De Broglie's Phase Waves of Electrons in Hydrogen-Like Atoms as Parts of Steady-State Forced Vibrations

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Abstract

One of the consequences of de Broglie's hypothetical internal clock of an electron is the existence of an associated phase wave that is locally in phase with the electron's internal clock. For a classical electron that is moving on a closed Bohr-Sommerfeld orbit in the electric potential of a positively charged nucleus, de Broglie showed that the requirement of a continuous phase wave on the closed orbit is in agreement with Einstein's quantization condition. The present work explores the hypothesis that this phase wave is part of a forced vibration in the space around the nucleus, which is driven by the electron's internal clock. Results of this work include new quantization conditions for closed orbits that are consistent with Einstein's quantization condition but include additional (yet unexplored) conditions. Furthermore, a method is presented for computing phase waves on closed orbits of electrons, which could help visualizing several important features of de Broglie's phase waves.

1 Introduction

Atomic orbitals are sometimes described as three-dimensional analogs to Chladni figures, which are generated by forced vibrations of metal plates [JPM+20]. However, such comparisons usually do not include a discussion of any mechanism driving the vibration of an electronic wave function. One possible mechanism might be provided by de Broglie's hypothetical concept of a phase wave that is in phase with the internal clock of an electron. In his doctoral thesis, de Broglie compared a bound electron's phase wave on a closed orbit in a hydrogen-like atom to "a liquid wave in a channel closed on itself but of variable depth" where "the length of the channel must be resonant with the wave" [dB25, page 28]. This comparison with a vibrating macroscopic system does not clarify what drives the vibration, but since the electron's internal clock is always in phase with the associated phase wave, it is an obvious candidate. The objective of the present work is, therefore, to explore the scenario of a bound electron with de Broglie's internal clock and an associated phase wave on a closed Bohr-Sommerfeld orbit in a hydrogen-like atom [Som23], where the electron drives a vibrating field in the space around the nucleus [Kra24b, Section 4.2]. Of particular interest is the resulting uniform frequency of this forced vibration. The results include a method for computing phase waves on electronic orbits and new quantization conditions, which imply Einstein's quantization condition [Ein17] and additional conditions, which have not been explored yet.

2 Phase Waves on Closed Bohr-Sommerfeld Orbits

This section presents the physical scenario that the present work is concerned with and analyzes a few of its most interesting features. The notation resembles the notation employed in de Broglie's doctoral thesis [dB25] with a few exceptions, e.g., the use of "$|\mathbf{v}|$" instead of "v" for an electron's speed in order to distinguish the symbol more clearly from "ν," which denotes various frequencies.

2.1 Scenario and Assumptions

In part, this work is based on the Bohr-Sommerfeld model of hydrogen-like atoms [Som23], specifically on the scenario of a single relativistic spinless electron of invariant mass m_0 and (negative) charge e moving with velocity $|\mathbf{v}| = \beta c$ in the electric Coulomb field of a nucleus with positive charge $-Ze$ at

a fixed position, which is the origin of a fixed observer's coordinate system. Effects of spin, magnetic moments, and magnetic fields are being ignored since they are not essential for the concepts presented in this work. In this scenario, Bohr-Sommerfeld orbits of classical electrons are closed except for a small rotation of elliptical orbits due to relativistic effects, which is ignored in this work. Thus, all Bohr-Sommerfeld orbits are assumed to be closed.

Additionally, it is assumed that the electron features an internal clock of the Compton frequency $\nu_0 \stackrel{\text{def}}{=} h^{-1}m_0c^2$ in its rest frame as proposed by de Broglie [dB25], where h is Planck's constant and c is the speed of light. Based on this assumption, de Broglie concluded that the internal clock "appears constantly in phase with a wave" [dB25, page 9], which "is spread out over an extended space" [dB25, page 8]. While de Broglie did not specify which physical field is carrying the phase wave, his comparison with "a liquid wave in a channel closed on itself" [dB25, page 28] suggests that he imagined a continuous wave in a field, which is spread out over at least several Compton wave lengths around the internal clock's position.

The present work further assumes that the electron's internal clock is causing this phase wave by means of some form of interaction between the internal clock and the vibrating field. One possibility is that the vibrating field is a non-linear generalization of the classical electromagnetic field as provided, for example, by Born-Infeld field theory [BIF34]. However, the physical nature of the vibrating field and the details of the interaction with the internal clock are beyond the scope of the present work.

The concept of a forced vibration driven by an electron's internal clock on a closed orbit leads to the assumption that a steady state of this forced vibration is characterized by a uniform frequency that depends on the frequency $\nu_0 \stackrel{\text{def}}{=} h^{-1}m_0c^2$ of the electron's internal clock. In the case of a free electron at constant velocity $\mathbf{v}$ and constant speed $|\mathbf{v}| = \beta c$, the constant (time-dilated) frequency of the electron's internal clock in the rest frame of a fixed observer is $\nu_1 \stackrel{\text{def}}{=} h^{-1}m_0c^2\sqrt{1-\beta^2}$, which leads to a constant frequency of the associated phase wave of $\nu \stackrel{\text{def}}{=} h^{-1}m_0c^2/\sqrt{1-\beta^2}$ [dB25]. In the present work, it is assumed that the uniform frequency of the forced vibration of the field around the free electron is also given by ν.

In the case of a bound electron on a circular Bohr orbit, the electron's speed $|\mathbf{v}| = \beta c$ is still constant, and, therefore, the frequency of the electron's internal clock in the rest frame of a fixed observer is still ν_1, and the uniform frequency of the phase wave and vibrating field is still ν. It should be noted that this conclusion disagrees with de Broglie's conception of this case [dB25, pages 23-25].

For elliptical Bohr-Sommerfeld orbits, the electron's speed is not constant. However, the time-dependent frequency of the electron's internal clock in the rest frame of a fixed observer is still given by $\nu_1 \stackrel{\text{def}}{=} h^{-1}m_0c^2\sqrt{1-\beta^2}$. Since the uniform frequency of the forced vibration must not depend on time, it cannot be given by the time-dependent value ν. A suitable generalization for a time-dependent speed of the electron on closed orbits appears to be the time average $\overline{\nu}$ over one period τ of the orbit, i.e.,

$$\overline{\nu} \stackrel{\text{def}}{=} \frac{1}{\tau}\int_0^\tau \nu dt = \frac{1}{\tau}\int_0^\tau \frac{m_0c^2}{h}\cdot\frac{1}{\sqrt{1-\beta^2}}dt. \tag{1}$$

In the present work, it is assumed that $\overline{\nu}$ is in fact the correct uniform frequency of the forced vibration in the case of elliptical Bohr-Sommerfeld orbits; again in contradiction to de Broglie's conception [dB25, pages 25-26].

2.2 Computation of Phase Waves on Closed Orbits

Based on the assumptions of the previous section, this section shows how to compute the phase $\phi(t, \mathbf{x})$ of the vibrating field at any point $\mathbf{x}$ of a closed Bohr-Sommerfeld orbit for any time t in the rest frame of a fixed observer.

First, let $\mathbf{y}(t)$ denote the known positions of the electron and its internal clock on a closed Bohr-Sommerfeld orbit at time t. ($\mathbf{y}(t)$ could be computed numerically or based on a closed form of an orbit [Som23].) For $\phi(0, \mathbf{y}(0)) = \phi_0$, the phase $\phi(t, \mathbf{y}(t))$ can be computed with the help of a numerical integration of the time-dilated frequency ν_1 of the moving internal clock as:

$$\phi(t, \mathbf{y}(t)) = \phi_0 + 2\pi\int_0^t \nu_1 dt' = \phi_0 + 2\pi\int_0^t \frac{m_0c^2}{h}\sqrt{1-\beta^2}dt', \tag{2}$$

where β depends on the time-dependent velocity of the internal clock, i.e., $\beta = \beta(t') = |\mathbf{v}(t')|/c$.

Second, consider a point $\mathbf{x}$ that is part of the orbit. There exists a time t_x such that $\mathbf{x} = \mathbf{y}(t_x)$, i.e., t_x denotes the time when the electron's internal clock passes through the point $\mathbf{x}$. (One way of computing t_x is to store t with each point $\mathbf{y}(t)$ when computing the orbit such that t_x may be looked up for each computed point of the orbit.) For given t_x, Equation 2 allows us to compute the phase at time t_x as $\phi(t_x, \mathbf{x}) = \phi(t_x, \mathbf{y}(t_x)$. For a time $t \neq t_x$, the phase $\phi(t, \mathbf{x})$ is different since the field at position $\mathbf{x}$ vibrates with constant frequency $\overline{\nu}$, which allows us to compute the resulting phase difference as $2\pi\overline{\nu} \cdot (t - t_x)$. Thus, the phase $\phi(t, \mathbf{x})$ may be computed as

$$\begin{aligned} \phi(t, \mathbf{x}) &= 2\pi\overline{\nu} \cdot (t - t_x) + \phi(t_x, \mathbf{x}) && (3) \\ &= 2\pi\overline{\nu} \cdot (t - t_x) + \phi(t_x, \mathbf{y}(t_x)) && (4) \\ &= 2\pi\overline{\nu} \cdot (t - t_x) + \phi_0 + 2\pi \int_0^{t_x} \frac{m_0 c^2}{h} \sqrt{1 - \beta^2} dt'. && (5) \end{aligned}$$

Some examples of $\phi(t, \mathbf{x})$ (for a fixed time t) on various Bohr-Sommerfeld orbits are visualized in Figure 1. The effects of the non-constant speed of electrons on elliptic orbits are visible as local shortenings of the wave length, which occur even for non-relativistic speed. (On the other hand, a visible rotation of the elliptic orbit occurs only if the electron reaches relativistic speed.) Also note that none of these phase waves are standing waves. In fact, their phase velocity is always faster than the speed of light. An animated visualization of $\phi(t, \mathbf{x})$ would be preferable to convey this feature.

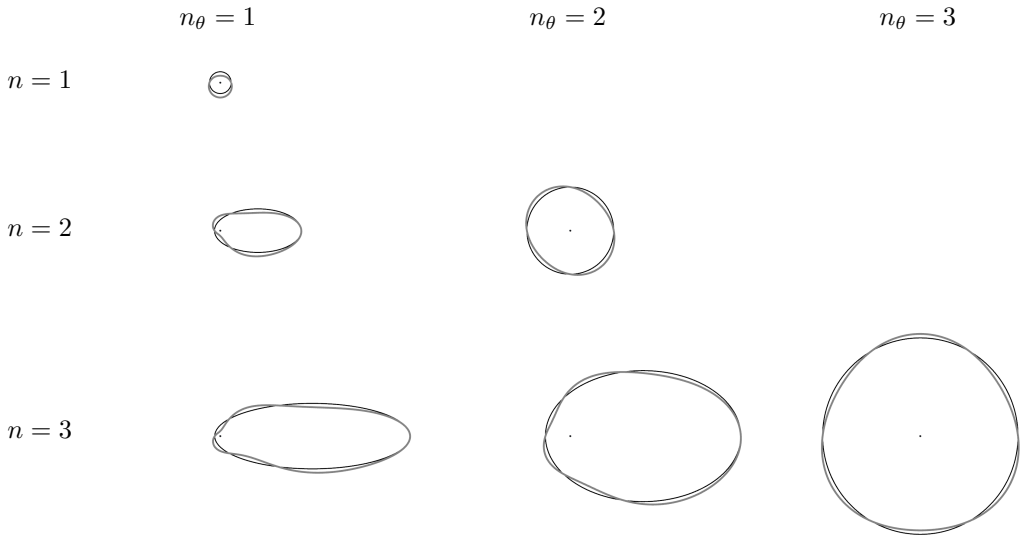

Figure 1: Bohr-Sommerfeld orbits *(black lines)* with helices around them *(gray lines)* representing snapshots of phases on each orbit at a specific time. See Section 2.3 for the definitions of n and n_θ.

2.3 Quantization Conditions

For a steady-state forced vibration, the electron's internal clock has to be moving on a closed orbit. If the phase of the vibrating field and the phase of the internal clock are part of the state of the system, the assumption of a periodic orbit implies that both have to return to exactly the same values after each period τ of the closed orbit.

First, consider the case of the internal clock's phase in the rest frame of a fixed observer. The phase difference after period τ of the closed orbit has to be an integer multiple of 2π, say $2\pi n_{\text{ic}}$, where n_{ic} denotes the integer number of cycles that the *internal clock* goes through during period τ. Since the (time-dependent and time-dilated) frequency of the internal clock is given by ν_1, this number of cycles n_{ic} is given by

$$n_{\text{ic}} \stackrel{\text{def}}{=} \int_0^\tau \nu_1 dt = \int_0^\tau \frac{m_0 c^2}{h} \sqrt{1 - \beta^2} dt. \quad (6)$$

Second, consider the case of the oscillating field's phase at a single point of the orbit fixed in the rest frame of a fixed observer. The phase difference after period τ has to be an integer multiple of 2π,

say $2\pi n_{\text{of}}$, where n_{of} denotes the integer number of cycles that the *oscillating field* at a fixed point goes through during period τ. Since the uniform frequency of the vibrating field is assumed to be $\overline{\nu}$, this number of cycles n_{of} is given by

$$n_{\text{of}} \stackrel{\text{def}}{=} \tau\overline{\nu} = \tau\frac{1}{\tau}\int_0^\tau \nu dt = \int_0^\tau \frac{m_0c^2}{h}\cdot\frac{1}{\sqrt{1-\beta^2}}dt. \tag{7}$$

In this form, the conditions on n_{of} and n_{ic} appear to be two new quantization conditions. For the ground state of hydrogen, n_{of} and n_{ic} are close to $\alpha^{-2} \approx 137^2$; thus, the conditions that n_{of} and n_{ic} must be integers can be satisfied by a relatively small "correction" of the electron's orbit. Presumably, this "correction" is only relevant for the fine-structure of an atom's spectral lines; thus, a quantitative analysis might reveal that these quantization conditions are, in fact, disguised correction terms of the atomic fine-structure. Unfortunately, this analysis is beyond the scope of the present work.

However, the difference $n \stackrel{\text{def}}{=} n_{\text{of}} - n_{\text{ic}}$ is of particular interest and deserves a brief analysis. Obviously, a difference of integers must be an integer. Furthermore, $n_{\text{of}} > n_{\text{ic}}$ due to the way the term $\sqrt{1-\beta^2} < 1$ (for $|\mathbf{v}| > 0$) appears in the definitions of n_{of} and n_{ic}. Therefore, $n \geq 1$. For more insights, we can just evaluate n by inserting the definitions of n_{of} and n_{ic}:

$$\begin{aligned}
n &\stackrel{\text{def}}{=} n_{\text{of}} - n_{\text{ic}} && (8)\\
&= \int_0^\tau \frac{m_0c^2}{h}\cdot\frac{1}{\sqrt{1-\beta^2}}dt - \int_0^\tau \frac{m_0c^2}{h}\sqrt{1-\beta^2}dt && (9)\\
&= \int_0^\tau \frac{m_0c^2}{h}\cdot\frac{1}{\sqrt{1-\beta^2}}\left(1-\left(1-\beta^2\right)\right)dt && (10)\\
&= \int_0^\tau \frac{m_0c^2}{h}\cdot\frac{\beta^2}{\sqrt{1-\beta^2}}dt && (11)\\
&= \frac{1}{h}\int_0^\tau \frac{m_0|\mathbf{v}|^2}{\sqrt{1-\beta^2}}dt && (12)
\end{aligned}$$

With the internal clock's time-dependent position $\mathbf{y}$ in Cartesian coordinates (y_1, y_2, y_3), the velocity is $\mathbf{v} = \dot{\mathbf{y}} = (\dot{y}_1, \dot{y}_2, \dot{y}_3)$. Combined with the relativistic momentum $\mathbf{P} \stackrel{\text{def}}{=} (P_1, P_2, P_3) \stackrel{\text{def}}{=} m_0\dot{\mathbf{y}}/\sqrt{1-\beta^2}$, the expression for n becomes:

$$\begin{aligned}
n &= \frac{1}{h}\int_0^\tau \frac{m_0|\mathbf{v}|^2}{\sqrt{1-\beta^2}}dt && (13)\\
&= \frac{1}{h}\int_0^\tau \frac{m_0}{\sqrt{1-\beta^2}}\left(\sum_{i=1}^3 \dot{y}_i^2\right)dt && (14)\\
&= \frac{1}{h}\int_0^\tau \sum_{i=1}^3 \frac{m_0\dot{y}_i}{\sqrt{1-\beta^2}}\frac{dy_i}{dt}dt && (15)\\
&= \frac{1}{h}\oint \sum_{i=1}^3 P_i dy_i. && (16)
\end{aligned}$$

Thus, for this choice of coordinates, the condition on $n \stackrel{\text{def}}{=} n_{\text{of}} - n_{\text{ic}}$ is just Einstein's condition [Ein17][dB25, pages 27-29].

Bohr-Sommerfeld orbits may also be described by polar coordinates r and θ with the corresponding relativistic momenta $p_r \stackrel{\text{def}}{=} m_0\dot{r}/\sqrt{1-\beta^2}$ and $p_\theta \stackrel{\text{def}}{=} m_0r^2\dot{\theta}/\sqrt{1-\beta^2}$ [Som23, page 608]. Using these

polar coordinates, the expression for n becomes:

$$n = \frac{1}{h}\int_0^\tau \frac{m_0|\mathbf{v}|^2}{\sqrt{1-\beta^2}}dt \tag{17}$$

$$= \frac{1}{h}\int_0^\tau \frac{m_0}{\sqrt{1-\beta^2}}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)dt \tag{18}$$

$$= \frac{1}{h}\int_0^\tau \left(\frac{m_0\dot{r}}{\sqrt{1-\beta^2}}\cdot\frac{dr}{dt} + \frac{m_0 r^2\dot{\theta}}{\sqrt{1-\beta^2}}\cdot\frac{d\theta}{dt}\right)dt \tag{19}$$

$$= \frac{1}{h}\oint \left(p_r dr + p_\theta d\theta\right), \tag{20}$$

which is equivalent to Einstein's condition for polar coordinates. The relation between Einstein's quantization condition and Sommerfeld's quantization conditions [Som23] have been discussed, for example, by Einstein [Ein17] and de Broglie [dB25, pages 29-31].

Note that the principal quantum number n may be written as a sum of other quantum numbers based on the employed coordinates:

$$n = \underbrace{\frac{1}{h}\oint p_r dr}_{n_r} + \underbrace{\frac{1}{h}\oint p_\theta d\theta}_{n_\theta} \tag{21}$$

$$= n_r + n_\theta. \tag{22}$$

Here, p_θ is the orbital angular momentum, and the associated quantum number n_θ (or $n_\theta - 1$; see the discussion by Bucher [Buc08]) is usually denoted by l or L. (Sommerfeld [Som23] uses the symbol n instead of n_θ and writes the principal quantum number as the "quantum sum" $n + n'$ where n' corresponds to n_r.)

The main observation, however, is that the conditions on n_{of} and n_{ic} imply (for suitable coordinates) a condition on the difference $n \stackrel{\text{def}}{=} n_{\text{of}} - n_{\text{ic}}$ that is equivalent to Einstein's condition with the principal quantum number n.

3 Discussion and Future Work

One of the contributions of the present work is a method for computing de Broglie's phase waves on closed orbits. Even static visualizations of these phase waves show de Broglie's quantization condition, which states that "the points of a wave located at whole multiples of the wave length l, must be in phase" [dB25, page 28]. In fact, such visualizations even show how this condition is fulfilled on elliptic orbits where wave length (and phase velocity) is not constant. Animated visualizations of these phase waves show that they are not standing waves; in fact, such visualizations can convey that the phase velocity of these waves is greater than the speed of light. Furthermore, animated visualizations may be used to explain the new quantization condition on the vibrating field introduced in Section 2.3.

If a visualization of the internal clock's phase is included, de Broglie's "theorem of phase harmony" [dB25, page 9] may be visualized. Additionally, the new quantization condition on the internal clock presented in Section 2.3 could be explained with the help of this kind of visualization.

One specific way of visually representing the internal clock's phase is by means of a point-like electron that "spins" on a bent helix around the trajectory of a "spin center" with the (time-dilated) frequency ν_1. (Note that many models of spinning electrons [Bec23] are based on the frequency of Zitterbewegung, which is twice the Compton frequency ν_0. However, at least one recent model [Kra24a] spins with the Compton frequency ν_0 in the rest frame of the internal clock and ν_1 in the rest frame of a fixed observer.) In many models, the point-like electron moves with the speed of light. This movement can provide a visual comparison to show that the phase velocity is faster than the speed of light. Furthermore, the helical trajectory of spinning electrons visually supports concepts that include orbits with zero orbital angular momentum [Buc08] while such orbits were regarded as impossible by Sommerfeld [Som23, page 238].

The other main contribution of the present work might be the new quantization conditions on n_{of} and n_{ic} that are presented in Section 2.3 and the role that $\overline{\nu}$ plays in those conditions. At the time of

writing, however, these conditions and the role of $\overline{\nu}$ are only hypothetical assumptions. Fortunately, quantitative comparisons with existing experimental data appear to be possible even though they are beyond the scope of the present work. Thus, instead of engaging in further speculation, this section is closing with the proposal to test the assumptions of Section 2 against experimental data in future work.

4 Conclusion

This work presents a method for computing phase waves on closed Bohr-Sommerfeld orbits and discusses how (animated) visualizations of the results could help conveying various features of phase waves.

The method is based on the concept of phase waves on closed orbits as part of a larger field that vibrates with a uniform frequency $\overline{\nu}$. This and other assumptions lead to new quantization conditions that are shown to imply Einstein's quantization condition [Ein17] and, therefore, are closely related to Sommerfeld's and de Broglie's quantization conditions [Som23, dB25]. Whether the assumptions are justified is an open question, which hopefully will be answered by future comparisons with existing experimental data.

This work started as part of numerical simulations of trajectories formed by a specific classical model of spinning electrons [Kra24a], which were developed to understand how this model might be related to single-electron wave functions [Kra24b]. From this point of view, the appearance of $\overline{\nu}$ (instead of the electron's energy divided by Planck's constant) raised more questions than it answered; but at least these new questions are as intriguing as the original ones.

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A Revisions

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