Theoretical Derivation of Bohr's Postulate for the Charge in a Hydrogen Atom. Coulomb's Law in Logarithmic Form with Corrections for Strong Interactions at Small Distances. The Physical Meaning of Planck's Constant.

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Abstract

This article proposes a revolutionary theoretical model that introduces a fifth spatial dimension—"space density"—as a fundamental property governing gravitational, electromagnetic, strong, and weak interactions. The model is based on the hypothesis that changes in space density can lead to phenomena analogous to known fundamental forces. Through a series of mathematical derivations, it is shown how the distribution of space density around spherical objects influences classical field theories. The main results include:

1. **Theoretical Proof of Bohr's Postulate**: For the first time, a theoretical justification for Bohr's postulate on the quantization of the electron's angular momentum in a hydrogen atom is proposed, which is key to quantum mechanics.
2. **Relationship Between Charge and Mass**: A novel relationship between charge and mass is established, allowing mass to be interpreted as the energy required to compress a clump of space density.
3. **Complex Solution and Imaginary Energy**: It is shown that the interaction of two clumps of space density has only a complex solution, where the imaginary part determines the resonant frequency of the system.
4. **Strong and Weak Interactions**: The model offers an explanation for strong and weak interactions through the properties of space density, opening new possibilities for understanding nuclear forces.

This work not only reproduces known physical patterns but also provides a new perspective on the nature of fundamental interactions, linking them to the intrinsic properties of space.

I Introduction

Electromagnetic and gravitational forces are among the most fundamental interactions known in physics. These forces govern the behavior of matter and energy at scales ranging from subatomic particles to the cosmos. Despite extensive empirical data and theoretical models describing the behavior of these forces, their true nature and the material essence from which they arise remain subjects of deep investigation.

From a physical standpoint, we understand how these forces act and can predict their effects with high accuracy. However, questions remain: What exactly are these forces? How are they interconnected? And most importantly, what is the proto-matter, the fundamental substance from which these forces emerge? These questions touch not only on physical principles but also on philosophical reflections on the nature of reality.

In this article, we propose a theoretical model that introduces a fifth spatial dimension called "space density." We suggest that this dimension plays a critical role in the formation of gravitational and electric fields. Our model posits that traditional three-dimensional space combined with time is insufficient to fully explain the origin of these forces. Instead, space itself may possess intrinsic properties that contribute to the formation of these fields. By expanding our understanding of space to include an additional dimension, we explore the potential for new interpretations of gravitational and electromagnetic interactions.

II Hypothesis

We propose that electromagnetic and gravitational fields are manifestations of a more fundamental property of space, which we call ”space density.” This property is defined in a five-dimensional system, where the fifth dimension is orthogonal to the traditional three spatial and one temporal dimensions.

In this model, ”space density” represents a measure of how space itself can be compressed or expanded independently of its metric. This density is not analogous to the density of matter as we know it in three-dimensional space but rather reflects a fundamental characteristic of space that influences the formation of gravitational and electric fields.

Our hypothesis is based on several key postulates:

- **Space Density**: In five-dimensional space, the density $\rho(r)$ characterizes the state of space and can change, thereby allowing us to speak of the curvature of space without curving its metric. Let us call this phenomenon first-order space curvature. A similar term is used in the Theory of Relativity, but within this theory, it will have a slightly different context.
- **Spherical Symmetry of Perturbations**: The distribution of space density upon perturbation assumes spherical symmetry. The distribution of space density $\rho(r)$ is assumed to be symmetric relative to the point that is the center of the perturbation.
- **Conservation of Space Density Quantity**: When a region of space is perturbed, the surrounding space can change its density such that the total density of the entire space remains unchanged. In other words, in a certain approximation, it can be said that the total ”density” of space over an infinite volume must remain constant.
- **Postulate of Maximum Entropy in Space Density Distribution**: Space tends toward states of maximum entropy, striving for a uniform distribution of space density. This principle defines the natural tendency of space to return to a uniform density distribution after perturbations, analogous to thermodynamic principles governing physical systems.

By exploring these postulates within a five-dimensional space, we aim to provide a deeper understanding of the origin of gravitational and electromagnetic fields. This model challenges the traditional view that these fields are independent and instead suggests that they are interconnected through the intrinsic properties of space itself. In the course of our research, we obtain entirely unexpected results: Coulomb’s law in logarithmic form, containing an expression accounting for the correction of elementary charge interactions at distances comparable to their ”classical” physical sizes (this phenomenon is well-studied in QED—screening). And the most unexpected result is the connection of this mathematical model to the foundation of Quantum Mechanics—Bohr’s Postulate on the quantization of electron states in a hydrogen atom. The solution for the interaction quantity of two clumps of space density in logarithmic form has only a complex solution, and as it turns out, the complex part of the solution determines the resonant frequency of the interaction quantity of the two clumps. Using the complex part of the solution as an expression for the resonant frequency of the two clumps of space density, we obtained the resonance condition for one clump orbiting another, which fully corresponds to the quantization condition of orbits derived from Bohr’s Postulate on the quantization of the electron’s angular momentum in a hydrogen atom. When analyzing the obtained formulas, we attempt to explain the physical meaning of such an empirically obtained quantity as Planck’s constant, which has two values within the presented mathematical model: the size of the electron and the ratio of the total energy of the electron in the atom to its imaginary energy. If you are interested in how all this follows from simple ideas about space density and its tendency toward maximum entropy, I will begin to outline the main approaches that form the basis of my theory presented in this article.

III Methodology

3.1 Distribution of Space Density Around a Single Compressed Spherical Region of Space

We have two states of the universe: in the first state, the density throughout space is ρ_0 and is a constant. In the second state of the system, we have a certain region of space bounded by a sphere $S(R_1)$, which we compress to $S(R'_1)$. We need to find the distribution of space density inside the sphere and outside it, based on the laws established in our hypothetical universe.

3.1.1 Density Distribution After Compression

The density after compression inside the sphere is defined as $\rho_{\text{inside}} = \rho_0 + \rho_1$, where ρ_1 is the added density, determined from the ratio of volumes before and after compression:

$$\rho_0 V(R_1) = \rho_{\text{inside}} V(R'_1)$$

Substituting the volumes of the spheres:

$$\rho_0 \frac{4}{3}\pi R_1^3 = (\rho_0 + \rho_1)\frac{4}{3}\pi R_1'^3$$

Simplifying:

$$\rho_0 R_1^3 = (\rho_0 + \rho_1) R_1'^3$$

$$\rho_1 = \rho_0 \left(\frac{R_1^3}{R_1'^3} - 1 \right)$$

3.1.2 Density Distribution Outside the Sphere

We assume that outside the sphere, the amount of removed space density must equal the amount added inside it, $\rho_1 \cdot V(R_1')$. Therefore, when integrating the perturbation from the surface of the compressed sphere to infinity, the integral must yield a finite number, i.e., converge, and accordingly, the integrand must be convergent. In three-dimensional space, such a function is $1/r^4$. Suppose the distribution of reduced density outside the compressed region of space satisfies this dependence on the distance from the center of the perturbation. Then we obtain the following dependence for the distribution of space density outside the compressed sphere:

$$\Delta\rho_{\text{decrease}}(r) = \frac{A}{r^4}$$

3.1.3 Normalization Coefficient A

To satisfy the law of conservation of space density, the integral of $\Delta\rho_{\text{decrease}}(r)$ over the volume from R_1' to infinity must equal the added density inside the sphere:

$$\rho_1 V(R_1') = \int_{R_1'}^{\infty} \Delta\rho_{\text{decrease}}(r) \cdot dV$$

Or, considering the law of spherical symmetry, in spherical coordinates, the integral simplifies to:

$$\rho_1 V(R_1') = \int_{R_1'}^{\infty} \Delta\rho_{\text{decrease}}(r) \cdot 4\pi r^2 \, dr$$

Substituting:

$$\rho_1 \frac{4}{3}\pi R_1'^3 = 4\pi \int_{R_1'}^{\infty} \frac{A}{r^4} r^2 \, dr$$

Solving the integral:

$$4\pi A \int_{R_1'}^{\infty} \frac{1}{r^2} \, dr = 4\pi A \left[-\frac{1}{r} \right]_{R_1'}^{\infty} = 4\pi A \left(\frac{1}{R_1'} - 0 \right) = \frac{4\pi A}{R_1'}$$

Equating the quantities of density:

$$\rho_1 \frac{4}{3}\pi R_1'^3 = \frac{4\pi A}{R_1'}$$

Finding A:

$$A = \rho_1 \frac{R_1'^4}{3}$$

The final formula for $\Delta\rho_{\text{decrease}}(r)$:

$$\Delta\rho_{\text{decrease}}(r) = \frac{A}{r^4} = \frac{\rho_1 \frac{R_1'^4}{3}}{r^4}$$

Now multiplying the numerator and denominator by 4π:

$$\Delta\rho_{\text{decrease}}(r) = \frac{4\pi\rho_1 \frac{R_1'^4}{3}}{4\pi r^4} = \frac{\rho_1 \frac{4}{3}\pi R_1'^4}{4\pi r^4} = \frac{\rho_1 \cdot R_1' \cdot V(R_1')}{4\pi r^4}$$

Thus, we obtain the following formula for the distribution of density outside the sphere $\Delta\rho_{\text{decrease}}(r)$:

$$\boxed{\Delta\rho_{\text{decrease}}(r) = \frac{\rho_1 \cdot R_1' \cdot V(R_1')}{4\pi r^4} \qquad (1)}$$

Also, considering that the amount of added density in the volume of the compressed sphere is expressed by the formula:

$$Q = (V(R_1) - V(R_1')) \cdot \rho_0$$

where $V(R_1)$ and $V(R_1')$ are the volumes of spheres with radii R_1 and R_1', respectively. And also considering the formula for ρ_1—the density of the added density inside the sphere:

$$\rho_1 = \frac{Q}{V(R_1')}$$

where $V(R_1')$ is the volume of the sphere after compression.

We can express the obtained formula for the distribution of space density $\Delta\rho_{\text{decrease}}(r)$ as:

$$\boxed{\Delta\rho_{\text{decrease}}(r) = \frac{Q \cdot R_1'}{4\pi r^4} \qquad (2)}$$

Where Q is the amount of density added to the volume of the sphere $S(R1')$, $R1'$ is the radius of the compressed sphere, and r is the distance from the center of the sphere to a point in space in spherical coordinates.

3.1.4 Verification of Conservation of Space Density Quantity

To satisfy the third law established in our system, the following equality must hold:

$$\int_{R_1'}^{\infty} \Delta\rho_{\text{decrease}}(r) \cdot \, dV = \int_{R_1'}^{\infty} \Delta\rho_{\text{decrease}}(r) \cdot 4\pi r^2 \, dr = \rho_1 V(R_1')$$

Substituting the expression for $\Delta\rho_{\text{decrease}}(r)$:

$$\int_{R_1'}^{\infty} \frac{\rho_1 \cdot R_1' \cdot V(R_1')}{4\pi r^4} \cdot 4\pi r^2 \, dr = \rho_1 \cdot R_1' \cdot V(R_1') \int_{R_1'}^{\infty} \frac{1}{r^2} \, dr$$

Integrating and substituting the limits of integration:

$$\rho_1 \cdot R_1' \cdot V(R_1') \left[-\frac{1}{r} \right]_{R_1'}^{\infty} = \rho_1 \cdot R_1' \cdot V(R_1') \left(\frac{1}{R_1'} - 0 \right) = \frac{\rho_1 \cdot R_1' \cdot V(R_1')}{R_1'}$$

We obtain:

$$\int_{R_1'}^{\infty} \Delta\rho_{\text{decrease}}(r) \cdot \, dV = \rho_1 V(R_1') = \rho_1 \frac{4}{3} \pi R_1'^3$$

Thus, we have verified that our distribution of space density outside the compressed sphere, proportional to $1/r^4$, is consistent with our third law of conservation of space density in the system, taking into account the normalization coefficient A.

IV Expression for the Total Distribution of Space Density for a Single Compressed Sphere.

Let us write our distribution taking into account boundary conditions using the Heaviside function. This representation of space density distribution will be needed to find the total interaction quantity of two clumps, taking into account the space density added to the first clump, as well as the gradient at the transition boundary—the sphere limiting the first clump. Why this is important will become clear in the next section of my article.

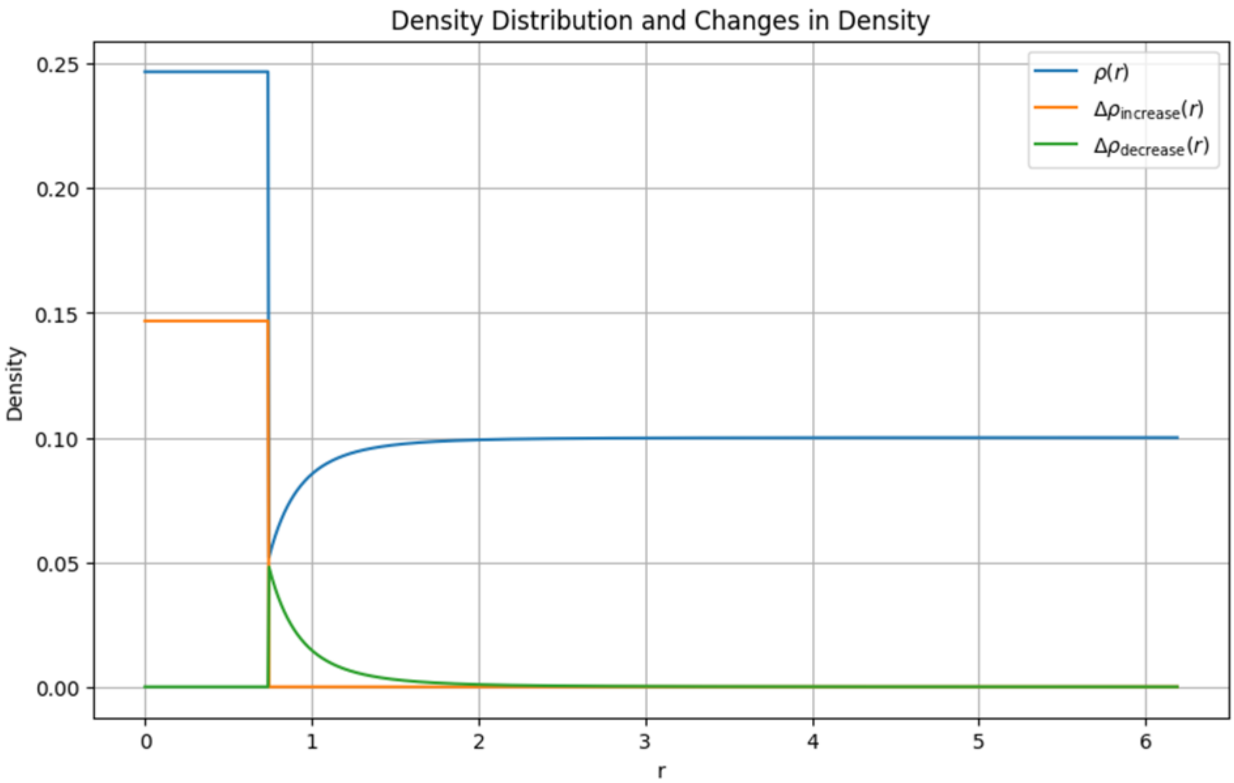

Figure 1: Graphs of space density distribution along a line passing through the center of the compressed sphere.

4.1 Representation of Space Density Distribution Using the Heaviside Function

The space density distribution, $\rho(r)$, for a single sphere can be expressed using the Heaviside function $H(x)$ to accurately describe the density inside and outside the compressed sphere. The main density distribution is defined as:

$$\rho(r) = \begin{cases} \rho_0 + \rho_1, & \text{if } r \leq R_1' \\ \rho_0 - \frac{R_1' \cdot \rho_1 \cdot V(R_1')}{4\pi r^4}, & \text{if } r > R_1' \end{cases}$$

The increase in density $\Delta\rho_{\text{increase}}(r)$ inside the compressed region can be expressed as:

$$\Delta\rho_{\text{increase}}(r) = \begin{cases} \rho_1, & \text{if } r \leq R_1' \\ 0, & \text{if } r > R_1' \end{cases}$$

Similarly, the decrease in density $\Delta\rho_{\text{decrease}}(r)$ outside the sphere:

$$\Delta\rho_{\text{decrease}}(r) = \begin{cases} 0, & \text{if } r \leq R_1' \\ \frac{R_1' \cdot \rho_1 \cdot V(R_1')}{4\pi r^4}, & \text{if } r > R_1' \end{cases}$$

Now we can rewrite these expressions in terms of the Heaviside function $H(x)$:

$$\Delta\rho_{\text{increase}}(r) = \rho_1 H(R_1' - r)$$

$$\Delta\rho_{\text{decrease}}(r) = \frac{R_1' \cdot \rho_1 \cdot V(R_1')}{4\pi r^4} H(r - R_1')$$

Thus, the total change in density $\Delta\rho(r)$:

$$\Delta\rho(r) = \rho_1 H(R_1' - r) - \frac{R_1' \cdot \rho_1 \cdot V(R_1')}{4\pi r^4} H(r - R_1')$$

4.1.1 Verification of Boundary Conditions

Now let us verify the boundary conditions:

1. For $r \leq R_1'$:

$$\Delta\rho(r) = \rho_1 H(R_1' - r) - \frac{R_1' \cdot \rho_1 \cdot V_{R1'}}{4\pi r^4} H(r - R_1')$$

Since $H(R_1' - r) = 1$ and $H(r - R_1') = 0$:

$$\Delta\rho(r) = \rho_1 - 0 = \rho_1$$

2. For $r > R_1'$:

$$\Delta\rho(r) = \rho_1 H(R_1' - r) - \frac{R_1' \cdot \rho_1 \cdot V_{R1'}}{4\pi r^4} H(r - R_1')$$

Since $H(R_1' - r) = 0$ and $H(r - R_1') = 1$:

$$\Delta\rho(r) = 0 - \frac{R_1' \cdot \rho_1 \cdot V_{R1'}}{4\pi r^4}$$

Now substitute $V_{R1'} = \frac{4}{3}\pi(R_1')^3$:

$$\Delta\rho(r) = -\frac{R_1' \cdot \rho_1 \cdot \frac{4}{3}\pi(R_1')^3}{4\pi r^4} = -\frac{\rho_1 \cdot R_1'^4}{3r^4}$$

Thus, we arrive at the following expression for $\Delta\rho(r)$ in terms of the Heaviside function:

$$\boxed{\Delta\rho(r) = \rho_1 H(R_1' - r) - \frac{\rho_1 \cdot R_1'^4}{3r^4} H(r - R_1') \qquad (3)}$$

4.2 Verification of Compliance with the Space Density Conservation Condition

To verify, let us take the integral of $\Delta\rho(r)$. Let us integrate $\Delta\rho(r)$ over the entire volume. Recall that $\Delta\rho(r)$ is represented as:

$$\Delta\rho(r) = \rho_1 \left[H(R_1' - r) - \frac{R_1'^4}{3r^4} H(r - R_1') \right]$$

Compute the integral:

$$\int_0^\infty \Delta\rho(r) \cdot 4\pi r^2 \, dr$$

Split the integral into two parts corresponding to $\Delta\rho_{\text{increase}}(r)$ and $\Delta\rho_{\text{decrease}}(r)$:

$$\int_0^\infty \Delta\rho(r) \cdot 4\pi r^2 \, dr = \int_0^\infty \left[\rho_1 H(R_1' - r) - \frac{\rho_1 \cdot R_1'^4}{3r^4} H(r - R_1') \right] \cdot 4\pi r^2 \, dr$$

Split into two separate integrals:

$$\int_0^\infty \rho_1 H(R_1' - r) \cdot 4\pi r^2 \, dr - \int_0^\infty \frac{\rho_1 \cdot R_1'^4}{3r^4} H(r - R_1') \cdot 4\pi r^2 \, dr$$

Consider the first integral:

$$\int_0^{R_1'} \rho_1 \cdot 4\pi r^2 \, dr = 4\pi\rho_1 \int_0^{R_1'} r^2 \, dr = 4\pi\rho_1 \left[\frac{r^3}{3} \right]_0^{R_1'} = 4\pi\rho_1 \cdot \frac{(R_1')^3}{3} = \frac{4\pi\rho_1 (R_1')^3}{3}$$

Now consider the second integral:

$$\int_{R_1'}^\infty \frac{\rho_1 \cdot R_1'^4}{3r^4} \cdot 4\pi r^2 \, dr = \frac{4\pi\rho_1 R_1'^4}{3} \int_{R_1'}^\infty \frac{1}{r^2} \, dr = \frac{4\pi\rho_1 R_1'^4}{3} \left[-\frac{1}{r} \right]_{R_1'}^\infty$$

Compute the limits:

$$\frac{4\pi\rho_1 R_1'^4}{3} \left(-\frac{1}{\infty} + \frac{1}{R_1'} \right) = \frac{4\pi\rho_1 R_1'^4}{3} \cdot \frac{1}{R_1'} = \frac{4\pi\rho_1 R_1'^3}{3}$$

Now add both results:

$$\int_0^\infty \Delta\rho(r) \cdot 4\pi r^2 \, dr = \frac{4\pi\rho_1 (R_1')^3}{3} - \frac{4\pi\rho_1 (R_1')^3}{3} = 0$$

Thus, the integral of $\Delta\rho(r)$ over the entire volume equals zero:

$$\int_0^\infty \Delta\rho(r) \cdot 4\pi r^2 \, dr = 0$$

We obtained the expected result, but this was necessary for verification.

V Quantity of Space Density Perturbation Created by Two Compressed Spheres at Distance D. Interaction Quantity. Coulomb's Law for Two Charges in Logarithmic Form

In this section, we investigate the interaction between two compressed spherical regions of space. By analyzing the space density distribution around these spheres, we derive the influence of one sphere on the density distribution of the other. This analysis is important for understanding the nature of their interaction arising from variations in space density.

5.1 Illustration of Space Density Distribution

Before proceeding to the mathematical derivation of the influence of space density distribution created by two spheres on each other, I present a graphical representation of the space density distribution around two compressed spheres, constructed based on the mathematical model using the formula for $\Delta\rho_{\text{decrease}}(r)$ (formula (2)). This figure allows us to visually understand how the density distribution created by each sphere changes depending on the distance between them.

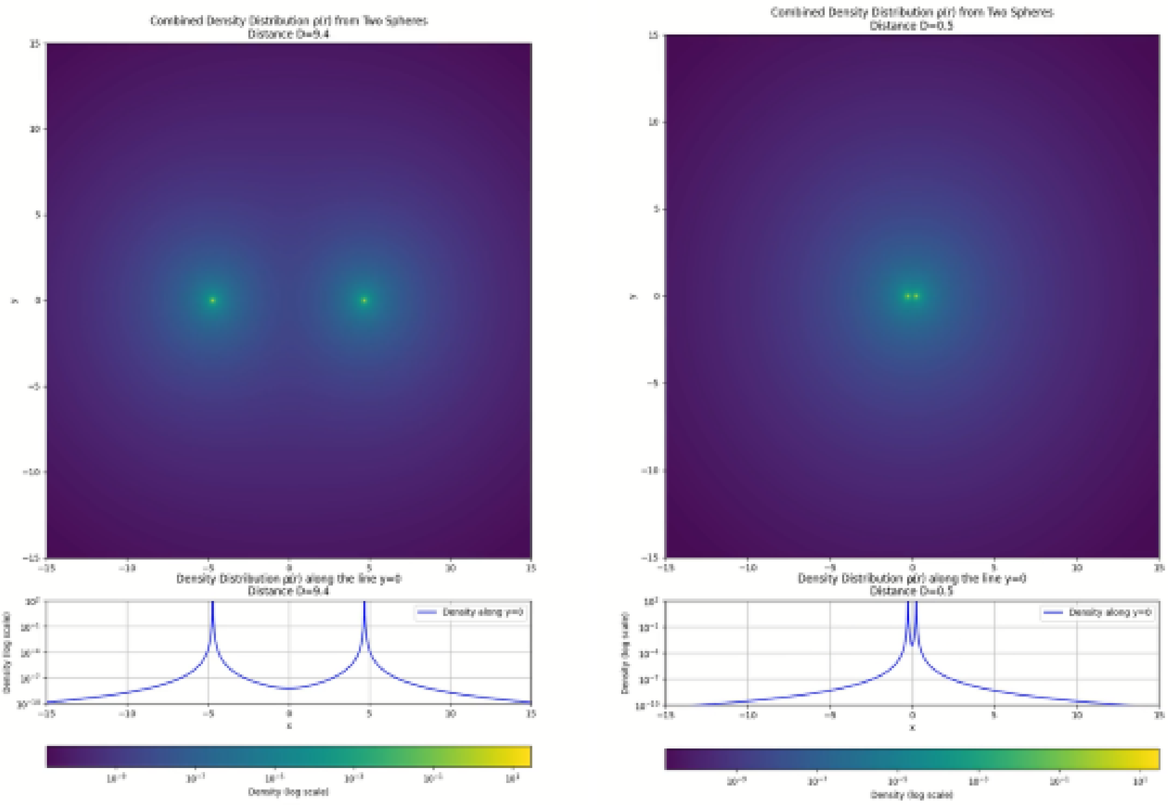

Figure 2: Space density distribution around two compressed spheres. The graph illustrates how space density changes along the line connecting the centers of the spheres as they approach each other.

5.2 Problem Formulation, Integral Expression for the Total Perturbation Created by Two Clumps of Space Density

To solve the problem, we start by writing the initial expression for the total perturbation W_{total}, using Fubini's theorem and gradient properties. We write the expression for the gradient of the product of functions:

$$\Delta\rho_1(r_1) \quad \text{and} \quad \Delta\rho_2(r_1' - D).$$

5.2.1 Initial Expression for Total Perturbation

The total perturbation W_{total} is defined as the integral over the entire space of the modulus of the gradient of the product of functions $\Delta\rho_1(r_1)$ and $\Delta\rho_2(r_1' - D)$. According to the assumption, the total quantity of space density perturbation created by two clumps relative to the reference system associated with the center of the first clump is determined by the formula:

$$W_{\text{total}} = \int_{V_1} \int_{V_1'} dV_1\, dV_1' \left| \nabla_{r_1} \nabla_{r_1'} \left(\Delta\rho_1(r_1) \cdot \Delta\rho_2(r_1' - D) \right) \right| .$$

5.2.2 Substitution of Function Expressions

Substitute the expressions for $\Delta\rho_1(r_1)$ and $\Delta\rho_2(r_1' - D)$:

$$\Delta\rho_1(r_1) = \rho_1 \left[H(R_1' - r_1) - \frac{R_1'^4}{3r_1^4} H(r_1 - R_1') \right],$$

$$\Delta\rho_2(r_1' - D) = \frac{R_2'\, \rho_2\, V(R_2')}{4\pi\, (r_1' - D)^4}.$$

Now substitute these expressions under the gradient sign:

$$W_{\text{total}} = \int_{V_1} \int_{V_1'} dV_1\, dV_1' \left| \nabla_{r_1} \nabla_{r_1'} \left(\rho_1 \left[H(R_1' - r_1) \right.\right.\right.$$
$$\left.\left.\left. - \frac{R_1'^4}{3r_1^4} H(r_1 - R_1') \right] \cdot \frac{R_2'\, \rho_2\, V(R_2')}{4\pi\, (r_1' - D)^4} \right) \right| .$$

Note that the gradient must be taken with respect to both V_1 (variable r_1) and V_1' (variable r_1'). Rewrite the expression taking into account gradients in both spaces.

5.2.3 Simplification of the Expression

For convenience, represent the expression in the following form:

$$W_{\text{total}} = \nabla_{r_1} \nabla_{r_1'} \left(\frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{4\pi\, (r_1' - D)^4} \cdot H(R_1' - r_1) \right)$$

$$- \nabla_{r_1} \nabla_{r_1'} \left(\frac{\rho_1 \, R_2' \, \rho_2 \, V(R_2') \, R_1'^4}{12\pi \, (r_1' - D)^4 \, r_1^4} \cdot H(r_1 - R_1') \right).$$

5.2.4 Simplification Considering Function Independence

Note that:

1. $\Delta\rho_1(r_1) = \rho_1 \left[H(R_1' - r_1) - \frac{R_1'^4}{3r_1^4} H(r_1 - R_1') \right]$ **does not depend** on r_1'.
2. $\Delta\rho_2(r_1' - D) = \frac{R_2' \, \rho_2 \, V(R_2')}{4\pi \, (r_1' - D)^4}$ **does not depend** on r_1.

Thus, the gradients can be separated:

- ∇_{r_1} acts only on $\Delta\rho_1(r_1)$.
- $\nabla_{r_1'}$ acts only on $\Delta\rho_2(r_1' - D)$.

5.2.5 Separation of Gradients

Now the expression can be rewritten as the product of gradients:

$$\nabla_{r_1} \left(\Delta\rho_1(r_1) \right) \cdot \nabla_{r_1'} \left(\Delta\rho_2(r_1' - D) \right).$$

Substitute the expressions for $\Delta\rho_1(r_1)$ and $\Delta\rho_2(r_1' - D)$:

$$\nabla_{r_1} \left(\rho_1 \left[H(R_1' - r_1) - \frac{R_1'^4}{3r_1^4} H(r_1 - R_1') \right] \right) \cdot \nabla_{r_1'} \left(\frac{R_2' \, \rho_2 \, V(R_2')}{4\pi \, (r_1' - D)^4} \right).$$

5.2.6 Final Expression

Now the expression takes the form:

$$\boxed{\begin{aligned} &\nabla_{r_1} \left(H(R_1' - r_1) \right) \cdot \nabla_{r_1'} \left(\frac{\rho_1 \, R_2' \, \rho_2 \, V(R_2')}{4\pi \, (r_1' - D)^4} \right) - \\ &\qquad - \nabla_{r_1} \left(\frac{1}{r_1^4} \cdot H(r_1 - R_1') \right) \cdot \nabla_{r_1'} \left(\frac{\rho_1 \, R_2' \, \rho_2 \, V(R_2') \, R_1'^4}{12\pi \, (r_1' - D)^4} \right). \quad (4) \end{aligned}}$$

5.2.7 Justification of the Approach

Here the question arises as to why the Leibniz rule is not applied when taking the gradient of the function. The reason is that we utilize the fact that the functions $\Delta\rho_1(r_1)$ and $\Delta\rho_2(r_1' - D)$ are completely independent of each other. To ensure this complete independence, we place them in different spaces V_1 and V_1'. We need to determine the total amount of perturbation, which is equal to the product of the density perturbation created by each of the clusters. For this purpose, in each space, for its corresponding function, we construct a continuous matrix of the absolute values of the gradient of each function and then compute the integral of each function over its respective space. In doing so, we must take into account that since the desired function is the product of two continuous sets of values (each corresponding to the integrals of the modulus of the gradient in the regions where one of the functions is nonzero), it does not make sense to integrate the second function in regions where the first is zero (and vice versa) because, when multiplying, a region where either function is zero will yield zero. In other words, the limits of integration, when representing the integral as a sum of products of several integrals with different limits for the first and second functions, must coincide. This is very important for constructing the total perturbation produced by the second cluster relative to the total perturbation of the first cluster.

5.2.8 Analogy with a Three-Dimensional Array

This approach can be compared to the methodology used when working with arrays of values obtained as the product of two functions. We can construct a three-dimensional array of values separately for each function with the same metric (i.e. the same array dimensions along all coordinates), and then multiply the cells of each array with matching indices to obtain the desired array of the product of the two functions. This is a kind of sublimation of a six-dimensional space, but it is not a six-dimensional space in the full sense of the word. We are seeking the projection onto our three-dimensional space of the product of the values of the two functions in two other independent spaces.

5.2.9 Justification for the Impossibility of the Solution in a Three-Dimensional Space

Next, I will explain why this problem cannot be solved in the usual three-dimensional space. The gradient of the function $1/r^4$ is given by $1/r^5$. For such a function, Gauss's theorem holds only in a space with $L = 6$, because in spaces of lower dimensionality the vector field in the form of the gradient of $1/r^4$ does not have a source.

5.3 Calculation of the Gradient and the Integral of the Gradient for the Total Perturbation Distribution of the Two-Cluster System Relative to the Reference Frame Associated with the Origin of the First Cluster

At this stage, we bring the gradient operators outside the parentheses for each expression, taking into account that:

1. ∇_{r_1} acts only on functions that depend on r_1.
2. $\nabla_{r_1'}$ acts only on functions that depend on r_1'.

5.3.1 Original Expression

The original expression is:

$$\nabla_{r_1}\nabla_{r_1'}\left(\frac{\rho_1\,R_2'\,\rho_2\,V(R_2')}{4\pi\,(r_1'-D)^4}\cdot H(R_1'-r_1)\right) \\ -\nabla_{r_1}\nabla_{r_1'}\left(\frac{\rho_1\,R_2'\,\rho_2\,V(R_2')\,R_1'^4}{12\pi\,(r_1'-D)^4\,r_1^4}\cdot H(r_1-R_1')\right).$$

5.3.2 Bringing the Gradient Operators Outside the Parentheses

First Term:

$$\nabla_{r_1}\nabla_{r_1'}\left(\frac{\rho_1\,R_2'\,\rho_2\,V(R_2')}{4\pi\,(r_1'-D)^4}\cdot H(R_1'-r_1)\right).$$

Here:

- ∇_{r_1} acts only on $H(R_1'-r_1)$ since $\frac{\rho_1\,R_2'\,\rho_2\,V(R_2')}{4\pi\,(r_1'-D)^4}$ does not depend on r_1.
- $\nabla_{r_1'}$ acts only on $\frac{\rho_1\,R_2'\,\rho_2\,V(R_2')}{4\pi\,(r_1'-D)^4}$ since $H(R_1'-r_1)$ does not depend on r_1'.

Thus, we can write:

$$\nabla_{r_1}\left(H(R_1'-r_1)\right)\cdot\nabla_{r_1'}\left(\frac{\rho_1\,R_2'\,\rho_2\,V(R_2')}{4\pi\,(r_1'-D)^4}\right).$$

Second Term:

$$\nabla_{r_1}\nabla_{r_1'}\left(\frac{\rho_1\,R_2'\,\rho_2\,V(R_2')\,R_1'^4}{12\pi\,(r_1'-D)^4\,r_1^4}\cdot H(r_1-R_1')\right).$$

Here:

- ∇_{r_1} acts only on $\frac{1}{r_1^4}\cdot H(r_1-R_1')$ since $\frac{\rho_1\,R_2'\,\rho_2\,V(R_2')\,R_1'^4}{12\pi\,(r_1'-D)^4}$ does not depend on r_1.
- $\nabla_{r_1'}$ acts only on $\frac{\rho_1\,R_2'\,\rho_2\,V(R_2')\,R_1'^4}{12\pi\,(r_1'-D)^4}$ since $H(r_1-R_1')$ does not depend on r_1'.

Thus, we have:

$$\nabla_{r_1}\left(\frac{1}{r_1^4}\cdot H(r_1-R_1')\right)\cdot\nabla_{r_1'}\left(\frac{\rho_1\,R_2'\,\rho_2\,V(R_2')\,R_1'^4}{12\pi\,(r_1'-D)^4}\right).$$

5.3.3 Final Expression after Bringing the Gradients Outside the Parentheses

Now the expression becomes:

$$\nabla_{r_1}\left(H(R_1'-r_1)\right)\cdot\nabla_{r_1'}\left(\frac{\rho_1\,R_2'\,\rho_2\,V(R_2')}{4\pi\,(r_1'-D)^4}\right) \\ -\nabla_{r_1}\left(\frac{1}{r_1^4}\cdot H(r_1-R_1')\right)\cdot\nabla_{r_1'}\left(\frac{\rho_1\,R_2'\,\rho_2\,V(R_2')\,R_1'^4}{12\pi\,(r_1'-D)^4}\right).$$

5.3.4 Calculation of the Gradients

1. Calculation of $\nabla_{r_1}(H(R'_1 - r_1))$: The function $H(R'_1 - r_1)$ is the Heaviside step function. Its gradient can be expressed in terms of the Dirac delta function:

$$\nabla_{r_1}(H(R'_1 - r_1)) = -\delta(R'_1 - r_1) \cdot \hat{r}_1,$$

where:

- $\delta(R'_1 - r_1)$ is the Dirac delta function,
- $\hat{r}_1$ is the unit vector in the direction of r_1.

2. Calculation of $\nabla_{r'_1}\left(\frac{\rho_1 R'_2 \rho_2 V(R'_2)}{4\pi (r'_1 - D)^4}\right)$: We compute the gradient of the function $\frac{1}{(r'_1 - D)^4}$. The gradient of the scalar function $f(r) = \frac{1}{r^4}$ is given by:

$$\nabla\left(\frac{1}{r^4}\right) = -\frac{4}{r^5}\hat{r}.$$

Applying this to our function:

$$\nabla_{r'_1}\left(\frac{1}{(r'_1 - D)^4}\right) = -\frac{4}{(r'_1 - D)^5}\hat{r}'_1.$$

Thus:

$$\nabla_{r'_1}\left(\frac{\rho_1 R'_2 \rho_2 V(R'_2)}{4\pi (r'_1 - D)^4}\right) = \frac{\rho_1 R'_2 \rho_2 V(R'_2)}{4\pi} \cdot \left(-\frac{4}{(r'_1 - D)^5}\hat{r}'_1\right) = -\frac{\rho_1 R'_2 \rho_2 V(R'_2)}{\pi (r'_1 - D)^5}\hat{r}'_1.$$

3. Calculation of $\nabla_{r_1}\left(\frac{1}{r_1^4} \cdot H(r_1 - R'_1)\right)$: Here we apply the product rule:

$$\nabla_{r_1}\left(\frac{1}{r_1^4} \cdot H(r_1 - R'_1)\right) = \nabla_{r_1}\left(\frac{1}{r_1^4}\right) \cdot H(r_1 - R'_1) + \frac{1}{r_1^4} \cdot \nabla_{r_1}(H(r_1 - R'_1)).$$

We compute each term:

1. The gradient of $\frac{1}{r_1^4}$:

$$\nabla_{r_1}\left(\frac{1}{r_1^4}\right) = -\frac{4}{r_1^5}\hat{r}_1.$$

2. The gradient of the Heaviside function:

$$\nabla_{r_1}(H(r_1 - R'_1)) = \delta(r_1 - R'_1) \cdot \hat{r}_1.$$

Thus:

$$\nabla_{r_1}\left(\frac{1}{r_1^4} \cdot H(r_1 - R'_1)\right) = -\frac{4}{r_1^5}\hat{r}_1 \cdot H(r_1 - R'_1) + \frac{1}{r_1^4} \cdot \delta(r_1 - R'_1) \cdot \hat{r}_1.$$

4. Calculation of $\nabla_{r_1'}\left(\frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{12\pi\,(r_1'-D)^4}\right)$: This gradient is analogous to the second case:

$$\nabla_{r_1'}\left(\frac{1}{(r_1'-D)^4}\right) = -\frac{4}{(r_1'-D)^5}\hat{r}_1'.$$

Thus:

$$\nabla_{r_1'}\left(\frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{12\pi\,(r_1'-D)^4}\right) = \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{12\pi} \cdot \left(-\frac{4}{(r_1'-D)^5}\hat{r}_1'\right) = -\frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{3\pi\,(r_1'-D)^5}\hat{r}_1'.$$

5.3.5 Final Integrand Expression

Now substitute the computed gradients back into the original expression:

$$\nabla_{r_1}\left(H(R_1'-r_1)\right)\cdot\nabla_{r_1'}\left(\frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{4\pi\,(r_1'-D)^4}\right) - \nabla_{r_1}\left(\frac{1}{r_1^4}\cdot H(r_1-R_1')\right)\cdot\nabla_{r_1'}\left(\frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{12\pi\,(r_1'-D)^4}\right).$$

Substituting the results:

$$\left(-\delta(R_1'-r_1)\cdot\hat{r}_1\right)\cdot\left(-\frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{\pi\,(r_1'-D)^5}\hat{r}_1'\right) - \left(-\frac{4}{r_1^5}\hat{r}_1\cdot H(r_1-R_1') + \frac{1}{r_1^4}\cdot\delta(r_1-R_1')\cdot\hat{r}_1\right)\cdot\left(-\frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{3\pi\,(r_1'-D)^5}\hat{r}_1'\right).$$

5.3.6 Simplification of the Expression

We simplify the expression, taking into account that $\hat{r}_1\cdot\hat{r}_1' = 1$ (if the directions coincide):

$$\delta(R_1'-r_1)\cdot\frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{\pi\,(r_1'-D)^5} - \left(-\frac{4}{r_1^5}H(r_1-R_1') + \frac{1}{r_1^4}\delta(r_1-R_1')\right)\cdot\frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{3\pi\,(r_1'-D)^5}.$$

5.3.7 Final Integrand Expression

Now we substitute this expression into the integral:

$$\boxed{\begin{aligned} W_{\text{total}} = \int_{V_1}\int_{V_1'} dV_1\, dV_1' \Bigg| \delta(R_1' - r_1) \cdot \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{\pi\,(r_1' - D)^5} - \\ - \left(-\frac{4}{r_1^5} H(r_1 - R_1') + \frac{1}{r_1^4}\delta(r_1 - R_1') \right) \cdot \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{3\pi\,(r_1' - D)^5} \Bigg|. \quad (5) \end{aligned}}$$

5.3.8 Consideration of the Regions Where the Functions are Non-Zero

a) The Heaviside Function $H(R_1' - r_1)$:

- $H(R_1' - r_1) = 1$ for $r_1 \leq R_1'$,
- $H(R_1' - r_1) = 0$ for $r_1 > R_1'$.

b) The Heaviside Function $H(r_1 - R_1')$:

- $H(r_1 - R_1') = 1$ for $r_1 \geq R_1'$,
- $H(r_1 - R_1') = 0$ for $r_1 < R_1'$.

c) The Dirac Delta Function $\delta(R_1' - r_1)$:

- $\delta(R_1' - r_1)$ "selects" the value $r_1 = R_1'$.

d) The Dirac Delta Function $\delta(r_1 - R_1')$:

- $\delta(r_1 - R_1')$ also "selects" the value $r_1 = R_1'$.

5.3.9 Matching the Limits of Integration

To ensure that the domains of the functions coincide, the integration limits for r_1 and r_1' must be identical. This means that:

1. The integration limits for r_1 and r_1' are set the same.
2. The regions where the Heaviside and Delta functions are non-zero are taken into account.

5.3.10 Adjustment of the Integration Limits

Now we rewrite the integrals, specifying **identical integration limits** for r_1 and r_1':

First Integral:

$$\int_{r_1=0}^{R_1'}\int_{r_1'=0}^{R_1'} \left| \delta(R_1' - r_1) \cdot \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{\pi\,(r_1' - D)^5} \right| dV_1\, dV_1'.$$

Here:

- r_1 ranges from 0 to R_1' (where $H(R_1' - r_1) = 1$),
- r_1' ranges from 0 to R_1' (to ensure matching limits).

Second Integral:

$$\int_{r_1=R_1'}^{\infty} \int_{r_1'=R_1'}^{\infty} \left| \frac{4}{r_1^5} \cdot \frac{\rho_1 \, R_2' \, \rho_2 \, V(R_2') \, R_1'^4}{3\pi \, (r_1' - D)^5} \right| \, dV_1 \, dV_1'.$$

Here:

- r_1 ranges from R_1' to ∞ (where $H(r_1 - R_1') = 1$),
- r_1' ranges from R_1' to ∞ (to ensure matching limits).

Third Integral:

$$\int_{r_1=R_1'}^{\infty} \int_{r_1'=R_1'}^{\infty} \left| \frac{1}{r_1^4} \delta(r_1 - R_1') \cdot \frac{\rho_1 \, R_2' \, \rho_2 \, V(R_2') \, R_1'^4}{3\pi \, (r_1' - D)^5} \right| \, dV_1 \, dV_1'.$$

Here:

- r_1 ranges from R_1' to ∞ (where $\delta(r_1 - R_1')$ "selects" $r_1 = R_1'$),
- r_1' ranges from R_1' to ∞ (to ensure matching limits).

5.4 Transition to Surface Integrals Using the Properties of the Dirac Delta Function

5.4.1 Properties of the Dirac Delta Function

The Dirac delta function $\delta(R_1' - r_1)$ has the following property:

$$\int_{V_1} f(r_1) \, \delta(R_1' - r_1) \, dV_1 = f(R_1'),$$

where $f(r_1)$ is an arbitrary function, and the integral is taken over the volume V_1. This means that the delta function "selects" the value of the function $f(r_1)$ on the surface of the sphere with radius R_1'.

5.4.2 Transition to Surface Integrals

a) First Integral: The original integral:

$$\int_{V_1} \int_{V_1'} \left| \delta(R_1' - r_1) \cdot \frac{\rho_1 \, R_2' \, \rho_2 \, V(R_2')}{\pi \, (r_1' - D)^5} \right| \, dV_1 \, dV_1'.$$

The delta function $\delta(R_1' - r_1)$ "selects" the value on the surface of the sphere with radius R_1'. Thus, the integral over dV_1 reduces to a surface integral over the sphere of radius R_1':

$$\int_{V_1} f(r_1) \, \delta(R_1' - r_1) \, dV_1 = f(R_1'),$$

where S_1 denotes the surface of the sphere with radius R'_1.

Similarly, the integral over dV'_1 also reduces to a surface integral over the sphere with radius R'_1:

$$\int_{V'_1} \frac{\rho_1 \, R'_2 \, \rho_2 \, V(R'_2)}{\pi \, (r'_1 - D)^5} \, dV'_1 = \int_{S'_1} \frac{\rho_1 \, R'_2 \, \rho_2 \, V(R'_2)}{\pi \, (r'_1 - D)^5} \, dS'_1.$$

Thus, the first integral takes the form:

$$\int_{S_1} \int_{S'_1} \left| \frac{\rho_1 \, R'_2 \, \rho_2 \, V(R'_2)}{\pi \, (r'_1 - D)^5} \right| \, dS_1 \, dS'_1.$$

b) Third Integral: The original integral:

$$\int_{V_1} \int_{V'_1} \left| \frac{1}{r_1^4} \delta(r_1 - R'_1) \cdot \frac{\rho_1 \, R'_2 \, \rho_2 \, V(R'_2) \, R'^4_1}{3\pi \, (r'_1 - D)^5} \right| \, dV_1 \, dV'_1.$$

The delta function $\delta(r_1 - R'_1)$ also "selects" the value on the surface of the sphere with radius R'_1. Thus, the integral over dV_1 reduces to a surface integral over the sphere with radius R'_1:

$$\int_{V_1} \frac{1}{r_1^4} \delta(r_1 - R'_1) \, dV_1 = \int_{S_1} \frac{1}{R'^4_1} \, dS_1.$$

Similarly, the integral over dV'_1 reduces to a surface integral over the sphere with radius R'_1:

$$\int_{V'_1} \frac{\rho_1 \, R'_2 \, \rho_2 \, V(R'_2) \, R'^4_1}{3\pi \, (r'_1 - D)^5} \, dV'_1 = \int_{S'_1} \frac{\rho_1 \, R'_2 \, \rho_2 \, V(R'_2) \, R'^4_1}{3\pi \, (r'_1 - D)^5} \, dS'_1.$$

Thus, after applying the delta function $\delta(r_1 - R'_1)$ and transitioning to surface integrals, the third integral takes the form:

$$\int_{S_1} \int_{S'_1} \left| \frac{\rho_1 \, R'_2 \, \rho_2 \, V(R'_2) \, R'^4_1}{3\pi \, R'^4_1 \, (r'_1 - D)^5} \right| \, dS_1 \, dS'_1.$$

Here, the R'^4_1 factors in the numerator and denominator cancel:

$$\int_{S_1} \int_{S'_1} \left| \frac{\rho_1 \, R'_2 \, \rho_2 \, V(R'_2)}{3\pi \, (r'_1 - D)^5} \right| \, dS_1 \, dS'_1.$$

5.4.3 Application of Fubini's Theorem

Fubini's theorem allows us to separate double integrals into a product of integrals. We apply it to the first and third integrals.

a) First Integral:

$$\int_{S_1} \int_{S'_1} \left| \frac{\rho_1 \, R'_2 \, \rho_2 \, V(R'_2)}{\pi \, (r'_1 - D)^5} \right| \, dS_1 \, dS'_1.$$

This integral can be represented as the product:

$$\left(\int_{S_1} dS_1\right) \cdot \left(\int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{\pi\, (r_1' - D)^5}\, dS_1'\right).$$

b) Third Integral:

$$\int_{S_1} \int_{S_1'} \left| \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{3\pi\, (r_1' - D)^5} \right| dS_1\, dS_1'.$$

This integral can similarly be written as:

$$\left(\int_{S_1} dS_1\right) \cdot \left(\int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{3\pi\, (r_1' - D)^5}\, dS_1'\right).$$

5.4.4 Calculation of the Integrals over dS_1

The integral over the surface of a sphere S_1 with radius R_1' is equal to the surface area of the sphere:

$$\int_{S_1} dS_1 = 4\pi R_1'^2.$$

Thus, the first and third integrals become:

a) First Integral:

$$4\pi R_1'^2 \cdot \int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{\pi\, (r_1' - D)^5}\, dS_1'.$$

b) Third Integral:

$$4\pi R_1'^2 \cdot \int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{3\pi\, (r_1' - D)^5}\, dS_1'.$$

5.4.5 Final Expressions

Thus, after applying Fubini's theorem and computing the integrals over dS_1, we obtain the following expressions for the first and third integrals:

a) First Integral:

$$4\pi R_1'^2 \cdot \int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{\pi\, (r_1' - D)^5}\, dS_1'.$$

b) Third Integral:

$$4\pi R_1'^2 \cdot \int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{3\pi\, (r_1' - D)^5}\, dS_1'.$$

Thus, we have transitioned from volume integrals to surface integrals using the properties of the Dirac delta function and Fubini's theorem. This simplifies the calculations and allows us to focus on integrating over the surfaces of the spheres.

5.5 Calculation of the Second Integral

The original second integral is:

$$\int_{r_1=R_1'}^{\infty} \int_{r_1'=R_1'}^{\infty} \left| \frac{4}{r_1^5} \cdot \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{3\pi\,(r_1' - D)^5} \right| dV_1\, dV_1'.$$

We apply Fubini's theorem to separate the integral into the product:

$$\left(\int_{r_1=R_1'}^{\infty} \frac{4}{r_1^5}\, dV_1 \right) \cdot \left(\int_{r_1'=R_1'}^{\infty} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{3\pi\,(r_1' - D)^5}\, dV_1' \right).$$

5.5.1 Calculation of the Integral over dV_1

The integral over dV_1 is given by:

$$\int_{r_1=R_1'}^{\infty} \frac{4}{r_1^5}\, dV_1.$$

In spherical coordinates, $dV_1 = 4\pi r_1^2\, dr_1$, so:

$$\int_{r_1=R_1'}^{\infty} \frac{4}{r_1^5} \cdot 4\pi r_1^2\, dr_1 = 16\pi \int_{r_1=R_1'}^{\infty} \frac{1}{r_1^3}\, dr_1.$$

Evaluating this integral:

$$16\pi \int_{r_1=R_1'}^{\infty} \frac{1}{r_1^3}\, dr_1 = 16\pi \left[-\frac{1}{2r_1^2} \right]_{R_1'}^{\infty} = 16\pi \left(0 + \frac{1}{2R_1'^2} \right) = \frac{8\pi}{R_1'^2}.$$

5.5.2 Final Formula for the Second Integral

Thus, the final formula for the second integral becomes:

$$\frac{8\pi}{R_1'^2} \cdot \int_{r_1'=R_1'}^{\infty} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{3\pi\,(r_1' - D)^5}\, dV_1'.$$

5.6 Final Expression for the Total Perturbation

The original expression for the total perturbation integral was:

$$
\begin{aligned}
W_{\text{total}} = &\left(\int_{V_1} \delta(R_1' - r_1)\, dV_1\right) \cdot \left(\int_{V_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{\pi\, (r_1' - D)^5}\, dV_1'\right) \\
&+ \left(\int_{V_1} \frac{4}{r_1^5} H(r_1 - R_1')\, dV_1\right) \cdot \left(\int_{V_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{3\pi\, (r_1' - D)^5}\, dV_1'\right) \\
&- \left(\int_{V_1} \frac{1}{r_1^4} \delta(r_1 - R_1')\, dV_1\right) \cdot \left(\int_{V_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{3\pi\, (r_1' - D)^5}\, dV_1'\right). \quad (6)
\end{aligned}
$$

5.6.1 Calculation of the Integrals with Respect to the Coordinate r_1

First Integral:

$$\int_{V_1} \delta(R_1' - r_1)\, dV_1 = 4\pi R_1'^2.$$

The second integral in the first term is taken over the surface S_1':

$$\int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{\pi\, (r_1' - D)^5}\, dS_1'.$$

Substituting into the first term:

$$4\pi R_1'^2 \cdot \int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{\pi\, (r_1' - D)^5}\, dS_1'.$$

Second Integral:

$$\int_{V_1} \frac{4}{r_1^5} H(r_1 - R_1')\, dV_1 = \frac{8\pi}{R_1'^2}.$$

The second integral in the second term is taken over the volume V_1':

$$\int_{V_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{3\pi\, (r_1' - D)^5}\, dV_1'.$$

Substituting into the second term:

$$\frac{8\pi}{R_1'^2} \cdot \int_{V_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{3\pi\, (r_1' - D)^5}\, dV_1'.$$

Third Integral:

$$\int_{V_1} \frac{1}{r_1^4} \delta(r_1 - R_1')\, dV_1 = 4\pi R_1'^2.$$

The second integral in the third term is taken over the surface S_1':

$$\int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{3\pi\, (r_1' - D)^5}\, dS_1'.$$

Substituting into the third term:

$$4\pi R_1'^2 \cdot \int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{3\pi\, (r_1' - D)^5}\, dS_1'.$$

Final Expression for the Total Perturbation

Now, substituting the calculated integrals into the original expression:

$$\boxed{\begin{aligned} W_{\text{total}} = 4\pi R_1'^2 \cdot \int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{\pi\, (r_1' - D)^5}\, dS_1' \\ + \frac{8\pi}{R_1'^2} \cdot \int_{V_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')\, R_1'^4}{3\pi\, (r_1' - D)^5}\, dV_1' \\ - 4\pi R_1'^2 \cdot \int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{3\pi\, (r_1' - D)^5}\, dS_1'. \quad (7) \end{aligned}}$$

5.6.2 Simplification of the First and Third Terms

The first and third integrals share a common factor $4\pi R_1'^2$. We factor this out:

$$4\pi R_1'^2 \cdot \left(\int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{\pi\, (r_1' - D)^5}\, dS_1' - \int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{3\pi\, (r_1' - D)^5}\, dS_1' \right).$$

Simplify the expression inside the parentheses:

$$\int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{\pi\, (r_1' - D)^5}\, dS_1' - \int_{S_1'} \frac{\rho_1\, R_2'\, \rho_2\, V(R_2')}{3\pi\, (r_1' - D)^5}\, dS_1' = \int_{S_1'} \frac{2\rho_1\, R_2'\, \rho_2\, V(R_2')}{3\pi\, (r_1' - D)^5}\, dS_1'.$$

Now, recall that

$$\rho_1 = \frac{Q_1}{\frac{4}{3}\pi R_1'^3},$$

and also $\rho_2\, V(R_2') = Q_2$. Thus, we obtain:

$$4\pi R_1'^2 \cdot \int_{S_1'} \frac{2 \cdot \frac{Q_1}{\frac{4}{3}\pi R_1'^3} \cdot R_2' \cdot Q_2}{3\pi\, (r_1' - D)^5}\, dS_1'.$$

Simplifying the coefficients:

$$4\pi R_1'^2 \cdot \frac{2Q_1 Q_2 R_2'}{3\pi \cdot \frac{4}{3}\pi R_1'^3} \int_{S_1'} \frac{dS_1'}{(r_1' - D)^5} = \frac{8 Q_1 Q_2 R_2'}{\pi R_1'} \int_{S_1'} \frac{dS_1'}{(r_1' - D)^5}.$$

5.6.3 Simplification of the Second Term

The second integral is

$$\frac{8\pi}{R_1'^2} \cdot \int_{r_1'=R_1'}^{\infty} \frac{\rho_1 \, R_2' \, \rho_2 \, V(R_2') \, R_1'^4}{3\pi \, (r_1' - D)^5} \, dV_1'.$$

Substitute $\rho_1 = \frac{Q_1}{\frac{4}{3}\pi R_1'^3}$ and $\rho_2 \, V(R_2') = Q_2$:

$$\frac{8\pi}{R_1'^2} \cdot \int_{r_1'=R_1'}^{\infty} \frac{\frac{Q_1}{\frac{4}{3}\pi R_1'^3} \cdot R_2' \cdot Q_2 \cdot R_1'^4}{3\pi \, (r_1' - D)^5} \, dV_1'.$$

Simplify the coefficients:

$$\frac{8\pi}{R_1'^2} \cdot \frac{Q_1 Q_2 R_2' R_1'}{\frac{4}{3}\pi \cdot 3\pi} \int_{r_1'=R_1'}^{\infty} \frac{dV_1'}{(r_1' - D)^5} = \frac{2 Q_1 Q_2 R_2'}{\pi R_1'} \int_{r_1'=R_1'}^{\infty} \frac{dV_1'}{(r_1' - D)^5}.$$

5.6.4 Final Expression

Now, combining all the terms we obtain:

$$\boxed{W_{\text{total}} = \frac{8 Q_1 Q_2 R_2'}{\pi R_1'} \int_{S_1'} \frac{dS_1'}{(r_1' - D)^5} + \frac{2 Q_1 Q_2 R_2'}{\pi R_1'} \int_{r_1'=R_1'}^{\infty} \frac{dV_1'}{(r_1' - D)^5}. \quad (8)}$$

5.7 Let Us Now Compute the First Integral:

$$\frac{8 Q_1 Q_2 R_2'}{\pi R_1'} \int_{S_1'} \frac{dS_1'}{|\mathbf{r}_1' - \mathbf{D}|^5}.$$

Where:

- S_1' is the sphere centered at the origin with radius R_1', i.e. for any point on the sphere its position vector $\mathbf{r}_1'$ satisfies $|\mathbf{r}_1'| = R_1'$;
- $\mathbf{D}$ is a fixed vector, whose modulus we denote by $D = |\mathbf{D}|$;
- The notation $|\mathbf{r}_1' - \mathbf{D}|$ denotes the distance between the point $\mathbf{r}_1'$ and the point specified by the vector $\mathbf{D}$.

5.7.1 Calculation of the Surface Integral

Choose the coordinate system such that the z-axis is directed along $\mathbf{D}$. Then a point on the sphere can be written in spherical coordinates as:

$$\mathbf{r}_1' = R_1' \, (\sin\theta \cos\phi, \; \sin\theta \sin\phi, \; \cos\theta),$$

and the surface element is

$$dS'_1 = R'^2_1 \sin\theta \, d\theta \, d\phi \, .$$

The modulus of the difference of the vectors is:

$$|\mathbf{r}'_1 - \mathbf{D}| = \sqrt{R'^2_1 + D^2 - 2R'_1 D \cos\theta} \, .$$

Then the integral becomes

$$I = \int_{S'_1} \frac{dS'_1}{|\mathbf{r}'_1 - \mathbf{D}|^5} = \int_0^{2\pi} \int_0^{\pi} \frac{R'^2_1 \sin\theta \, d\theta \, d\phi}{\left(R'^2_1 + D^2 - 2R'_1 D \cos\theta\right)^{5/2}} \, .$$

Integration over ϕ gives a factor of 2π:

$$I = 2\pi R'^2_1 \int_0^{\pi} \frac{\sin\theta \, d\theta}{\left(R'^2_1 + D^2 - 2R'_1 D \cos\theta\right)^{5/2}} \, .$$

5.7.2 Change of Variable $u = \cos\theta$

Let $u = \cos\theta$, then $du = -\sin\theta \, d\theta$. When $\theta = 0$, we have $u = 1$, and when $\theta = \pi$, $u = -1$. Thus, we obtain:

$$I = 2\pi R'^2_1 \int_{u=1}^{-1} \frac{-du}{\left(R'^2_1 + D^2 - 2R'_1 D \, u\right)^{5/2}} = 2\pi R'^2_1 \int_{-1}^{1} \frac{du}{\left(R'^2_1 + D^2 - 2R'_1 D \, u\right)^{5/2}} \, .$$

Let us denote:

$$A = R'^2_1 + D^2, \quad B = 2R'_1 D \, .$$

Then the integral takes the form

$$I = 2\pi R'^2_1 \int_{-1}^{1} \frac{du}{(A - Bu)^{5/2}} \, .$$

5.7.3 Evaluation of the Integral with Respect to u

Perform the substitution: $v = A - Bu$ so that $dv = -B \, du$ or $du = -\frac{dv}{B}$. When $u = -1$ we get $v = A + B$, and when $u = 1$ we have $v = A - B$. Then:

$$\begin{aligned}\int_{-1}^{1} \frac{du}{(A - Bu)^{5/2}} &= \frac{1}{B} \int_{v=A-B}^{A+B} \frac{dv}{v^{5/2}} = -\frac{1}{B} \int_{v=A+B}^{A-B} \frac{dv}{v^{5/2}} \\ &= -\frac{2}{3B} \left[v^{-3/2} \right]_{v=A+B}^{A-B} = \frac{2}{3B} \left[(A - B)^{-3/2} - (A + B)^{-3/2} \right] .\end{aligned}$$

Thus,

$$I = 2\pi R_1'^2 \cdot \frac{1}{B} \cdot \frac{2}{3}\left[(A-B)^{-3/2} - (A+B)^{-3/2}\right]$$

$$= \frac{4\pi R_1'^2}{3B}\left[(A-B)^{-3/2} - (A+B)^{-3/2}\right].$$

Returning to the original notations:

$$A \pm B = R_1'^2 + D^2 \mp 2R_1'D = (R_1' \mp D)^2 .$$

Therefore,

$$(A \mp B)^{-3/2} = \frac{1}{\left[(R_1' \mp D)^2\right]^{3/2}} = \frac{1}{(R_1' \mp D)^3} .$$

Also, $B = 2R_1'D$. Then,

$$I = \frac{4\pi R_1'^2}{3 \cdot 2R_1'D}\left[\frac{1}{(R_1'-D)^3} - \frac{1}{(R_1'+D)^3}\right]$$

$$= \frac{2\pi R_1'}{3D}\left[\frac{1}{(R_1'-D)^3} - \frac{1}{(R_1'+D)^3}\right] .$$

5.7.4 Final Result

Substituting the obtained I into the original expression, we have:

$$\frac{8Q_1Q_2R_2'}{\pi R_1'} I = \frac{8Q_1Q_2R_2'}{\pi R_1'} \cdot \frac{2\pi R_1'}{3D} \cdot \left[\frac{1}{(R_1'-D)^3} - \frac{1}{(R_1'+D)^3}\right] .$$

Canceling π and R_1', we finally obtain:

$$\frac{16Q_1Q_2R_2'}{3D}\left[\frac{1}{(R_1'-D)^3} - \frac{1}{(R_1'+D)^3}\right] .$$

$$\boxed{\frac{8Q_1Q_2R_2'}{\pi R_1'}\int_{S_1'} \frac{dS_1'}{|\mathbf{r}_1' - \mathbf{D}|^5} = \frac{16Q_1Q_2R_2'}{3D}\left[\frac{1}{(R_1'-D)^3} - \frac{1}{(R_1'+D)^3}\right] \quad (9)}$$

5.8 Calculation of the Second Integral in the Expression for the Total Perturbation of the Two-Cluster Spatial Density System Relative to the Reference Frame $\mathbf{r}_1'$ Associated with the Center of the First Cluster's Sphere

In the obtained solution for the total perturbation, the second integral has the form:

$$I = \frac{2Q_1Q_2R_2'}{\pi R_1'}\int_{r_1'=R_1'}^{\infty} \frac{dV_1'}{|\mathbf{r}_1' - \mathbf{D}|^5},$$

where:

- $\mathbf{r_1'}$ is the position vector in the $\mathbf{r_1'}$ system,
- $\mathbf{D}$ is a vector lying along the Z-axis, directed from the origin of the $\mathbf{r_1'}$ system to the origin of the $\mathbf{r_2'}$ system,
- dV_1' is the volume element in the $\mathbf{r_1'}$ system.

5.8.1 Transition to the $\mathbf{r_2'}$ System

The transformation to the $\mathbf{r_2'}$ system is performed via the relation:

$$\mathbf{r_2'} = \mathbf{r_1'} - \mathbf{D}.$$

In this case,

$$r_2' = |\mathbf{r_2'}| = |\mathbf{r_1'} - \mathbf{D}|.$$

Since $\mathbf{D}$ lies along the Z-axis, the X and Y axes coincide, and the angle between the vector $\mathbf{r_1'}$ and $\mathbf{D}$ is the same as the angle θ in the $\mathbf{r_2'}$ system.

5.8.2 Writing the Integral in the Spherical Coordinate System $\mathbf{r_1'}$

In the $\mathbf{r_1'}$ system, the volume element in spherical coordinates is given by:

$$dV_1' = (r_1')^2 \sin\theta \, dr_1' \, d\theta \, d\phi.$$

Thus, the integral is written as:

$$I = \frac{2Q_1Q_2R_2'}{\pi R_1'} \int_{r_1'=R_1'}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{(r_1')^2 \sin\theta}{|\mathbf{r_1'} - \mathbf{D}|^5} \, d\phi \, d\theta \, dr_1'.$$

5.8.3 Transition to the Spherical Coordinate System $\mathbf{r_2'}$

Consider the integral in the original spherical coordinate system $\mathbf{r_1'}$:

$$I = \frac{2Q_1Q_2R_2'}{\pi R_1'} \int_{r_1'=R_1'}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{(r_1')^2 \sin\theta}{|\mathbf{r_1'} - \mathbf{D}|^5} \, d\phi \, d\theta \, dr_1'.$$

We need to change to another spherical coordinate system $\mathbf{r_2'}$, recalculate the integration limits and the volume element, and write the integral in the new coordinate system.

5.8.4 Calculate the Jacobian of the Transformation and Compare the Volume Elements in the $\mathbf{r_1'}$ and $\mathbf{r_2'}$ Systems, Bearing in Mind that $\mathbf{r_2'} = \mathbf{r_1'} - \mathbf{D}$.

Relationship Between the Coordinate Systems Transformation Conditions:

Given:

$$r_2' = |\mathbf{r_1'} - \mathbf{D}|,$$

where:

- $\mathbf{r_1'} = (r_1', \theta_1', \phi_1')$ is the position vector in the first coordinate system,
- $\mathbf{D} = (0, 0, D)$ is a fixed vector lying along the Z-axis,

• $\theta_2' = \theta_1'$ and $\phi_2' = \phi_1'$ (the angular coordinates coincide).

The norm of the difference of the vectors is expressed as:

$$r_2' = \sqrt{r_1'^2 \sin^2\theta_1' + (r_1'\cos\theta_1' - D)^2}.$$

The Jacobian of the transformation is determined solely by the partial derivative $\frac{\partial r_2'}{\partial r_1'}$, since the angular coordinates remain the same:

$$J = \frac{\partial r_2'}{\partial r_1'}.$$

Differentiating r_2' with Respect to r_1' We compute the derivative:

$$\frac{\partial r_2'}{\partial r_1'} = \frac{1}{2\sqrt{r_1'^2 \sin^2\theta_1' + (r_1'\cos\theta_1' - D)^2}} \cdot \frac{\partial}{\partial r_1'}\left(r_1'^2 \sin^2\theta_1' + (r_1'\cos\theta_1' - D)^2\right).$$

First Part:

$$\frac{\partial}{\partial r_1'}(r_1'^2 \sin^2\theta_1') = 2r_1'\sin^2\theta_1'.$$

Second Part:

$$\frac{\partial}{\partial r_1'}(r_1'\cos\theta_1' - D)^2 = 2(r_1'\cos\theta_1' - D)\cos\theta_1'.$$

Combining the Parts Now, the full derivative is:

$$\frac{\partial r_2'}{\partial r_1'} = \frac{2r_1'\sin^2\theta_1' + 2(r_1'\cos\theta_1' - D)\cos\theta_1'}{2\sqrt{r_1'^2 \sin^2\theta_1' + (r_1'\cos\theta_1' - D)^2}}.$$

We can write:

$$\frac{\partial r_2'}{\partial r_1'} = \frac{r_1'\sin^2\theta_1' + (r_1'\cos^2\theta_1' - D\cos\theta_1')}{\sqrt{r_1'^2 \sin^2\theta_1' + r_1'^2\cos^2\theta_1' - 2Dr_1'\cos\theta_1' + D^2}}.$$

Using the identity $\sin^2\theta_1' + \cos^2\theta_1' = 1$:

$$\frac{\partial r_2'}{\partial r_1'} = \frac{r_1' - D\cos\theta_1'}{\sqrt{r_1'^2 - 2Dr_1'\cos\theta_1' + D^2}}.$$

Note that:

$$r_1'^2 - 2Dr_1'\cos\theta_1' + D^2 = (r_1' - D\cos\theta_1')^2.$$

Thus, we have:

$$\frac{\partial r_2'}{\partial r_1'} = \frac{r_1' - D\cos\theta_1'}{r_1' - D\cos\theta_1'} = 1.$$

Since the Jacobian of the Transformation is 1, the Following Holds in Our System:

$$dV_1' = dV_2'.$$

5.8.5 Recalculation of the Integration Limits

The Original Integral

In the original $\mathbf{r_1'}$ coordinate system, the integral is defined by:

$$I = \frac{2Q_1 Q_2 R_2'}{\pi R_1'} \int_{r_1'=R_1'}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{(r_1')^2 \sin\theta}{|\mathbf{r_1'} - \mathbf{D}|^5} \, d\phi \, d\theta \, dr_1'.$$

The Lower Limit of Integration

In the new $\mathbf{r_2'}$ coordinate system, the lower limit for r_2' is determined by the condition $r_1' = R_1'$:

$$r_2' = \sqrt{(R_1')^2 + D^2 - 2R_1' D \cos\theta}.$$

Thus, the lower limit for r_2' is:

$$r_2' \geq \sqrt{(R_1')^2 + D^2 - 2R_1' D \cos\theta}.$$

The Transformed Integral

After changing to the $\mathbf{r_2'}$ coordinate system, the integral becomes:

$$I = \frac{2Q_1 Q_2 R_2'}{\pi R_1'} \int_{r_2'=\sqrt{(R_1')^2 + D^2 - 2R_1' D \cos\theta}}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{(r_2')^2 \sin\theta}{(r_2')^5} \, d\phi \, d\theta \, dr_2'.$$

5.8.6 Integration with Respect to r_2'

We compute the inner integral with respect to r_2':

$$\begin{aligned} \int_{r_2'=\sqrt{(R_1')^2 + D^2 - 2R_1' D \cos\theta}}^{\infty} \frac{dr_2'}{(r_2')^3} &= \left[-\frac{1}{2(r_2')^2} \right]_{r_2'=\sqrt{(R_1')^2 + D^2 - 2R_1' D \cos\theta}}^{\infty} \\ &= \frac{1}{2\left((R_1')^2 + D^2 - 2R_1' D \cos\theta\right)}. \end{aligned}$$

Thus, we have:

$$I = \frac{2Q_1 Q_2 R_2'}{\pi R_1'} \int_0^{\pi} \int_0^{2\pi} \frac{\sin\theta}{2\left((R_1')^2 + D^2 - 2R_1' D \cos\theta\right)} \, d\phi \, d\theta.$$

5.8.7 Integration with Respect to ϕ

The integral over ϕ is:

$$\int_0^{2\pi} d\phi = 2\pi.$$

Thus,

$$I = \frac{2Q_1 Q_2 R_2'}{\pi R_1'} \cdot 2\pi \cdot \frac{1}{2} \int_0^{\pi} \frac{\sin\theta}{(R_1')^2 + D^2 - 2R_1' D \cos\theta} \, d\theta.$$

Simplifying:

$$I = \frac{2Q_1 Q_2 R_2'}{R_1'} \int_0^{\pi} \frac{\sin\theta}{(R_1')^2 + D^2 - 2R_1' D \cos\theta} \, d\theta.$$

5.8.8 Integration with Respect to θ

Make the substitution:

$$u = \cos\theta, \quad du = -\sin\theta \, d\theta.$$

The integration limits become:

- When $\theta = 0$: $u = 1$.
- When $\theta = \pi$: $u = -1$.

Thus, we obtain:

$$\begin{aligned} I &= \frac{2Q_1Q_2R_2'}{R_1'} \int_{u=1}^{-1} \frac{-du}{(R_1')^2 + D^2 - 2R_1'Du} \\ &= \frac{2Q_1Q_2R_2'}{R_1'} \int_{-1}^{1} \frac{du}{(R_1')^2 + D^2 - 2R_1'Du}. \end{aligned}$$

5.8.9 Evaluation of the Integral with Respect to u

We use the standard integral:

$$\int_{-1}^{1} \frac{du}{A - Bu} = \frac{1}{B} \ln \left| \frac{A+B}{A-B} \right|,$$

where:

$$A = (R_1')^2 + D^2, \quad B = 2R_1'D.$$

Thus:

$$I = \frac{2Q_1Q_2R_2'}{R_1'} \cdot \frac{1}{2R_1'D} \ln \left| \frac{(R_1')^2 + D^2 + 2R_1'D}{(R_1')^2 + D^2 - 2R_1'D} \right|.$$

5.8.10 Simplification of the Expression Inside the Logarithm

The expression inside the logarithm is:

$$\frac{(R_1')^2 + D^2 + 2R_1'D}{(R_1')^2 + D^2 - 2R_1'D}.$$

Notice that:

$$\begin{aligned} (R_1')^2 + D^2 + 2R_1'D &= (R_1' + D)^2, \\ (R_1')^2 + D^2 - 2R_1'D &= (R_1' - D)^2. \end{aligned}$$

Thus:

$$\frac{(R_1')^2 + D^2 + 2R_1'D}{(R_1')^2 + D^2 - 2R_1'D} = \frac{(R_1' + D)^2}{(R_1' - D)^2}.$$

Since both $(R_1' + D)^2$ and $(R_1' - D)^2$ are positive, we can drop the absolute value:

$$\ln \left| \frac{(R_1' + D)^2}{(R_1' - D)^2} \right| = \ln \left(\frac{(R_1' + D)^2}{(R_1' - D)^2} \right) = 2 \ln \left(\frac{R_1' + D}{R_1' - D} \right).$$

5.8.11 Final Result

Substitute the simplified expression into the integral:

$$I = \frac{2Q_1Q_2R_2'}{R_1'} \cdot \frac{1}{2R_1'D} \cdot 2\ln\left(\frac{R_1'+D}{R_1'-D}\right).$$

Simplifying, we obtain:

$$\boxed{I = \frac{2Q_1Q_2R_2'}{(R_1')^2D}\ln\left(\frac{R_1'+D}{R_1'-D}\right) \qquad (10)}$$

5.9 Consider the Case When $D > R_1'$

In this case the expression inside the logarithm,

$$\left(\frac{R_1'+D}{R_1'-D}\right),$$

becomes negative because $R_1' - D < 0$ while $R_1' + D > 0$. To work with the logarithm of a negative number, we use the formula for the logarithm of a complex number.

5.9.1 Formula for the Logarithm of a Complex Number

The logarithm of a complex number $z = x + iy$ (where x and y are real numbers) is defined as:

$$\ln(z) = \ln|z| + i\arg(z),$$

where:

- $|z| = \sqrt{x^2+y^2}$ is the modulus of the complex number,
- $\arg(z)$ is the argument of the complex number (the angle in the complex plane).

For a negative real number $z = -a$ (with $a > 0$):

$$\ln(-a) = \ln(a) + i\pi,$$

since the modulus $|z| = a$ and the argument $\arg(z) = \pi$ (because a negative number lies on the negative real axis).

5.9.2 Application to Our Case

Consider the expression inside the logarithm:

$$\frac{R_1'+D}{R_1'-D}.$$

When $D > R_1'$, the denominator $R_1' - D$ is negative while the numerator $R_1' + D$ is positive. Thus, the expression inside the logarithm is negative:

$$\frac{R_1' + D}{R_1' - D} = -\frac{R_1' + D}{D - R_1'}.$$

Now applying the formula for the logarithm of a negative number:

$$\ln\left(\frac{R_1' + D}{R_1' - D}\right) = \ln\left(-\frac{R_1' + D}{D - R_1'}\right) = \ln\left(\frac{R_1' + D}{D - R_1'}\right) + i\pi.$$

5.9.3 Substitution into the Integral

Now substitute this expression into our integral:

$$I = \frac{2Q_1Q_2R_2'}{R_1'^2 D} \ln\left(\frac{R_1' + D}{R_1' - D}\right).$$

For $D > R_1'$, we have:

$$I = \frac{2Q_1Q_2R_2'}{R_1'^2 D} \left(\ln\left(\frac{R_1' + D}{D - R_1'}\right) + i\pi\right).$$

Thus, the integral takes a complex value:

$$I = \frac{2Q_1Q_2R_2'}{R_1'^2 D} \ln\left(\frac{R_1' + D}{D - R_1'}\right) + i\frac{2Q_1Q_2R_2'\pi}{R_1'^2 D}.$$

5.9.4 Final Result for $D > R_1'$

For the region $D > R_1'$, the solution of the integral in its complex form is:

$$\boxed{I = \frac{2Q_1Q_2R_2'}{R_1'^2 D} \ln\left(\frac{R_1' + D}{D - R_1'}\right) + i\frac{2Q_1Q_2R_2'\pi}{R_1'^2 D}. \qquad (11)}$$

5.9.5 Physical Interpretation

- **Real Part**:

$$\boxed{\mathrm{Re}(I) = \frac{2Q_1Q_2R_2'}{R_1'^2 D} \ln\left(\frac{R_1' + D}{D - R_1'}\right) \qquad (12)}$$

  describes the physical quantity related to the interaction between the systems.

- **Imaginary Part**:

$$\boxed{\mathrm{Im}(I) = \frac{2Q_1Q_2R_2'\pi}{R_1'^2 D}} \tag{13}$$

arises due to the sign change inside the logarithm and can be interpreted as a phase or additional energy associated with the geometry of the problem.

Thus, for $D > R_1'$, the solution becomes complex, reflecting a change in the physical nature of the problem in this region.

5.10 Taylor Series Expansion for $D > R_1'$

For the case $D > R_1'$, we can expand the logarithm in a Taylor series. Consider the expression inside the logarithm:

$$\ln\left(\frac{R_1' + D}{D - R_1'}\right).$$

Let $x = \frac{R_1'}{D}$. Then the expression inside the logarithm can be rewritten as:

$$\frac{R_1' + D}{D - R_1'} = \frac{1 + x}{1 - x}.$$

Now expand $\ln\left(\frac{1+x}{1-x}\right)$ in a Taylor series. Note that:

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x).$$

The Taylor series for $\ln(1+x)$ and $\ln(1-x)$ are well-known:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots,$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots.$$

Subtracting the second series from the first, we obtain:

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \dots.$$

Keeping only the first two terms, we have:

$$\ln\left(\frac{1+x}{1-x}\right) \approx 2x + \frac{2x^3}{3}.$$

Now substitute $x = \frac{R_1'}{D}$:

$$\ln\left(\frac{R_1' + D}{D - R_1'}\right) \approx 2\left(\frac{R_1'}{D}\right) + \frac{2}{3}\left(\frac{R_1'}{D}\right)^3.$$

5.10.1 Substitution into the Formula for I

Now substitute this approximation into our formula for I:

$$I = \frac{2Q_1Q_2R_2'}{R_1'^2 D}\left(\ln\left(\frac{R_1'+D}{D-R_1'}\right)+i\pi\right).$$

Using the logarithm approximation, we get:

$$I \approx \frac{2Q_1Q_2R_2'}{R_1'^2 D}\left(2\left(\frac{R_1'}{D}\right)+\frac{2}{3}\left(\frac{R_1'}{D}\right)^3+i\pi\right).$$

Simplify the expression:

- **Real Part**:

$$\frac{2Q_1Q_2R_2'}{R_1'^2 D}\cdot 2\left(\frac{R_1'}{D}\right) = \frac{4Q_1Q_2R_2'}{R_1' D^2},$$

$$\frac{2Q_1Q_2R_2'}{R_1'^2 D}\cdot \frac{2}{3}\left(\frac{R_1'}{D}\right)^3 = \frac{4Q_1Q_2R_2'}{3R_1' D^4}.$$

- **Imaginary Part**:

$$\frac{2Q_1Q_2R_2'}{R_1'^2 D}\cdot i\pi = i\frac{2Q_1Q_2R_2'\pi}{R_1'^2 D}.$$

5.10.2 Final Result

For $D > R_1'$, expanding the logarithm in a Taylor series up to the second term, we obtain:

$$\boxed{I \approx \frac{4Q_1Q_2R_2'}{R_1' D^2}+\frac{4Q_1Q_2R_2'}{3R_1' D^4}+i\frac{2Q_1Q_2R_2'\pi}{R_1'^2 D}. \quad (14)}$$

5.10.3 Physical Interpretation

- **Real Part**:

$$\boxed{\mathrm{Re}(I) \approx \frac{4Q_1Q_2R_2'}{R_1' D^2}+\frac{4Q_1Q_2R_2'}{3R_1' D^4} \quad (15)}$$

  describes the physical quantity related to the interaction between the systems, including higher-order corrections.

- **Imaginary Part**:

$$\boxed{\mathrm{Im}(I) = \frac{2Q_1Q_2R'_2\pi}{R'^2_1D} \qquad (16)}$$

remains unchanged and is associated with the phase or additional energy arising from the geometry of the problem.

Thus, the Taylor series expansion allows us to obtain an approximate expression for I in the region $D > R'_1$, which simplifies the analysis and interpretation of the result.

VI Graph of the Total Perturbation Based on the Obtained Approximation Formulas for the Real Part of the Integral Solution

Substitute the solutions for each of the integrals into the original expression for the total perturbation W_{total}. Recall that:

$$\boxed{W_{\text{total}} = \frac{8Q_1Q_2R_2'}{\pi R_1'} \int_{S_1'} \frac{dS_1'}{(r_1' - D)^5} + \frac{2Q_1Q_2R_2'}{\pi R_1'} \int_{r_1'=R_1'}^{\infty} \frac{dV_1'}{(r_1' - D)^5}}.$$

6.1 Solution for the First Integral

The first integral equals:

$$\frac{8Q_1Q_2R_2'}{\pi R_1'} \int_{S_1'} \frac{dS_1'}{(r_1' - D)^5} = \frac{16Q_1Q_2R_2'}{3D} \left[\frac{1}{(R_1' - D)^3} - \frac{1}{(R_1' + D)^3} \right].$$

6.2 Solution for the Second Integral

The second integral in complex form equals:

$$\frac{2Q_1Q_2R_2'}{\pi R_1'} \int_{r_1'=R_1'}^{\infty} \frac{dV_1'}{(r_1' - D)^5} \approx \frac{4Q_1Q_2R_2'}{R_1'D^2} + \frac{4Q_1Q_2R_2'}{3R_1'D^4} + i\frac{2Q_1Q_2R_2'\pi}{R_1'^2 D}.$$

6.3 Substitution into the Expression for W_{total}

Now substitute the solutions for each integral into the original expression for W_{total}:

$$\begin{aligned} W_{\text{total}} = \frac{16Q_1Q_2R_2'}{3D} \left[\frac{1}{(R_1' - D)^3} - \frac{1}{(R_1' + D)^3} \right] + \\ + \left(\frac{4Q_1Q_2R_2'}{R_1'D^2} + \frac{4Q_1Q_2R_2'}{3R_1'D^4} + i\frac{2Q_1Q_2R_2'\pi}{R_1'^2 D} \right). \end{aligned} \tag{17}$$

6.4 Graph of the Real Part of the Obtained Expression for the Total Spatial Density Perturbation W_{total} and of Each Term for Detailed Analysis

Let us plot four graphs for each term of the real part of the total perturbation in logarithmic scale for the function values and in a linear scale for the variable D.

Breaking Down the Expression

The full real part is given by:

$$\text{Re}(W_{\text{total}}) = A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 = \frac{16Q_1Q_2R_2'}{3D} \cdot \frac{1}{(R_1' - D)^3}, \quad A_2 = -\frac{16Q_1Q_2R_2'}{3D} \cdot \frac{1}{(R_1' + D)^3},$$

$$A_3 = \frac{4Q_1Q_2R_2'}{R_1'D^2}, \quad A_4 = \frac{4Q_1Q_2R_2'}{3R_1'D^4}.$$

Parameters Adopted

We set:

$$Q_1 = Q_2 = R_1' = R_2' = 5,$$

and the variable D varies in the interval:

$$D \in [0.1, 5].$$

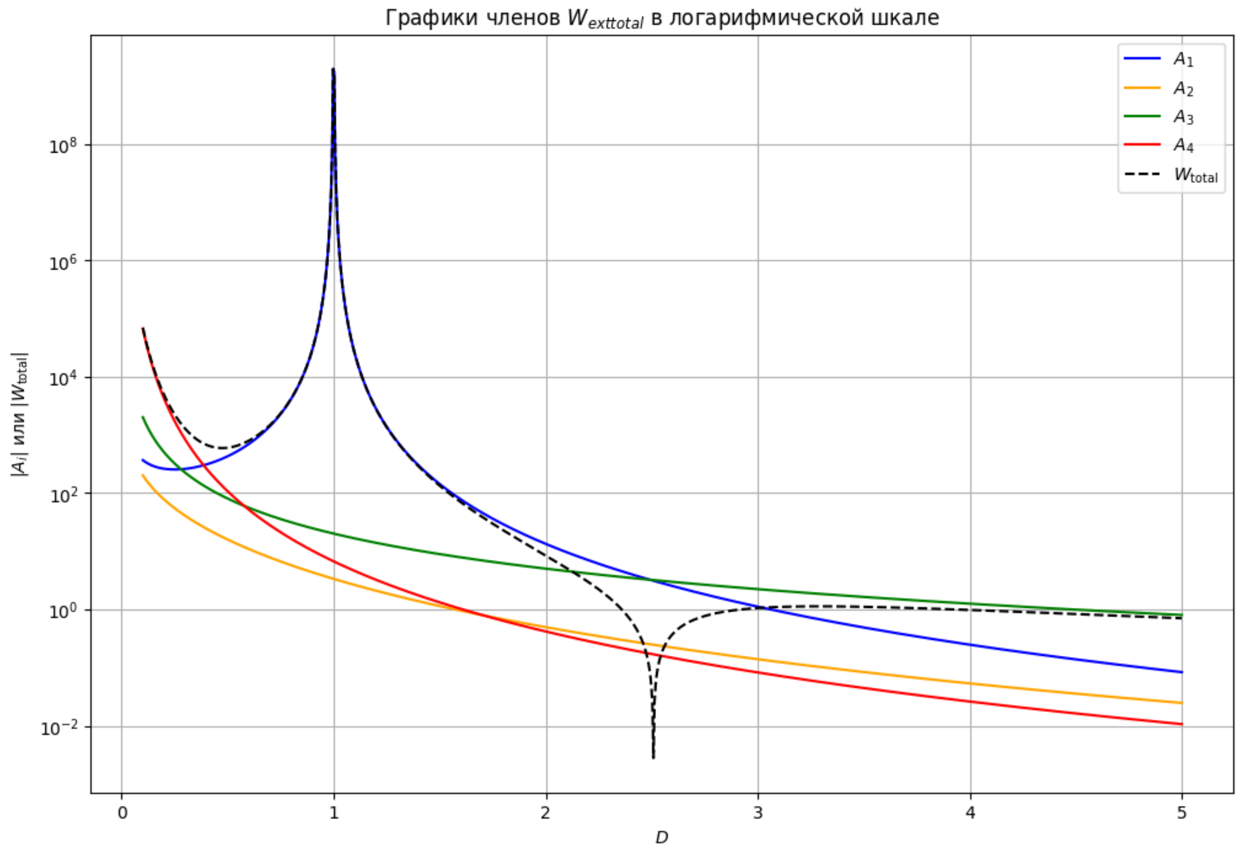

Figure 3: Graph of the Real Part of the Obtained Expression for the Total Spatial Density Perturbation W_{total}

6.5 Analysis of the Result of the Interaction Quantity W_{total} Based on the Representation of the Spatial Density

The obtained result, based on the analysis of the graph of the total interaction W_{total}, is very interesting and allows several important conclusions about the behavior of the spatial density cluster system. Let us examine in detail what is happening on the graph and how it is related to the physical interactions.

6.5.1 Graph Analysis

a) Region $D > 2.5$
In this region, W_{total} is positive, indicating **repulsion** between the spatial density clusters.

- The dependence $W_{\text{total}}(D) \sim \frac{1}{D^2}$ resembles **Coulomb repulsion** of like charges.
- As the distance D increases, the magnitude of the repulsion decreases, which is consistent with classical electrostatic interaction.

b) Region $1 < D < 2.5$
In this range of distances, the function W_{total} becomes negative, indicating **attraction**.

- The attraction in this range is significantly stronger than the repulsion at larger distances.
- The behavior of the function resembles the **strong interaction** in nuclear physics.
- The maximum attraction is reached at some value $D_{\text{min}} \approx 1.5$.

c) Region $D < 1$
In this region, the function becomes positive again, indicating **repulsion** at very short distances.

- The repulsion may be related to the overlapping of spatial density clusters.
- At $D = R_1' = 1$ a **singularity** is observed, which may be associated with a transition of the system between two interaction regimes.

—

6.5.2 Comparison with Known Physical Interactions

a) Coulomb Repulsion ($D > 2.5$**)**
At large distances, the interaction resembles classical Coulomb repulsion between like charges:

$$W_{\text{total}}(D) \sim \frac{1}{D^2}.$$

This is consistent with the hypothesis that spatial density clusters create a field analogous to the electric field of charges.
b) Strong Interaction ($1 < D < 2.5$**)**
The strong attraction in the range $1 < D < 2.5$ in form resembles nuclear forces:

$$W_{\text{total}}(D) \sim -\frac{1}{D^n}, \quad n \approx 6.$$

Such behavior may be associated with resonant effects in the spatial density model, where a stable interaction arises at a certain distance.

c) Repulsion at Short Distances ($D < 1$)
At very short distances, rapidly increasing repulsion arises:

$$W_{\text{total}}(D) \sim \frac{1}{D^4}.$$

This interaction may be related to the overlapping of spatial density and resembles exchange interactions arising from the Pauli exclusion principle in quantum mechanics.
—

6.5.3 Interpretation of the Point $D = 1$

The point $D = 1$ is of particular significance in the model since it corresponds to the scale of the spatial density cluster:

$$D = R_1'.$$

At this point, a singularity occurs, which may indicate a transition of the system from a state of strong attraction to a state of repulsion.
—

6.5.4 Physical Interpretation

The results obtained allow us to propose the following interpretation of the behavior of spatial density clusters:
- Repulsion at large distances resembles electrostatic interaction.
- Strong attraction at intermediate distances may be analogous to the strong interaction between hadrons.
- Repulsion at short distances is associated with the overlapping of spatial density and may be related to exchange effects.

Maximum Entropy of the System The cluster system tends toward minimal energy and maximum entropy. This leads to the formation of stable configurations at certain values of D.
—

6.5.5 Conclusion

The obtained result shows that the spatial density cluster model exhibits complex behavior that includes both repulsive and attractive components depending on the distance.

The main conclusions are:

- At large distances the interaction resembles Coulomb repulsion.
- At intermediate distances strong attraction, similar to nuclear forces, arises.
- At very short distances the interaction becomes repulsive due to the overlapping of spatial density.

VII Solution of the Gradient Integral over the Entire Volume for the Spatial Density Distribution Equation of a Single Sphere

In this section, we solve the gradient integral over the entire volume for the spatial density distribution equation of a single sphere. The approach utilizes the Heaviside function, which effectively describes the boundary conditions and sharp transitions in the spatial density distribution. This detailed derivation ensures that conservation laws are satisfied and provides insight into the nature of the spatial density perturbations.

In Section IV of our study, we obtained that the formula for the total spatial density distribution around a sphere in terms of the Heaviside function is given by:

$$\Delta\rho(r) = \rho_1 H(R_1' - r) - \frac{\rho_1 \cdot R_1'^4}{3r^4} H(r - R_1'),$$

where $r = |\mathbf{r}|$.

Now, we replace r with the norm of the difference of vectors $|\mathbf{r} - \mathbf{R}_1'|$:

$$\Delta\rho(|\mathbf{r} - \mathbf{R}_1'|) = \rho_1 H(R_1' - |\mathbf{r} - \mathbf{R}_1'|) - \frac{\rho_1 \cdot R_1'^4}{3|\mathbf{r} - \mathbf{R}_1'|^4} H(|\mathbf{r} - \mathbf{R}_1'| - R_1').$$

7.1 Calculation of the Gradient

The gradient of the function $\Delta\rho(|\mathbf{r} - \mathbf{R}_1'|)$ is calculated as:

$$\nabla\Delta\rho(|\mathbf{r} - \mathbf{R}_1'|) = \frac{\partial \Delta\rho}{\partial|\mathbf{r} - \mathbf{R}_1'|} \cdot \nabla|\mathbf{r} - \mathbf{R}_1'|.$$

7.1.1 Calculation of $\frac{\partial\Delta\rho}{\partial|\mathbf{r}-\mathbf{R}_1'|}$

The function $\Delta\rho(|\mathbf{r} - \mathbf{R}_1'|)$ consists of two parts:

1. $\rho_1 H(R_1' - |\mathbf{r} - \mathbf{R}_1'|)$,
2. $-\frac{\rho_1 \cdot R_1'^4}{3|\mathbf{r}-\mathbf{R}_1'|^4} H(|\mathbf{r} - \mathbf{R}_1'| - R_1')$.

The derivative of the Heaviside function $H(x)$ is the Dirac delta function $\delta(x)$. Thus:

$$\frac{\partial}{\partial|\mathbf{r} - \mathbf{R}_1'|}\left(\rho_1 H(R_1' - |\mathbf{r} - \mathbf{R}_1'|)\right) = -\rho_1 \delta(R_1' - |\mathbf{r} - \mathbf{R}_1'|),$$

$$\frac{\partial}{\partial|\mathbf{r} - \mathbf{R}_1'|}\left(-\frac{\rho_1 \cdot R_1'^4}{3|\mathbf{r} - \mathbf{R}_1'|^4} H(|\mathbf{r} - \mathbf{R}_1'| - R_1')\right) = \frac{4\rho_1 \cdot R_1'^4}{3|\mathbf{r} - \mathbf{R}_1'|^5} H(|\mathbf{r} - \mathbf{R}_1'| - R_1') - \frac{\rho_1 \cdot R_1'^4}{3|\mathbf{r} - \mathbf{R}_1'|^4} \delta(|$$

7.1.2 Calculation of $\nabla|\mathbf{r} - \mathbf{R}_1'|$

The gradient of the norm of the vector difference is:

$$\nabla|\mathbf{r} - \mathbf{R}_1'| = \frac{\mathbf{r} - \mathbf{R}_1'}{|\mathbf{r} - \mathbf{R}_1'|}.$$

7.1.3 Final Gradient

Combining the results, we have:

$$\nabla\Delta\rho(|\mathbf{r}-\mathbf{R}_1'|) = \left(-\rho_1\delta(R_1' - |\mathbf{r}-\mathbf{R}_1'|) + \frac{4\rho_1 \cdot R_1'^4}{3|\mathbf{r}-\mathbf{R}_1'|^5}H(|\mathbf{r}-\mathbf{R}_1'| - R_1') - \frac{\rho_1 \cdot R_1'^4}{3|\mathbf{r}-\mathbf{R}_1'|^4}\delta(|\mathbf{r}$$

7.2 Modulus of the Gradient

Now, we calculate the modulus of the gradient:

$$|\nabla\Delta\rho(|\mathbf{r}-\mathbf{R}_1'|)| = \left|-\rho_1\delta(R_1' - |\mathbf{r}-\mathbf{R}_1'|) + \frac{4\rho_1 \cdot R_1'^4}{3|\mathbf{r}-\mathbf{R}_1'|^5}H(|\mathbf{r}-\mathbf{R}_1'| - R_1') - \frac{\rho_1 \cdot R_1'^4}{3|\mathbf{r}-\mathbf{R}_1'|^4}\delta(|\mathbf{r}\right.$$

7.3 Integral of the Modulus of the Gradient

Now, we compute the integral of the modulus of the gradient over the entire volume:

$$\int_0^\infty |\nabla\Delta\rho(|\mathbf{r}-\mathbf{R}_1'|)|\, dV.$$

7.3.1 Separation into Parts

The integral is separated into three parts:

1. The contribution from $-\rho_1\delta(R_1' - |\mathbf{r}-\mathbf{R}_1'|)$,
2. The contribution from $\frac{4\rho_1 \cdot R_1'^4}{3|\mathbf{r}-\mathbf{R}_1'|^5}H(|\mathbf{r}-\mathbf{R}_1'| - R_1')$,
3. The contribution from $-\frac{\rho_1 \cdot R_1'^4}{3|\mathbf{r}-\mathbf{R}_1'|^4}\delta(|\mathbf{r}-\mathbf{R}_1'| - R_1')$.

7.3.2 Calculation of Each Part

1. **Contribution from the delta function $-\rho_1\delta(R_1' - |\mathbf{r}-\mathbf{R}_1'|)$:**

$$\int_0^\infty -\rho_1\delta(R_1' - |\mathbf{r}-\mathbf{R}_1'|)\, dV = -\rho_1 \cdot 4\pi R_1'^2.$$

2. **Contribution from $\frac{4\rho_1 \cdot R_1'^4}{3|\mathbf{r}-\mathbf{R}_1'|^5}H(|\mathbf{r}-\mathbf{R}_1'| - R_1')$:**

$$\int_{R_1'}^\infty \frac{4\rho_1 \cdot R_1'^4}{3|\mathbf{r}-\mathbf{R}_1'|^5}\, dV = \frac{4\rho_1 \cdot R_1'^4}{3} \cdot 4\pi \int_{R_1'}^\infty \frac{1}{s^5} \cdot s^2\, ds = \frac{16\pi\rho_1 R_1'^4}{3} \int_{R_1'}^\infty \frac{1}{s^3}\, ds.$$

The integral:

$$\int_{R_1'}^\infty \frac{1}{s^3}\, ds = \left[-\frac{1}{2s^2}\right]_{R_1'}^\infty = \frac{1}{2R_1'^2}.$$

Therefore:

$$\frac{16\pi\rho_1 R_1'^4}{3} \cdot \frac{1}{2R_1'^2} = \frac{8\pi\rho_1 R_1'^2}{3}.$$

3. **Contribution from** $-\frac{\rho_1 \cdot R_1'^4}{3|\mathbf{r}-\mathbf{R}_1'|^4}\delta(|\mathbf{r}-\mathbf{R}_1'|-R_1')$**:**

$$\int_0^\infty -\frac{\rho_1 \cdot R_1'^4}{3|\mathbf{r}-\mathbf{R}_1'|^4}\delta(|\mathbf{r}-\mathbf{R}_1'|-R_1')\,dV = -\frac{\rho_1 \cdot R_1'^4}{3R_1'^4} \cdot 4\pi R_1'^2 = -\frac{4\pi\rho_1 R_1'^2}{3}.$$

7.3.3 Final Result

Now, combining all parts:

$$\int_0^\infty |\nabla \Delta\rho(|\mathbf{r}-\mathbf{R}_1'|)|\,dV = -\rho_1 \cdot 4\pi R_1'^2 + \frac{8\pi\rho_1 R_1'^2}{3} - \frac{4\pi\rho_1 R_1'^2}{3}.$$

Summing up:

$$-4\pi\rho_1 R_1'^2 + \frac{8\pi\rho_1 R_1'^2}{3} - \frac{4\pi\rho_1 R_1'^2}{3} = -4\pi\rho_1 R_1'^2 + \frac{4\pi\rho_1 R_1'^2}{3} = -\frac{8\pi\rho_1 R_1'^2}{3}.$$

7.4 Conclusion

The integral of the modulus of the gradient of the function $\Delta\rho(|\mathbf{r}-\mathbf{R}_1'|)$ is:

$$\int_0^\infty |\nabla \Delta\rho(|\mathbf{r}-\mathbf{R}_1'|)|\,dV = -\frac{8\pi\rho_1 R_1'^2}{3}.$$

Thus, the final expression for the integral of the modulus of the gradient is:

$$\boxed{\int_0^\infty |\nabla \Delta\rho(|\mathbf{r}-\mathbf{R}_1'|)|\,dV = \left|-\frac{8\pi\rho_1 R_1'^2}{3}\right| = \frac{8\pi\rho_1 R_1'^2}{3}. \qquad (18)}$$

Now, we express this in terms of Q_1. Recall that:

$$Q_1 = \rho_1 \cdot \frac{4}{3}\pi R_1'^3.$$

Express ρ_1 in terms of Q_1:

$$\rho_1 = \frac{Q_1}{\frac{4}{3}\pi R_1'^3}.$$

Substitute ρ_1 into the final expression:

$$\frac{8\pi\rho_1 R_1'^2}{3} = \frac{8\pi\left(\frac{Q_1}{\frac{4}{3}\pi R_1'^3}\right)R_1'^2}{3}.$$

Simplify:

$$\frac{8\pi Q_1 R_1'^2}{\frac{4}{3}\pi R_1'^3 \cdot 3} = \frac{8\pi Q_1 R_1'^2}{4\pi R_1'^3} = \frac{2Q_1}{R_1'}.$$

$$\int_0^\infty |\nabla\Delta\rho(|\mathbf{r}-\mathbf{R}_1'|)|\,dV = \frac{2Q_1}{R_1'}, \tag{19}$$

where $Q_1 = \rho_1 \cdot \frac{4}{3}\pi R_1'^3$.

We obtained the same dimensionality as in the formula for

$$\Delta W_{\text{total}}(D) \approx 2\frac{R_2'Q}{D^2},$$

so that, after canceling R_2' and D^2, we get the same dimensionality as for the amount of spatial density perturbation between two spheres, which by analogy with Coulomb's law has the physical meaning of force. This means that our reasoning is correct—the integral of the gradient for the spatial density distribution from 0 to infinity shows the force required to keep the spatial density in a compressed state.

We also see that, although the third postulate of our system—the conservation of the spatial density quantity—is satisfied, the system is not in equilibrium and remains perturbed. Thus, for fulfilling the fourth law of our universe—to maximize the entropy of the spatial density distribution—it is necessary that the total perturbation of the spatial density (from 0 to infinity) also tends to zero. However, if we make an additional change in the density distribution outside the sphere and somehow redistribute the spatial density outside the sphere, it will lead to a violation of the third law, which is associated with the conservation of spatial density.

In this regard, one may assume that in order to compensate for this perturbation, space will further curve, but now through the curvature of its metric. Only in this case will both the third and fourth postulates of our hypothetical universe be satisfied.

VIII The Relationship between Spatial Density and the Mass of a Compressed Sphere

In the previous section, we obtained that

$$\int_0^\infty \nabla\Delta\rho(r)\cdot dV = -\frac{2\,Q}{R_1'},$$

which is nonzero and characterizes the force that holds the spatial density sphere in its compressed state.

Now, let us calculate the energy required to compress this sphere from $S(R_1)$ to $S(R_1')$. If the magnitude of the integral of the gradient is a measure of the force, then by integrating this force along the path we obtain the work required for the compression of the sphere, i.e. its internal energy.

Next, we will find the relationship between the internal energy of the charge—equal to the integral of the force necessary to compress the sphere from its initial radius R_1 to the final radius R_1'. This relationship is crucial for understanding how the energy contained within the compressed sphere determines the curvature of space, and consequently, the gravitational field created by the compressed region of space in the form of a sphere, i.e. its mass.

8.1 Energy Required to Compress the Sphere from R_1 to R_1'

8.1.1 Initial Equation

We have:

$$\int_0^\infty \nabla\Delta\rho(r)\cdot 4\pi r^2\,dr = \frac{8\pi\rho_1(R_1')^2}{3},$$

where

$$\rho_1 = \rho_0\left(\frac{R_1^3}{R_1'^3} - 1\right).$$

Substituting the value of ρ_1, we obtain:

$$\int_0^\infty \nabla\Delta\rho(r)\cdot 4\pi r^2\,dr = \frac{8\pi\rho_0}{3}\left(\frac{R_1^3}{R_1'} - (R_1')^2\right).$$

8.1.2 Let Us Perform a Change of the Integration Variable from R_1' to t, so that our Expression Becomes:

$$F(t) = \frac{8\pi\rho_0}{3}\left(\frac{R_1^3}{t} - t^2\right).$$

Here, $F(t)$ has the physical meaning of the force that must be applied to compress the sphere $S(t)$ from $t = R_1$ to $t = R_1'$.

8.1.3 Calculation of the Energy Required to Compress the Sphere from R_1 to R_1'

Consider the sphere $S(t)$ with radius t, which is to be compressed from radius R_1 to radius R_1'. The force that holds the sphere in its compressed state $S(R_1')$ is given by the function:

$$F(t) = \frac{8\pi\rho_0}{3}\left(\frac{R_1^3}{t} - t^2\right).$$

We need to find the energy E expended to compress the sphere from R_1 to R_1'. To this end, we use the formula for work, which in this case is equal to the compression energy:

$$E = \int_{R_1}^{R_1'} F(t)\,dt.$$

Substituting the expression for $F(t)$:

$$E = \int_{R_1}^{R_1'} \frac{8\pi\rho_0}{3}\left(\frac{R_1^3}{t} - t^2\right)\,dt.$$

We split the integral into two terms:

$$E = \frac{8\pi\rho_0}{3}\left[\int_{R_1}^{R_1'} \frac{R_1^3}{t}\,dt - \int_{R_1}^{R_1'} t^2\,dt\right].$$

Integrating each term with respect to t, for the first term we obtain:

$$\int \frac{R_1^3}{t}\,dt = R_1^3 \ln t,$$

and for the second term:

$$\int t^2\,dt = \frac{t^3}{3}.$$

Substituting the integration results and the limits, we obtain:

$$\boxed{E = \frac{8\pi\rho_0}{3}\left[R_1^3 \ln\left(\frac{R_1'}{R_1}\right) + \frac{1}{3}\left((R_1')^3 - R_1^3\right)\right]. \quad (20)}$$

This expression represents the energy required to compress the sphere from R_1 to R_1'. This energy is equivalent to the energy contained within the compressed sphere, which causes the curvature of space along with its metric, thereby determining the mass of the sphere.

8.2 Mass of the Compressed Sphere

Using Einstein's famous equation $E = mc^2$, we can find the mass m of the compressed sphere:

$$m = \frac{E}{c^2}.$$

Substituting the expression for E:

$$\boxed{m = \frac{8\pi\rho_0}{3c^2}\left[R_1^3 \ln\left(\frac{R_1'}{R_1}\right) + \frac{1}{3}\left((R_1')^3 - R_1^3\right)\right]. \qquad (21)}$$

This expression determines the mass of the compressed sphere based on the energy required for its compression, which can also be interpreted as the energy that holds the sphere in a compressed state. This result illustrates how the energy associated with the compression of the sphere is converted into an equivalent mass, which (in order to satisfy our fourth postulate) creates curvature of space with respect to its metric and gives rise to effects such as mass and the gravitational field.

IX The Theoretical Justification of Bohr's Postulate for the Electron in the Hydrogen Atom Based on the Imaginary Part of the Solution for the Total Perturbation

When we obtained the expression for the internal energy of the spatial density cluster—which we interpret as the charge's mass—we can proceed to write the equation for the equilibrium of forces acting on a cluster moving around an opposite charge and check how this motion can be related to the imaginary solution of the expression for the total perturbation of the two-charge system. For this purpose, we restrict ourselves to the quadratic term of our expression W_{total}, since the other corrections are significant only when $D \sim R'_1$.

9.1 Problem Statement

Consider two clusters that create spatial density perturbations. The first cluster is in the reference frame $\mathbf{r_1}$, and the second is located at a distance D from the first. The total spatial density perturbation produced by the two clusters, relative to the coordinate system $\mathbf{r_1}$, is expressed as:

$$\boxed{W_{\text{total}_{r_1}} \approx \frac{Q_1 Q_2 R'_2}{R'_1 D^2} + i\frac{\pi Q_1 Q_2 R'_2}{2R'^2_1 D}, \qquad (22)}$$

where:

- $Q_1 = \rho_1 V(R'_1)$ is the "charge" of the first cluster,
- $Q_2 = \rho_2 V(R'_2)$ is the "charge" of the second cluster,
- R'_1 and R'_2 are the radii of the clusters,
- D is the distance between the clusters.

It is necessary to find the distance D_0 and the frequency ω_0 at which the centrifugal force is balanced by the attractive force determined by the real part of the perturbation, and the orbital frequency ω_0 is related to the imaginary part of the perturbation.

9.1.1 Force Balance

The equilibrium condition (balance between the centrifugal force and the attractive force) for the first cluster of mass m_1 is given by:

$$m_1 \omega_0^2 D_0 = \text{Re}(W_{\text{total}_{r_1}}).$$

Substituting the real part of $W_{\text{total}_{r_1}}$:

$$m_1 \omega_0^2 D_0 = \frac{Q_1 Q_2 R'_2}{R'_1 D_0^2}.$$

9.1.2 Relation Between the Frequency and the Imaginary Part of the Perturbation

The orbital frequency ω_0 is related to the imaginary part of the perturbation $\mathrm{Im}(W_{\mathrm{total}_{r_1}})$ through a proportionality constant H:

$$\omega_0 = H \cdot \mathrm{Im}(W_{\mathrm{total}_{r_1}}).$$

Substituting the imaginary part of $W_{\mathrm{total}_{r_1}}$:

$$\omega_0 = H \cdot \frac{\pi Q_1 Q_2 R'_2}{2R'^2_1 D_0}.$$

9.1.3 Substituting ω_0 into the Force Balance Equation

Substitute the expression for ω_0 into the force balance condition:

$$m_1 \left(H \cdot \frac{\pi Q_1 Q_2 R'_2}{2R'^2_1 D_0} \right)^2 D_0 = \frac{Q_1 Q_2 R'_2}{R'_1 D_0^2}.$$

Simplify the left-hand side:

$$m_1 \cdot \frac{H^2 \pi^2 Q_1^2 Q_2^2 R'^2_2}{4R'^4_1 D_0^2} \cdot D_0 = \frac{Q_1 Q_2 R'_2}{R'_1 D_0^2}.$$

Cancel the common factors:

$$\frac{m_1 H^2 \pi^2 Q_1 Q_2 R'_2}{4R'^4_1 D_0} = \frac{1}{R'_1 D_0^2}.$$

Multiply both sides by D_0^2:

$$\frac{m_1 H^2 \pi^2 Q_1 Q_2 R'_2}{4R'^4_1} D_0 = \frac{1}{R'_1}.$$

Solving for D_0:

$$D_0 = \frac{4R'^3_1}{m_1 H^2 \pi^2 Q_1 Q_2 R'_2}.$$

9.1.4 Expression for the Frequency ω_0

Substitute D_0 into the expression for ω_0:

$$\omega_0 = H \cdot \frac{\pi Q_1 Q_2 R'_2}{2R'^2_1 D_0}.$$

Substitute $D_0 = \frac{4R_1'^3}{m_1 H^2 \pi^2 Q_1 Q_2 R_2'}$:

$$\omega_0 = H \cdot \frac{\pi Q_1 Q_2 R_2'}{2R_1'^2} \cdot \frac{m_1 H^2 \pi^2 Q_1 Q_2 R_2'}{4R_1'^3}.$$

Simplify:

$$\omega_0 = \frac{m_1 H^3 \pi^3 Q_1^2 Q_2^2 R_2'^2}{8R_1'^5}.$$

9.1.5 Final Solution

1. **Distance D_0 at Force Balance:**

$$\boxed{D_0 = \frac{4R_1'^3}{m_1 H^2 \pi^2 Q_1 Q_2 R_2'}. \qquad (23)}$$

2. **Orbital Frequency ω_0:**

$$\boxed{\omega_0 = \frac{m_1 H^3 \pi^3 Q_1^2 Q_2^2 R_2'^2}{8R_1'^5}. \qquad (24)}$$

9.2 Introducing the Constant h and Finding D_n and ω_n for Lower Modes

9.2.1 Introducing the Constant h

For convenience, we introduce a constant h:

$$h = \frac{2R_1'}{H\pi}.$$

Then the coefficient H can be expressed in terms of h:

$$H = \frac{2R_1'}{h\pi}.$$

9.2.2 Rewriting the Formulas for D_0 and ω_0 Using h

Formula for D_0:

The original formula for D_0 is:

$$D_0 = \frac{4R_1'^3}{m_1 H^2 \pi^2 Q_1 Q_2 R_2'}.$$

Substitute $H = \frac{2R_1'}{h\pi}$:

$$D_0 = \frac{4R_1'^3}{m_1 \left(\frac{2R_1'}{h\pi}\right)^2 \pi^2 Q_1 Q_2 R_2'}.$$

Simplify:

$$D_0 = \frac{4R_1'^3 h^2 \pi^2}{4m_1 R_1'^2 \pi^2 Q_1 Q_2 R_2'} = \frac{R_1' h^2}{m_1 Q_1 Q_2 R_2'}.$$

Formula for ω_0:
The original formula for ω_0 is:

$$\omega_0 = \frac{m_1 H^3 \pi^3 Q_1^2 Q_2^2 R_2'^2}{8R_1'^5}.$$

Substitute $H = \frac{2R_1'}{h\pi}$:

$$\omega_0 = \frac{m_1 \left(\frac{2R_1'}{h\pi}\right)^3 \pi^3 Q_1^2 Q_2^2 R_2'^2}{8R_1'^5}.$$

Simplify:

$$\boxed{\omega_0 = \frac{m_1 Q_1^2 Q_2^2 R_2'^2}{h^3 R_1'^2}. \qquad (25)}$$

9.3 Finding D_n and ω_n for Lower Modes

For lower modes n, we assume that the orbital frequency ω_n is related to the imaginary part of the perturbation as follows:

$$\boxed{n\omega_n = H \cdot \mathrm{Im}(W_{\text{total}_{r_1}}(D_n)). \qquad (26)}$$

Substitute the imaginary part of $W_{\text{total}_{r_1}}$:

$$n\omega_n = H \cdot \frac{\pi Q_1 Q_2 R_2'}{2R_1'^2 D_n}.$$

Solve for ω_n:

$$\omega_n = \frac{H\pi Q_1 Q_2 R_2'}{2nR_1'^2 D_n}.$$

9.3.1 Force Balance Condition for the nth Mode

The force balance condition for the nth mode is:

$$m_1\omega_n^2 D_n = \mathrm{Re}(W_{\mathrm{total}_{r_1}}(D_n)).$$

Substitute $\mathrm{Re}(W_{\mathrm{total}_{r_1}}(D_n))$:

$$m_1\omega_n^2 D_n = \frac{Q_1 Q_2 R_2'}{R_1' D_n^2}.$$

Substitute $\omega_n = \frac{H\pi Q_1 Q_2 R_2'}{2nR_1'^2 D_n}$:

$$m_1\left(\frac{H\pi Q_1 Q_2 R_2'}{2nR_1'^2 D_n}\right)^2 D_n = \frac{Q_1 Q_2 R_2'}{R_1' D_n^2}.$$

Simplify the left-hand side:

$$m_1 \cdot \frac{H^2\pi^2 Q_1^2 Q_2^2 R_2'^2}{4n^2 R_1'^4 D_n^2} \cdot D_n = \frac{Q_1 Q_2 R_2'}{R_1' D_n^2}.$$

Cancel common factors:

$$\frac{m_1 H^2\pi^2 Q_1 Q_2 R_2'}{4n^2 R_1'^4 D_n} = \frac{1}{R_1' D_n^2}.$$

Multiply both sides by D_n^2:

$$\frac{m_1 H^2\pi^2 Q_1 Q_2 R_2'}{4n^2 R_1'^4} D_n = \frac{1}{R_1'}.$$

Solving for D_n:

$$D_n = \frac{4n^2 R_1'^3}{m_1 H^2 \pi^2 Q_1 Q_2 R_2'}.$$

Substitute $H = \frac{2R_1'}{h\pi}$:

$$D_n = \frac{4n^2 R_1'^3 h^2 \pi^2}{4m_1 R_1'^2 \pi^2 Q_1 Q_2 R_2'} = \frac{n^2 R_1' h^2}{m_1 Q_1 Q_2 R_2'}.$$

9.3.2 Expression for ω_n

Substitute D_n into the expression for ω_n:

$$\omega_n = \frac{H\pi Q_1 Q_2 R_2'}{2nR_1'^2 D_n}.$$

Substitute $D_n = \frac{n^2 R_1' h^2}{m_1 Q_1 Q_2 R_2'}$:

$$\omega_n = \frac{H\pi Q_1 Q_2 R_2'}{2nR_1'^2} \cdot \frac{m_1 Q_1 Q_2 R_2'}{n^2 R_1' h^2}.$$

Simplify:

$$\omega_n = \frac{H\pi m_1 Q_1^2 Q_2^2 R_2'^2}{2n^3 R_1'^3 h^2}.$$

Substitute $H = \frac{2R_1'}{h\pi}$:

$$\omega_n = \frac{\left(\frac{2R_1'}{h\pi}\right) \pi m_1 Q_1^2 Q_2^2 R_2'^2}{2n^3 R_1'^3 h^2} = \frac{m_1 Q_1^2 Q_2^2 R_2'^2}{n^3 R_1'^2 h^3}.$$

9.3.3 Final Solution

1. **Distance D_n for the nth Mode:**

$$D_n = \frac{n^2 R_1' h^2}{m_1 Q_1 Q_2 R_2'}.$$

2. **Orbital Frequency ω_n for the nth Mode:**

$$\omega_n = \frac{m_1 Q_1^2 Q_2^2 R_2'^2}{n^3 R_1'^2 h^3}.$$

3. **Relationship between D_0 and D_n:**

$$\boxed{D_n = n^2 D_0.} \tag{27}$$

4. **Relationship between ω_0 and ω_n:**

$$\boxed{\omega_n = \frac{\omega_0}{n^3}.} \tag{28}$$

9.4 Obtaining the Expression of Bohr's Postulate Based on the Expressions for the Resonant Orbit Radius, the Product $V_n \cdot D_n$, and the Angular Momentum L_n of the Two-Spatial-Density-Cluster System

9.4.1 Orbital Speed V_n

The orbital speed V_n of the first cluster on the nth orbit is defined as:

$$V_n = \omega_n \cdot D_n.$$

Substitute the expressions for ω_n and D_n:

$$\omega_n = \frac{m_1 Q_1^2 Q_2^2 R_2'^2}{n^3 R_1'^2 h^3},$$

$$D_n = \frac{n^2 R_1' h^2}{m_1 Q_1 Q_2 R_2'}.$$

Then:

$$V_n = \left(\frac{m_1 Q_1^2 Q_2^2 R_2'^2}{n^3 R_1'^2 h^3}\right) \cdot \left(\frac{n^2 R_1' h^2}{m_1 Q_1 Q_2 R_2'}\right).$$

Simplifying:

$$V_n = \frac{m_1 Q_1^2 Q_2^2 R_2'^2 \cdot n^2 R_1' h^2}{n^3 R_1'^2 h^3 \cdot m_1 Q_1 Q_2 R_2'} = \frac{Q_1 Q_2 R_2' \cdot n^2 R_1' h^2}{n^3 R_1'^2 h^3}.$$

Canceling common factors:

$$V_n = \frac{Q_1 Q_2 R_2'}{n R_1' h}.$$

9.4.2 The Product $V_n \cdot D_n$

Now, find the product $V_n \cdot D_n$:

$$V_n \cdot D_n = \left(\frac{Q_1 Q_2 R_2'}{n R_1' h}\right) \cdot \left(\frac{n^2 R_1' h^2}{m_1 Q_1 Q_2 R_2'}\right).$$

Simplify:

$$V_n \cdot D_n = \frac{Q_1 Q_2 R_2' \cdot n^2 R_1' h^2}{n R_1' h \cdot m_1 Q_1 Q_2 R_2'} = \frac{nh}{m_1}.$$

Thus:

$$V_n \cdot D_n = \frac{nh}{m_1}.$$

9.4.3 Angular Momentum L_n

The angular momentum L_n of the first cluster on the nth orbit is defined as:

$$L_n = m_1 V_n D_n.$$

Substitute $V_n \cdot D_n = \frac{nh}{m_1}$:

$$L_n = m_1 \cdot \frac{nh}{m_1} = nh.$$

9.4.4 The Relation Between $V_n \cdot D_n$ and L_n

From the obtained expressions, it is clear that:

$$V_n \cdot D_n = \frac{nh}{m_1},$$

so that

$$L_n = nh = m_1 \cdot (V_n \cdot D_n).$$

Thus, the angular momentum L_n is directly proportional to the product $V_n \cdot D_n$:

$$\boxed{L_n = m_1 \cdot (V_n \cdot D_n) = nh. \qquad (29)}$$

9.4.5 Result Summary

- The product $V_n \cdot D_n$ characterizes the ”torque” of the system, related to the orbital speed and the orbit radius. It is proportional to the mode number n and the constant h.
- The angular momentum L_n is directly proportional to nh, which corresponds to the quantization of angular momentum in the system exactly as postulated empirically by Bohr for the electron in the hydrogen atom based on the emission spectrum.

Thus, the obtained expressions confirm that the angular momentum of the system is quantized and is related to the orbital speed and orbit radius through the constant h. This provides theoretical proof of Bohr’s postulate based on simple considerations of the system of spatial density clusters striving for maximum entropy, suggesting that the approach and the underlying mathematical model deserve at least attention and discussion.

9.5 Total Energy E_n of the Two-Spatial-Density-Cluster System and Its Relation to the Rotational Frequency ω_n

9.5.1 The Total Energy of the First Cluster on the nth Orbit Consists of Kinetic and Potential Energy:

The total energy E_n of the first cluster on the nth orbit is the sum of kinetic and potential energy:

$$E_n = \frac{1}{2} m_1 V_n^2 - \int_{D_n}^{\infty} \mathrm{Re}(W_{\mathrm{total}_{r_1}})\, dD.$$

9.5.2 Kinetic Energy of the First Cluster:

$$T_n = \frac{1}{2} m_1 V_n^2.$$

Substitute $V_n = \frac{nh}{m_1 D_n}$:

$$T_n = \frac{1}{2} m_1 \left(\frac{nh}{m_1 D_n} \right)^2 = \frac{1}{2} m_1 \cdot \frac{n^2 h^2}{m_1^2 D_n^2} = \frac{n^2 h^2}{2 m_1 D_n^2}.$$

9.5.3 Potential Energy

The potential energy is defined by the integral of the real part of the perturbation $\mathrm{Re}(W_{\mathrm{total}_{r_1}})$:

$$U_n = -\int_{D_n}^{\infty} \mathrm{Re}(W_{\mathrm{total}_{r_1}})\, dD.$$

Substitute $\mathrm{Re}(W_{\mathrm{total}_{r_1}}) = \frac{Q_1 Q_2 R_2'}{R_1' D^2}$:

$$U_n = -\int_{D_n}^{\infty} \frac{Q_1 Q_2 R_2'}{R_1' D^2}\, dD.$$

Evaluate the integral:

$$U_n = -\frac{Q_1 Q_2 R_2'}{R_1'} \int_{D_n}^{\infty} \frac{1}{D^2}\, dD = -\frac{Q_1 Q_2 R_2'}{R_1'} \left[-\frac{1}{D} \right]_{D_n}^{\infty} = -\frac{Q_1 Q_2 R_2'}{R_1' D_n}.$$

9.5.4 Total Energy E_n

Substitute T_n and U_n into the expression for E_n:

$$E_n = \frac{n^2 h^2}{2 m_1 D_n^2} - \frac{Q_1 Q_2 R_2'}{R_1' D_n}.$$

Substitute $D_n = \frac{n^2 R_1' h^2}{m_1 Q_1 Q_2 R_2'}$:

$$E_n = \frac{n^2 h^2}{2 m_1} \cdot \left(\frac{m_1 Q_1 Q_2 R_2'}{n^2 R_1' h^2} \right)^2 - \frac{Q_1 Q_2 R_2'}{R_1'} \cdot \frac{m_1 Q_1 Q_2 R_2'}{n^2 R_1' h^2}.$$

Simplify:

$$E_n = \frac{n^2h^2}{2m_1} \cdot \frac{m_1^2Q_1^2Q_2^2R_2'^2}{n^4R_1'^2h^4} - \frac{m_1Q_1^2Q_2^2R_2'^2}{n^2R_1'^2h^2}.$$

Cancel common factors:

$$E_n = \frac{m_1Q_1^2Q_2^2R_2'^2}{2n^2R_1'^2h^2} - \frac{m_1Q_1^2Q_2^2R_2'^2}{n^2R_1'^2h^2} = -\frac{m_1Q_1^2Q_2^2R_2'^2}{2n^2R_1'^2h^2}.$$

9.5.5 Comparison with the Rotational Frequency ω_n

The rotational frequency ω_n is given by:

$$\omega_n = \frac{m_1Q_1^2Q_2^2R_2'^2}{n^3R_1'^2h^3}.$$

Express E_n in terms of ω_n:

$$E_n = -\frac{m_1Q_1^2Q_2^2R_2'^2}{2n^2R_1'^2h^2}.$$

Substitute ω_n:

$$E_n = -\frac{nh}{2} \cdot \frac{m_1Q_1^2Q_2^2R_2'^2}{n^3R_1'^2h^3} = -\frac{nh}{2} \cdot \omega_n.$$

Thus:

$$E_n = -\frac{nh}{2}\omega_n.$$

9.5.6 Final Solution

1. The total energy E_n:

$$\boxed{E_n = -\frac{m_1Q_1^2Q_2^2R_2'^2}{2n^2R_1'^2h^2}. \qquad (30)}$$

2. The relation between E_n and ω_n:

$$\boxed{E_n = -\frac{nh}{2}\omega_n. \qquad (31)}$$

Thus, we obtain the expected result: the total energy is proportional to the rotational frequency ω_n and the constant h. This relation is analogous to the connection between energy and frequency in quantum systems, confirming that the total energy of the system is related to the rotation frequency through the constant h and the mode number n.

9.6 Interpreting the Result: The Ratio of E_n to $\mathrm{Im}(W_{\textbf{total}_{r_1}}(D_n))$

9.6.1 Total Energy E_n

From the previous solution, the total energy E_n is given by:

$$E_n = -\frac{m_1 Q_1^2 Q_2^2 R_2'^2}{2n^2 R_1'^2 h^2}.$$

9.6.2 The Imaginary Part of the Perturbation $\mathrm{Im}(W_{\textbf{total}_{r_1}}(D_n))$

The imaginary part of the perturbation $W_{\text{total}_{r_1}}$ at a distance D_n is given by:

$$\mathrm{Im}(W_{\text{total}_{r_1}}(D_n)) = \frac{\pi Q_1 Q_2 R_2'}{2R_1'^2 D_n}.$$

Substitute $D_n = \frac{n^2 R_1' h^2}{m_1 Q_1 Q_2 R_2'}$:

$$\mathrm{Im}(W_{\text{total}_{r_1}}(D_n)) = \frac{\pi Q_1 Q_2 R_2'}{2R_1'^2} \cdot \frac{m_1 Q_1 Q_2 R_2'}{n^2 R_1' h^2} = \frac{\pi m_1 Q_1^2 Q_2^2 R_2'^2}{2n^2 R_1'^3 h^2}.$$

9.6.3 The Ratio $\frac{E_n}{\mathrm{Im}(W_{\textbf{total}_{r_1}}(D_n))}$

Now, find the ratio:

$$\frac{E_n}{\mathrm{Im}(W_{\text{total}_{r_1}}(D_n))} = \frac{-\frac{m_1 Q_1^2 Q_2^2 R_2'^2}{2n^2 R_1'^2 h^2}}{\frac{\pi m_1 Q_1^2 Q_2^2 R_2'^2}{2n^2 R_1'^3 h^2}}.$$

Simplifying:

$$\frac{E_n}{\mathrm{Im}(W_{\text{total}_{r_1}}(D_n))} = -\frac{\frac{m_1 Q_1^2 Q_2^2 R_2'^2}{2n^2 R_1'^2 h^2}}{\frac{\pi m_1 Q_1^2 Q_2^2 R_2'^2}{2n^2 R_1'^3 h^2}} = -\frac{R_1'}{\pi}.$$

9.6.4 Final Result

1. *The ratio of E_n to $\mathrm{Im}(W_{total_{r_1}}(D_n))$:*

$$\boxed{\frac{E_n}{\mathrm{Im}(W_{\text{total}_{r_1}}(D_n))} = -\frac{R_1'}{\pi}. \qquad (32)}$$

Thus, the ratio of E_n to $\mathrm{Im}(W_{\text{total}_{r_1}}(D_n))$ is expressed in terms of the radius of the first cluster R_1' and the constant π, which confirms our hypothesis that the imaginary part of the solution for the total perturbation represents the energy of the system that determines the resonant frequency of the two spatial-density clusters.

9.7 The Physical Meaning of Planck's Constant

In our model, we obtained that the constant h (analogous to Planck's constant) is expressed as:

$$h = \frac{2R_1'}{H\pi},$$

where:

- R_1' is the radius of the first cluster,
- H is the proportionality coefficient linking the rotational frequency ω_n and the imaginary part of the perturbation $\mathrm{Im}(W_{\text{total}_{r_1}})$.

For $H = 1$ (i.e. the resonant frequency equals the imaginary part of the perturbation), the constant h becomes:

$$h = \frac{2R_1'}{\pi}.$$

We also established that the ratio of the total energy E_n of the system to the imaginary part of the perturbation $\mathrm{Im}(W_{\text{total}_{r_1}})$ is:

$$\boxed{\frac{E_n}{\mathrm{Im}(W_{\text{total}_{r_1}})} = 2h. \qquad (33)}$$

This relation allows us to interpret h as the **ratio of the total energy of the system in the resonant state to its "imaginary energy"**. Thus, Planck's constant h acquires a deep physical meaning: it characterizes the connection between the energy of the system and its imaginary (resonant) component.

9.8 Analogy with the Quantization of Angular Momentum

In our model, the angular momentum L_n of the first cluster is quantized according to the rule:

$$L_n = nh,$$

where n is an integer (the mode number). This is directly analogous to Bohr's postulate for the quantization of the electron's angular momentum in the hydrogen atom, where the electron's angular momentum is quantized as $L = n\hbar$ (with $\hbar = h/2\pi$ being the reduced Planck's constant).

Thus, our model not only reproduces the well-known quantum-mechanical regularities but also offers a new perspective on the nature of Planck's constant, linking it to the resonant properties of the system.

The obtained result allows us to consider Planck's constant not as an abstract constant, but as a physical quantity that determines the connection between the imaginary and total energy of the two-cluster system.

Let us also recall that when we first encountered the need to introduce the normalization constant $\frac{R_1'}{4\pi}$ to satisfy our postulate of the conservation of the spatial density, as introduced in the third section of our study, it became clear that Planck's constant has a very deep physical meaning: it is both the normalization constant to fulfill the postulate of the spatial density conservation, and the constant in the quantization of the angular momentum of the two-cluster system during the rotation of one around the other, as well as the ratio of the total energy of the two-cluster system to its imaginary energy, and it also corresponds to the electron's size divided by $\frac{1}{2}\pi$.

Many might object that the size of the electron does not equal Planck's constant based on measurements. I would answer that the electron's size is beyond the limits of measurement. Thus, the electron's size is taken to be equal to the minimal possible experimental measurement. Theoretical calculations predict that its size is much smaller than commonly believed. The obtained electron size agrees well with the formulas for the masses of the electron and proton based on the internal energy formula derived in this article, if one considers the electron as a spatial density cluster and the proton, on the contrary, as a depletion (expansion). This indirectly confirms our assumptions about the nature of the internal energy of the charge, which determines its mass. In this way, our theory closes upon itself, which, as a researcher, I find deeply impressive.

My aim at the beginning of this investigation was to understand why the interaction between two charges falls off as $\sim \frac{1}{D^2}$, and I obtained the completely unexpected result that it actually has a logarithmic dependence, and the solution is only complex, the consequence of which is an imaginary energy of the two-cluster system that determines their resonant frequency. At this frequency the system is in resonance and, therefore, does not radiate energy during uniform circular motion around the first spatial density cluster.

X The Ratio of the Energy Required to Create Two Spatial Density Clusters (the Internal Energy of Two Charges) to the Potential Energy of Their Interaction. Internal Energy of Space.

Let us find the ratio of the energy expended to create two charges to the interaction energy that will arise between them if they are placed at a distance of $10R_1'$.

Recall that we have:

$$Q_1 = \big(V(R_1) - V(R_1')\big)\rho_0,$$

$$V(R_1) = \frac{4}{3}\pi R_1^3,$$

$$V(R_1') = \frac{4}{3}\pi (R_1')^3,$$

where:

- ρ_0 is the spatial density before perturbation,
- R_1' and R_1 are the radii of the spatial density spheres, respectively after and before the compression of the spatial density.

10.1 The Initial Expression for the Force $F(t)$ Holding the Cluster in Its Compressed State, as Derived in Section IV of This Article

Given:

$$F(t) = \frac{8\pi\rho_0}{3}\left(\frac{R_1^3}{t} - t^2\right).$$

This force describes the confinement of the spatial density in its compressed state.

10.2 Compression Energy E_{inside}

The energy expended to compress the sphere from radius R_1 to R_1' is calculated as the integral of $F(t)$ with respect to t from R_1 to R_1':

$$E_{\text{inside}} = \int_{R_1}^{R_1'} F(t)\,dt.$$

Substituting $F(t)$:

$$E_{\text{inside}} = \frac{8\pi\rho_0}{3}\int_{R_1}^{R_1'}\left(\frac{R_1^3}{t} - t^2\right)dt.$$

We split the integral into two terms:

$$E_{\text{inside}} = \frac{8\pi\rho_0}{3}\left(R_1^3\int_{R_1}^{R_1'}\frac{1}{t}\,dt - \int_{R_1}^{R_1'} t^2\,dt\right).$$

Evaluating the integrals:

$$\int \frac{1}{t}\,dt = \ln|t|, \quad \int t^2\,dt = \frac{t^3}{3}.$$

Substituting the limits:

$$E_{\text{inside}} = \frac{8\pi\rho_0}{3}\left(R_1^3\big(\ln|R_1'| - \ln|R_1|\big) - \frac{(R_1')^3 - R_1^3}{3}\right).$$

Simplifying:

$$E_{\text{inside}} = \frac{8\pi\rho_0}{3}\left(R_1^3 \ln\left(\frac{R_1'}{R_1}\right) - \frac{(R_1')^3 - R_1^3}{3}\right).$$

10.3 The Potential Energy of the Interaction $E_{Q1,Q2}(D)$

Given:

$$E_{Q1,Q2}(D) = \int_{\infty}^{10R_1'} W_{Q1,Q2}\,dD,$$

where

$$W_{Q1,Q2} = \frac{Q_1^2 R_1'}{R_1' D^2} = \frac{Q_1^2}{D^2}.$$

Thus, the integral is:

$$E_{Q1,Q2}(D) = Q_1^2 \int_{\infty}^{10R_1'} \frac{1}{D^2}\,dD.$$

Evaluating the integral:

$$\int \frac{1}{D^2}\,dD = -\frac{1}{D}.$$

Substituting the limits:

$$E_{Q1,Q2}(D) = Q_1^2\left(-\frac{1}{10R_1'} - \left(-\frac{1}{\infty}\right)\right) = -\frac{Q_1^2}{10R_1'}.$$

10.4 Expression for Q_1

Given:

$$Q_1 = \big(V(R_1) - V(R_1')\big)\rho_0,$$

with

$$V(R_1) = \frac{4}{3}\pi R_1^3, \quad V(R_1') = \frac{4}{3}\pi (R_1')^3.$$

Thus,

$$Q_1 = \frac{4}{3}\pi\rho_0\left(R_1^3 - (R_1')^3\right).$$

10.5 The Ratio $\frac{E_{\text{inside}}(R_1')}{E_{Q1,Q2}(D)}$

Now substitute the expressions for E_{inside} and $E_{Q1,Q2}(D)$:

$$\frac{E_{\text{inside}}(R_1')}{E_{Q1,Q2}(D)} = -\frac{\frac{8\pi\rho_0}{3}\left(R_1^3 \ln\left(\frac{R_1'}{R_1}\right) - \frac{(R_1')^3 - R_1^3}{3}\right)}{\frac{Q_1^2}{10R_1'}}.$$

Substitute Q_1:

$$Q_1^2 = \left(\frac{4}{3}\pi\rho_0(R_1^3 - (R_1')^3)\right)^2 = \frac{16}{9}\pi^2\rho_0^2(R_1^3 - (R_1')^3)^2.$$

Now substitute Q_1^2 in the denominator:

$$\frac{E_{\text{inside}}(R_1')}{E_{Q1,Q2}(D)} = -\frac{\frac{8\pi\rho_0}{3}\left(R_1^3 \ln\left(\frac{R_1'}{R_1}\right) - \frac{(R_1')^3 - R_1^3}{3}\right)}{\frac{16}{9}\pi^2\rho_0^2(R_1^3 - (R_1')^3)^2/(10R_1')}.$$

Simplify:

$$\frac{E_{\text{inside}}(R_1')}{E_{Q1,Q2}(D)} = -\frac{8\pi\rho_0}{3} \cdot \frac{10R_1' \cdot 9}{16\pi^2\rho_0^2(R_1^3 - (R_1')^3)^2}\left(R_1^3 \ln\left(\frac{R_1'}{R_1}\right) - \frac{(R_1')^3 - R_1^3}{3}\right).$$

10.6 Final Expression

$$\boxed{\frac{E_{\text{inside}}(R_1')}{E_{Q1,Q2}(D)} = -\frac{15R_1'}{2\pi\rho_0\left(R_1^3 - (R_1')^3\right)^2}\left(R_1^3 \ln\left(\frac{R_1'}{R_1}\right) - \frac{(R_1')^3 - R_1^3}{3}\right). \quad (34)}$$

10.7 Let Us Plot the Graph of the Function $\frac{E_{\text{inside}}(R_1')}{E_{Q1,Q2}(D)}$ Using the Following Parameters:

- $R_1 = 10$ — a fixed value of R_1.
- $\rho_0 = 1$ — a fixed value of ρ_0.
- R_1' varies from 1 to 20, i.e., $R_1' \in [1, 20]$.

To plot the graph, we use the range:

$$R_1' = \text{np.linspace}(1, 20, 500).$$

After the numerical study of the graph of the function $\frac{E_{\text{inside}}(R_1')}{E_{Q1,Q2}(D)}$, we found that there are two regions where $E_{\text{space}} = \left|\frac{E(R_1')}{E(D)}\right| < 1$, meaning that the ratio of the energy required to create a spatial density cluster to the interaction energy between two spatial

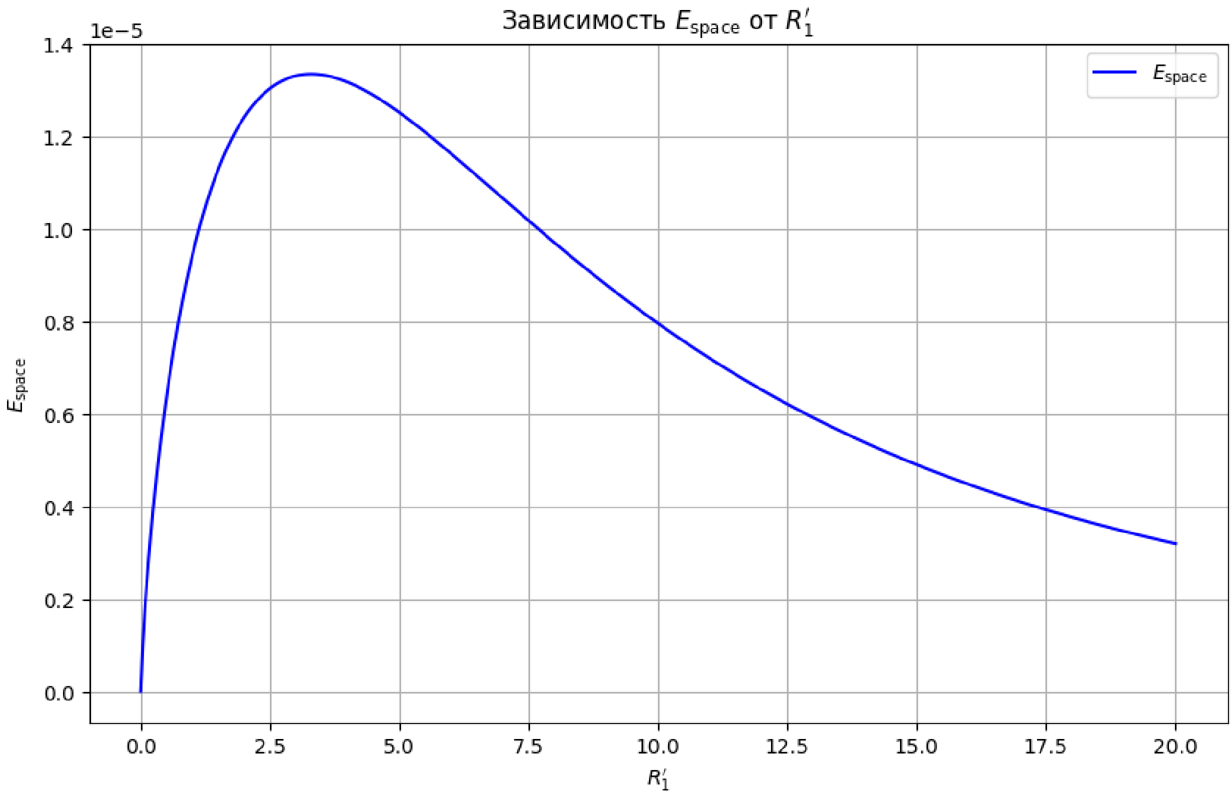

Figure 4: Graph of the Ratio of the Internal Energy of Two Spatial Density Clusters to Their Interaction Potential Energy $\frac{E_{\text{inside}}(R_1')}{E_{Q1,Q2}(D)}$

density clusters is less than 1. This implies that after their creation, the energy released from their interaction exceeds the energy spent on their creation. Although this is difficult to grasp intuitively, our space is structured in such a way that it can generate energy. We too might be able to do so if we understand how to compress spatial density and maintain it in a compressed state, as observed in elementary charges.

10.8 How to Compress Spatial Density

If space is a "quasi-medium" that has density but does not possess mass, friction, or viscosity—only a tendency toward maximum entropy—then, assuming Bernoulli's law applies, where higher flow speed corresponds to lower pressure, that region will compress and the density will increase until the pressure equalizes. However, once we compress the spatial density, it will cause a curvature in the spatial metric, leading to the emergence of mass. Thus, the compressed region of space will acquire kinetic energy (inertia) that will keep it in motion, i.e., in a compressed state. If we twist the spatial density into a torus (such structures can be stable and self-sustaining) that rotates in all degrees of freedom, we obtain a stable self-sustaining state—something akin to a negatively charged elementary particle (electron). If this torus is stretched, i.e., its radius is increased, we get something similar to a positively charged elementary particle (proton). If, under high pressure, an electron and a proton are combined, a neutron is formed; however, the mass of the neutron will be greater than the sum of the electron's and proton's masses, since its mass will also include the energy required to compress the electron and proton, and according to our model they repel each other at about 2.5 electron radii. Thus, we obtain the process for the formation of the basic building blocks of our Universe—electron, neutron, and proton. In their production, energy will be released that fills our Universe, transforming from one form to another. Hence, the law of energy conservation for spatial density and its derived clusters does not hold, as observed, for example, in the electron's tunneling effect in overcoming a potential barrier. During the formation of elementary charges, more energy is released than the energy expended to create them, thereby triggering a self-sustaining chain reaction for the production of matter and energy directly from space.

XI Conclusion and Summary

1. **Space Density as a Universal Property**: This article proposes a hypothesis that spatial density is a key property determining all fundamental interactions—gravitational, electromagnetic, strong, and weak. This property is described in a five-dimensional coordinate system, with the fifth dimension orthogonal to the conventional spatial and temporal dimensions.

2. **Theoretical Proof of Bohr's Postulate**: For the first time, a theoretical justification of Bohr's postulate regarding the quantization of the electron's angular momentum in the hydrogen atom is presented. The model shows that the angular momentum of the two-spatial-density-cluster system is quantized, in accordance with Bohr's postulate. This confirms that the quantization of angular momentum can be explained through the properties of spatial density—a significant step in understanding quantum mechanics.

3. **The Connection Between Charge and Mass**: A novel connection between charge and its mass is established. It is shown that the mass of a spatial density cluster is equivalent to the energy required to compress it. This allows mass to be interpreted as a measure of the energy that holds the cluster in its compressed state, consistent with Einstein's equation $E = mc^2$.

4. **Complex Solutions and Imaginary Energy**: The solution for the interaction of two spatial density clusters is found to be purely complex, with the imaginary part determining the system's resonant frequency. This opens new avenues for understanding the nature and stability of quantum systems.

5. **Strong and Weak Interactions**: The model offers an explanation for the strong and weak interactions through the properties of spatial density. It is shown that strong interaction at short distances can be associated with resonant effects in the spatial density model, while the weak interaction is related to the redistribution of density.

6. **The Physical Meaning of Planck's Constant**: In this model, Planck's constant h is interpreted as the ratio of the total energy of the system to its imaginary (resonant) component. This provides a new perspective on the nature of this fundamental constant, linking it with the resonant properties of the system.

7. **Connection with Quantum Mechanics**: The results demonstrate that the spatial density model can reproduce well-known quantum-mechanical phenomena such as angular momentum quantization and the energy-frequency relation. This confirms that the model can be used for further development of quantum theory.

8. **New Research Directions**: The model opens new avenues for research, such as the study of resonant phenomena in quantum systems, explanations for dark matter and dark energy, and the development of novel approaches to unifying fundamental interactions.

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