A Simple Approach to Understanding the Stress-Energy Tensor

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Abstract

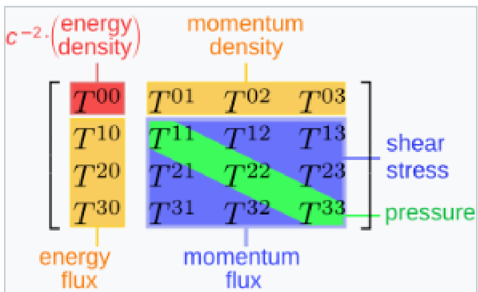
The common method of defining the stress-energy tensor is confusing and inconsistent. In this article, an alternative simple method for definition is presented. The connection between the stress-energy tensor used in general relativity and the stress tensor used in material mechanics contains errors, as all stresses occurring in the material cause an increase in the fields that bind the microscopic parts of the material and affect only the material's density.

Many students, not just beginners, find it very difficult to understand the meaning and significance of the stress-energy tensor. However, I do not believe that the difficulty arises from the concept itself. The stress-energy tensor is meant to describe the state of spacetime in a small region around a point in terms of the presence and motion of matter in that region. This is straightforward if approached correctly. However, due to historical reasons related to the emergence and first usage of the tensor concept, a particular way of defining the components of the stress-energy tensor became widespread. This approach, which is found in most references **[1]**, defines these components in the same way as the stress tensor used in the theory of mechanics of materials, known as Cauchy's stress tensor. This method is beautifully illustrated on Wikipedia:

Stress–energy tensor

Article Talk

From Wikipedia, the free encyclopedia



The stress–energy tensor is defined as the tensor $T^{\alpha\beta}$ of order two that gives the flux of the αth component of the momentum vector across a surface with constant x^{β} coordinate.

Unfortunately, this Method of Defining Stress-Energy Tensor is Confusing and not Consistent.

Unfortunately, this has led to much complexity and inconsistency.

The concept of the stress tensor used in the theory of mechanics of materials is well-suited for that field. In this context, time is treated separately from the three spatial dimensions as an independent variable. Stresses are then analyzed in terms of three spatial dimensions rather than four, with each stress component associated with two of these dimensions. This results in a 3×3 matrix of components. Each component is defined as follows: the component represents the stress calculated by dividing the force applied to the material in **i** direction by the area of the surface where is **j** dimension constant.

When we try to define another quantity in four dimensions using a similar approach, it cannot simply be done by extending the indices and from **3** to **4**, as in the common definition. This would create significant confusion. For example, defining a plane using a single index is meaningless. Consider the component, which is known to represent density. If the first index **t** indicates the direction of momentum, the second index **t** cannot be associated with a specific plane because fixing dimension **t** corresponds to three different planes: **xy**, **xz**, and **yz**.

The correct way to define the stress-energy tensor is by generalizing the concept of density. Density is composed of two quantities: mass and volume. In the framework of relativity, which does not distinguish between the four dimensions, the mass corresponds to the time component of **4-momentum**:

$$\boldsymbol{p^t, p^x, p^y, p^z.}$$

Similarly, volume corresponds to the time component of a quantity that we may call the **4-volume**:

$$\boldsymbol{v^t, v^x, v^y, v^z.}$$

In Cartesian coordinate we have.

$$\boldsymbol{v^t \equiv \Delta x \Delta y \Delta z.}$$

$$\boldsymbol{v^x \equiv \Delta t \Delta y \Delta z.}$$

$$\boldsymbol{v^y \equiv \Delta x \Delta t \Delta z.}$$

$$\boldsymbol{v^z \equiv \Delta x \Delta y \Delta t.}$$

Or in general:

$\boldsymbol{v^i = V/\Delta x^i}$ where $\boldsymbol{V \equiv \Delta t\, \Delta x \Delta y \Delta z.}$

In non-Cartesian coordinates, we multiply the volume by the square root of the determinant of the metric tensor in each case.

Thus, dividing **4-momentum** by **4-volume** results in a **4×4** matrix representing all possibilities. In this definition, the component $\boldsymbol{T^{ij}}$ represents the momentum in the **i**-direction divided by the volume associated with the **j**-direction:

$$\boldsymbol{T^{ij} \equiv p^i / v^j.}$$

This means that the stress-energy tensor represents density in its general sense: momentum divided by volume in its general sense.

This clear definition of the stress-energy tensor aligns with the common definition in that some components describe the flow of momentum across certain surfaces. However, its advantage is that it remains valid and consistent for all components.

Another important point regarding the common definition of the stress-energy tensor is how its components are interpreted based on theories from mechanics of materials. The component associated with the flow of momentum across surfaces perpendicular to momentum is interpreted as pressure stress. Meanwhile, components representing the flow of momentum parallel to a given surface are interpreted as shear stress. This is a clear mistake. It is true that the rates of momentum flow in perpendicular directions correspond to pressure, but stresses are a different matter. The similarity between pressure and stress is practical rather than fundamental, and this should not lead to a generalization in all cases and all aspects. For instance, the stress-energy tensor does not contain any component representing tensile stress. Similarly, shear stress cannot be determined solely from momentum quantities and their flow rates parallel to a given surface. All stresses are states associated with the fields that bind material parts together when exposed to external forces. These fields are represented in the stress-energy tensor by the component related to time (density).

Imagine a rod subjected to a torque that causes it to twist around its axis. In this case, the rod experiences shear stress at every point. In the theory of mechanics of materials, the value of the shear stress appears in the component representing shear stress in the Cauchy's stress tensor. While in stress-energy tensor used in general relativity, in this case, the only affected component in the stress-energy tensor would be the density. There would be no change in the momentum flowing in directions perpendicular to the rod's axis. For this reason, the best name for this quantity is the "material energy tensor"**[2]**. Including the concept of stress in its name only adds confusion.

References:

[1] See for example: *Gravitation* by Misner, Thorne, and Wheeler (1973)

[2] This term was used by Bertrand Russell in his book *The Analysis of Matter*.