the case of the three Geometries, the contradiction can be averted only by (1) formulating an additional model for Space with alternative assumptions that do not include the concept of point, and then (2) using this model to form a second model for Mathematics by avoiding the existence of limits (in the customary sense) altogether through choosing lines (in lieu of points) as fundamental entities[6] and in so doing enable he existence of null lines.

This model is therefore a continuous model for Mathematics because it is based on lines and not on points as fundamental spatial entities. The alternative model for space is given below and the model for Mathematics built on it is called Leibnizian Mathematics. It complements the existing two models called standard and non-standard Euclidean Mathematics (Evolved from Abstract Mathematics) and thus extends the paradigm of Mathematics.

The Riemann integral – and by implication all of Calculus – is therefore best described by Leibnizian Mathematics, named so in honour of Gottfried Wilhelm Leibniz[7] who was first to reach back and to propose, in the spirit of the ideas of Parmenides of Elea, the existence of infinitesimals to be the null lines that act as an alternative to the points used in Euclidean Mathematics.

BEWARE: Note that this means that whatever is true in one model – like the sum of the interior angles of a triangle in the three geometries - is not necessarily true in another model. For instance, without the use of EA3, the concept "more than countable" is superfluous and cannot even be defined. It also means that most of Calculus cannot easily be fitted into Euclidean Mathematics (hence the name "Non-standard Analysis" presently used as name for Analysis based on infinitesimals).

0.4 LEIBNIZIAN MATHEMATICS

Leibnizian Mathematics is based on an alternative set of assumptions about Space in which the concept of point, as a piece of space with zero extent, does not appear. The assumption that points exist is supplanted in this model by the assumption that lines (**all** with non-zero extent) exist and that the endpoints of lines merely indicate places in Space. To form this model, the Euclidean Assumptions EA1 … EA6 of paragraph 0.1.3 above are supplanted by the following assumptions:

[6] The focus thus moves from points to endpoints.

[7] Gottfried Wilhelm Leibniz 1-7-1646 to 14-11-1716

4.1 LEIBNIZIAN ASSUMPTIONS

- **L1**: **Axiom of Parmenides**: All spatial entities have non-zero extent.
- **L2**: Any solid, surface or line can always be divided.[iii]
- **L3**: When divided, the total extent of the resulting parts equals the extent of the original.
- **L4**: **The Real line**: There is an order-preserving one to one mapping of the real numbers onto lines from the origin; mapping the magnitudes of the numbers onto the lengths of the lines.

Rephrasing L4 results in the statement that there is a one-to-one correspondence between the real numbers and the endpoints of lines on the axis. These endpoints turn out to be a countable set. The Dedekind Cuts, the Axiom of Choice and Lebesgue Theory do not form part of this model – same as that the contradiction constructed above shows that the Riemann integral does not form part of the Euclidean model.

In Leibnizian Mathematics an endpoint is merely a property of a line indicating a place in space (the line being the spatial object). Lines therefore take over the fundamental place that points have in Euclidean Mathematics. The Leibnizian model and its properties are developed in the document "LEIBNIZIAN MATHEMATICS" [1] that is posted on viXra.

The current document aims at introducing the concept of the full Dedekind cut and also to facilitate the reading of the document [1] which is written in a philosophical style.

0.5 REFERENCES

NOTE TO THE READER: The documents referred to below were never critically read by any mathematician, hence they contain all my original mistakes – logical and otherwise. They report the evolution of my thinking over the span of forty years so that some of the ideas changed often as my insight evolved. The book [3] has been emended and appended to [2].

The model for Leibnizian Mathematics is developed in [1] from page 9 onwards (But please also take note of TO THE READER on page 8).

These documents can be accessed by using the following links and then downloading the text:

[1] LEIBNIZIAN MATHEMATICS 2022
http://viXra.org/abs/2201.0175

[2] AN ALTERNATIVE MODEL FOR SPACE 2022
http://viXra.org/abs/2201.0176

[3] CANTOR'S FALLACY
https://vixra.org/abs/1501.0153 2015

0.6 PRIMER

In Leibnizian Mathematics space is not synthesised from points, but is analysed by solids, surfaces and lines. It is never required to sum zeros to a non-zero total, and hence none of the conclusions of paragraph 0.1.4 are applicable to Leibnizian Mathematics. To ease access to the document “LEIBNIZIAN MATHEMATICS”, the following primer is an effort to alleviate paradigm shock by discussing some consequential differences in the meaning of words common to all models.

Mathematics

Euclidean (Abstract) Mathematics: Mathematics based on a discrete model for Space. It is built on the assumption that a piece of space, called a point, with zero extent exists and that all other spatial entities are formed through aggregation of points.

Leibnizian Mathematics: Mathematics based on a continuous model for Space. It is built on the assumption that all spatial entities have extent, and that space can be analysed by isolating and bounding pieces of continuous Space.

Zero

Euclidean: It is defined as a non-negative number less than all positive numbers. As a real number it is also the equivalence class of Cauchy sequences converging to the rational number zero.

Leibnizian: Same as in Euclidean Mathematics. As the Cauchy (infinitesimal) number zero it is the null Cauchy sequence (0. ; 0.0 ; 0.00 ; ….).

Infinity (Noun)

Euclidean: An integer larger than all other integers.

Leibnizian: An irrational number only; the equivalence class of divergent sequences. The individual divergent sequences in this class are the infinite Cauchy numbers.

Infinite (Adjective, Adverb)

Euclidean: Larger than all integers.

Leibnizian: Never ending, open ended, unbounded.

Infinite decimal fraction

Euclidean: A non-finite decimal fraction containing all its digits up to and including the last one.

Leibnizian: A never-ending decimal fraction.

Point/Endpoint

Euclidean: Point - A piece of space with zero extent.

Leibnizian: Endpoint - The endpoint of a line (A place in space like the point of a needle. As such it cannot have intrinsic extent because it is not a spatial entity but a property of a line.)

Limit

Euclidean: A way of handling continuity in a model of Mathematics based on a discrete model for space.

Leibnizian: Absent - in the sense of 'being not required' but useful in referring to an equivalence class.

(See the adaptation of the rule of L'Hospital on page 18 of the reference.)

Cauchy Number

Euclidean: Absent.

Leibnizian: The Cauchy numbers are the sequences that form the equivalence classes that define the real numbers. They are classified as:

Infinitesimal numbers: The Cauchy sequences that form the real number zero.

Infinite Numbers: The divergent Sequences that form the real number infinity.

Rated numbers: The Cauchy sequences that form all other real numbers.

More than countable

Euclidean: Cardinality of some sets of points and of the real numbers

Leibnizian: Absent

0.7 CONCLUSION

Like with Geometry, as mentioned in the prologue, Mathematics divides into two main sub models namely Euclidean Mathematics and Leibnizian Mathematics.

The first model, Euclidean Mathematics, in turn divides into two models. The first is Standard Euclidean Mathematics (Abstract Mathematics) that assumes the Dedekind cut as supplement to the basic assumptions. The second is Nonstandard Euclidean Mathematics that assumes the full Dedekind cut as supplement to the basic assumptions. Euclidean Mathematics is more suitable for discrete problems (like in Algebra, Lebesgue Theory and in the Theory of Probability for discrete events that are modelled as non-spatial entities).

The second main model is Leibnizian Mathematics, and it is more suitable for continuous problems (like in Calculus and in parts of Statistics).

Hence, there are three different Models, complementing each other and suitable to be used in different circumstances to support mathematical arguments.

-ooOOoo-

ENDNOTES

[i] It is therefore mandatory, whenever using Euclidean Mathematics, to make sure that the Axiom of Choice is only applied in arguments solely about abstract entities.

[ii] In Cantor's well-known proof that there are more than countable many real numbers, his argument assumes that these numbers are countable and therefore that it is possible to make a list of all infinite decimal fractions. He then showed that there existed an infinite decimal fraction that was not in the list, and from that he concluded that **all** infinite decimal fractions cannot be listed and thus there must be more than countable many real numbers. But an equally valid conclusion is that the real numbers cannot be listed **at all** – namely that a list of a single infinite decimal fraction cannot be made (as in perceived reality). The conclusion EA4 ensures the existence of such a symbol and validates the proof, albeit only in abstract Euclidean space.

[iii] The well-known rhyme about fleas can be adapted to Leibnizian Mathematics:

Big space has little space
That sum to what is in it,
And little space has lesser space,
And so on without limit.