Complex Extensions of Riemann Zeta Function and Complementary Formula of Gama Function

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Abstract This paper discusses the complex extensions of Riemann Zeta function and complementary formulas of Gama function. By re-writing the Zeta function equation, it is proved that the equation described a relation between the original Zeta function $\zeta(s)$ and a now function $\zeta'(s) = \zeta(1-s)$. But the domains of these two function does not the same and incompatible, so the Riemann Zeta function equation does not hold. It is also proved that the complex extension formula of the present complementary formula of Gama function is wrong. The correct formula is given by strict calculation. The condition $0 < \mathrm{Re}(s) < 1$ needs to be satisfied in order to make the residue integral finite for this formula. However, Riemann Zeta function itself requires $\mathrm{Re}(s) > 1$. Therefore, Riemann Zeta function equation does not hold at any point in the complex plane, and it is meaningless to discuss it.

Key Words： Riemann conjecture, Riemann Zeta function, Zeta function equation, Analytical extension, Complementary formula of Gama function, Residue theorem

1 Introduce

In 2019, the author of this paper published a paper to prove that Riemann's original paper in 1859 on Riemann hypothesis contained four fundamental mistakes that led to serious inconsistencies in Riemann Zeta function equations. Therefore, the Riemann hypothesis is meaningless, which reveals the essential reason why Riemann conjecture problem cannot be solved for a long time [1].

In 2020, the author published a paper again to prove that even without considering the problems existing in Riemann Zeta function equation and assuming that Riemann Zeta function equation is valid, Riemann conjecture is also not valid, and the Riemann Zeta function has no non-trivial zeros [2].

This paper discusses the complex extensions of Riemann Zeta function. By re-writing the Zeta function equation, it is proved that the equation described a relation between the original Zeta function $\zeta(s)$ and a now function $\zeta'(s) = \zeta(1-s)$. But the domains of these two function does not the same and incompatible, so the Riemann Zeta function equation does not hold.

It is also proved that the complex extension formula of the existing complementary formula of Gama function is wrong. The correct formula is given through strict calculation, and the condition $0 < \mathrm{Re}(s) = a < 1$ needs to be satisfied in order to make the residue integral be limited. However, the Zeta function itself requires $\mathrm{Re}(s) > 1$ so that Riemann Zeta function equations do not hold at any point in the complex plane.

Riemann Zeta function has two forms, the series summation form and the integral form. The summation form of series is initially defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \qquad (1)$$

Where $s = a + ib$ is a complex number. It is generally believed that when $Re(s) > 1$, Eq. (1) is convergent. When $Re(s) \le 1$, Eq.(1) is divergent and the function is meaningless.

In order to make the function meaningful in the region $Re(s) < 1$, Riemann used the Gama function to express Eq.(1) in the form of integral with ($x \in R$) [3,4]

$$\sum_{n=1}^{\infty} n^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \qquad Re(s) > 1 \qquad (2)$$

To calculate Eq.(2), Riemann extended it into the integral on the complex plane by $dx \to dz = d(x + iy)$ and get

$$\sum_{n=1}^{\infty} n^{-s} = \frac{\Gamma(1-s)}{2\pi i} \int_L \frac{(-z)^{s-1}}{e^z - 1} dz \qquad (3)$$

Using the method of residue, Riemann deduced following relation

$$\sum_{n=1}^{\infty} n^{-s} = 2(2\pi)^{s-1} \Gamma(1-s) \sin\left(\frac{s\pi}{2}\right) \sum_{n=1}^{\infty} n^{-(1-s)} \qquad (4)$$

By considering Eq.(1) again and let

$$\zeta(1-s) = \sum_{n=1}^{\infty} n^{-(1-s)} \qquad (5)$$

Eq.(4) is briefly written as

$$\zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{s\pi}{2}\right) \Gamma(1-s) \zeta(1-s) \qquad (6)$$

Eq.(6) is called the Riemann Zeta function equation. However, Riemann in his 1859′s paper had no any discussion on the domain of Eq.(6). He only directly wrote [3]

This equation gives the values of the function $\zeta(s)$ for all complex s and shows that it is single-valued and finite for all valves of s other than 1, and also that it vanishes when s is negative even integer.

What this sentence means that Eq.(6) holds for all finite points $s = a + ib$ including the points $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s) < 1$, except for the point $\mathrm{Re}(s) \neq 1$. However, this is not possible. Because Eq.(6) comes directly from Eq.(4), which is nothing more than a symbolic representation of Eq. (4). For example, according to Eq.(4), the right of Eq.(6) is $\zeta(1-s) = \zeta(-1.5)$ which is infinite and does not make sense.

Perhaps some mathematicians have seen the problem with Riemann's statement, and in the literature [3], $\zeta(s)$ in the left side of Eq. (6) is regarded as a newly defined function $\zeta'(s)$ and change its domain to the following statement [4]

Eq.(6) is for $\mathrm{Re}(s) < 0$. By uniqueness of analytic continuation, Eq.(5) is valid for all $s \neq 1$.

But this statement is obviously self-contradictory, since it is believed that Eq.(6) is only applicable to the region $\mathrm{Re}(s) < 0$, it cannot be applicable to the region $\mathrm{Re}(s) \ge 0$, and it cannot be valid in all cases with $s \neq 1$. In fact, if a function is infinite in a certain region, it is not an analytic function in that region.

The huge confusion caused by this contradictory definition is also one of the reasons to lead that Riemann conjecture does not make sense.

In the second chapter, the complex extension of Riemann Zeta function is discussed in detail. Let $\zeta'(s) = \zeta(1-s)$ represent a new function(It is actually not the Zeta function of original meaning.), and write Eq.(6) as

$$\zeta'(s) = (2\pi)^s \cos\left(\frac{s\pi}{2}\right)\Gamma(s)\zeta(s) \tag{7}$$

It is easy to prove that the definition domain of product of all terms on the right side of Eq.(7) is still $\mathrm{Re}(s) > 1$ 。 By considering Eq.(5), the domain of new function $\zeta'(s)$ on the left side of Eq.(7) is $\mathrm{Re}(s) < 1$. So the domains on the two sides of Eq.(7) are different. It is shown again that the Riemann Zeta function does not hold.

It is known that the complementary formula of Gama function for real numbers is [5]

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi} \qquad 0 < a < 1 \tag{7}$$

Where a is a real non-integer. Riemann used following complementary formula of Gama function of complex number when he deduced the Zeta function equation [5].

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin s\pi} \tag{8}$$

However, Eq. (8) is not a strict calculation result, but is obtained directly let $a \to a + ib = s$ in Eq.(7). It is proved in this paper that Eq. (8) is wrong. By the strict calculation, the correct result is

$$\Gamma(s)\Gamma(1-s) = \frac{\pi e^{b\pi}}{\sin a\pi} \qquad 0 < \mathrm{Re}(s) < 1 \tag{9}$$

It indicates that the product of two complementary complex Gama functions is a real number, so Eqs. (4) and (6) of Riemann Zeta function have to be changed accordingly, and the final result is:

$$\zeta(s) = \frac{(2\pi)^{s-1}\sin\pi a}{e^{b\pi}\cos(\pi s/2)}\Gamma(1-s)\zeta(1-s) \tag{10}$$

According to Eq.(10), the Riemann conjecture on the trivial and non-trivial zeros (if having any) of the Zeta function had to change accordingly.

More importantly, it is proved in this paper that in order to make the residue circumference integral absolutely convergent, Eq.(9) needs to satisfy the condition $0 < \mathrm{Re}(s) < 1$, so Eq.(10) must also satisfy this condition. On the other hand, according to the definition of Eq.(1), $\zeta(s)$ itself requires $\mathrm{Re}(s) > 1$. These two conditions are mutually exclusive. As a result, the Riemann Zeta function equation does not hold at any point oan the complex plane, and any discussion of it is meaningless. So this paper will not discuss the trivial and non-trivial zeros of Eq.(10) any more.

2. The existing problems in the complex extension of Zeta function

2.1 The definition of analytic extension of a function

As we known that general function has a domain of definition. Beyond the domain of definition, the function may be meaningless. In order to make the function sense in a larger region, it needs to be extended. Suppose that the function $f_1(x)$ is clearly defined in the region L_1, but it has no meaning in the region L_2. In order to make it sense in the region L_2, we need to extent and change it into $f_2(x)$. The analytic extension of a function needs to meet following three conditions [5].

1. In the extended region L_2, the form of $f_2(x)$ must be different from $f_1(x)$, otherwise the contradiction will be caused and the extension becomes meaningless.

2. In the original region L_1, the form of $f_2(x)$ should be completely the same with $f_1(x)$, otherwise, $f_2(x)$ can not be regarded as the extension of $f_1(x)$.

3. Since a function can be extended in many different ways, it will lead to different results. In order to guarantee uniqueness, the extension must be an analytic extension, or the extend function must be derivable everywhere in whole region. If it is a complex continuation, the Cauchy-Riemann equation must be satisfied.

2.2 The analytic continuations of real functions

The common example of a real function's continuation is [6]

$$f_1(x) = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1 \tag{11}$$

Where $x \in R$ is a real number. Eq. (11) is meaningful and limited in the field $|x| < 1$. when $|x| > 1$, the function $f_1(x) \to \infty$ and becomes meaningless. On the other hand, we define

$$f_2(x) = \frac{1}{1-x} \qquad x \neq 1 \tag{12}$$

$f_2(x)$ is meaningful on the whole number axis except at the point $x = 1$. At all points of the field with $|x| < 1$, we always have $f_1(x) = f_2(x)$. By developing $f_2(x)$ into the Taiwan's series when $|x| < 1$, we can prove $f_1(x) = f_2(x)$ completely.

However, when $|x| > 1$, Eqs. (2) and (12) can not be equal each other. For example, let $x = 2$, we have $f_1(2) \to \infty$ and $f_2(2) = -1$. Because the definition field of $f_2(x)$ is greater than that of $f_1(x)$, we consider $f_2(x)$ as the continuation of $f_1(x)$ in the field with $|x| > 1$ and write two functions in the unified form as below

$$f(x) = \begin{cases} 1 + x + x^2 + x^3 + \cdots & |x| < 1 \\ \dfrac{1}{1-x} & |x| \neq 1 \end{cases} \tag{13}$$

In general, a function can be extended in many different ways, resulting in different results. In order to ensure the uniqueness, the continuation of function needs to meet the continuity condition, so that the function can be differentiated everywhere. The continuation that meets this condition is called as the analytic continuation [3].

It is important to emphasize that at every point in the small field where the original function is meaningful, the value of the extended function should be exactly the same as the value of original function, otherwise it is not the continuation of the original function. Meanwhile, in the extended field, the form of the extended function must be different from the original function, otherwise the extension of the function is meaningless [3].

For example, for Eqs. (2) and (3), in the field $|x| < 1$ where the original function makes sense, the values of function $f_2(x)$ and $f_1(x)$ must be the same at every point. Although their forms look different on the surface, they are the same actually. In the field $|x|>1$ after the continuation, the forms of $f_2(x)$ and $f_1(x)$ must be different. Otherwise $f_2(x)$ is still equal to $f_1(x)$, which is no meaning.

It is difficult to find the form of analytic continuation of a function in practical problems. The analytic continuations of some functions in existing mathematics are actually unsuccessful. As we see below, the negative continuation of the Gama function violates the above principles. In the extended field, the form of the function is exactly the same as the original function's form, so it is still infinite and meaningless.

2.3 The analytic continuation of complex function

Let $z = x + iy \in C$ be a complex number, a similar example of analytic continuation of complex function is shown below [6].

$$F_1(z) = 1 - z^2 + z^4 - z^6 + \cdots \qquad |z| < 1 \tag{14}$$

$$F_2(z) = \frac{1}{1+z^2} \qquad z \neq \pm i \tag{15}$$

$F_1(z)$ is an analytic function and convergent inside the unit circle but is divergent outside the unit circle without meaning. $F_2(z)$ is an analytic function meaningful on the whole complex plane except at points $z \neq \pm i$.

To developing $F_2(z)$ into the Taylor's series of complex functions in the field $|z|<1$, we can obtain $F_1(z)$. That is to say, two functions are completely the same. Since the definition field of $F_2(z)$ is larger than $F_1(z)$, $F_2(z)$ can be regarded as an analytic continuation of $F_1(z)$ over the entire complex plane (except at points $z \neq \pm i$). Similar to Eq.(4), we write they in the unified form

$$F(z) = \begin{cases} 1 - z^2 + z^4 - z^6 + \cdots & (|z|<1) \\ \dfrac{1}{1+z^2} & (z \neq \pm i) \end{cases} \tag{16}$$

Similarly, within the extended field $|z|>1$ ($z \neq \pm i$), $F_1(z)$ and $F_2(z)$ are not the same functions, because $F_1(z)$ is meaningless in this case.

2.4 The existing problem of complex extension of Riemann Zeta function

According to common understanding at present, Eq.(6) is regarded as the analytic extension of Eq.(1) in the region $\text{Re}(s)<0$. The left side of Eq.(6) is considered as a newly defined Zeta function which is different from Eq.(1), i.e., $\zeta'(s) \neq \zeta(s)$. The right side of Eq.(6) is the concrete form of new Zeta function, and the domain for both sides of the formula is $\text{Re}(s)<0$. According to this understanding, Eq.(6) should be written as

$$\zeta'(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{s\pi}{2}\right)\zeta(1-s) \qquad \text{Re}(s)<0 \tag{17}$$

However, this is obviously impossible, because Eq.(6) comes from formula (4), and the one to the left is defined in terms of Eq.(1). If $\zeta(s)$ on the left side of Eq.(6) is regarded as a new function $\zeta'(s)$, $\zeta(1-s)$ on the right side should also be regarded as a new function $\zeta'(1-s)$. Eq.(6) becomes the relation between the two new functions $\zeta'(s)$ and $\zeta'(1-s)$ and it has nothing to do with the original Zeta function, so there is no so-called new Zeta function $\zeta'(s) \neq \zeta(s)$.

According to the definition of function extension in Section 3.1, if new function $\zeta'(s)$ shown in

Eq.(17) is meaningful in the region $\operatorname{Re}(s)<0$, it should be exactly the same as the original function $\zeta(s)$ in the original region $\operatorname{Re}(s)>0$, otherwise it cannot be considered as the extension of $\zeta(s)$. However, this is simply not possible. For example, let $s=2.5>1$, according to the definition of function extension, there should be $\zeta'(2.5)=\zeta(2.5)$. However, according to Eq.(17), this is obviously not possible, we have

$$\zeta'(2.5)=2(2\pi)^{1.5}\Gamma(-1.5)\sin\left(\frac{2.5\pi}{2}\right)\zeta(-1.5)\neq\zeta(2.5) \qquad (18)$$

Therefore, even if $\zeta'(s)$ on the left of Eq.(18) is regarded as the new Zeta function, it is not a complex extension of the Zeta function shown in Eq.(1).

By considering Eq.(9) and let $\zeta'(s)=\zeta(1-s)$, the Riemann Zeta function can be written into the form of Eq.(7)。Because $\zeta(1-s)$ is not the original Zeta function as shown in Eq.(1), it is rational to consider it as a now function $\zeta'(s)$. According to Eq.(1), the domain of Zeta function $\zeta(s)$ is $\operatorname{Re}(s)>1$。The domain of Gama function is $\operatorname{Re}(s)>0$. For the other part $(2\pi)^s\cos(\pi s/2)$ 中, the domain of s 的 can be arbitrary. So the domain of the functions on the right side of Eq.(7) is still $\operatorname{Re}(s)>1$。On the region of $\operatorname{Re}(s)\leq 1$，the right side of Eq.(7) becomes infinite and meaningless.

However, by considering Eq.(5), the domain of new function $\zeta'(s)$ on the left side of Eq.(7) is $\operatorname{Re}(s)<1$. Therefore, the domains on the two sides of Eq.(7) are different. This result indicates that the Riemann Zeta function does not hols. In fact, as the author proved in Reference [1] by concrete calculation that , On the real axis of the complex plane, the two sides of equation (5) are not equal to each other, which fully illustrates the problem

Besides, even if Eq.(17) is regarded as an extension of Zeta function on the region $\operatorname{Re}(s)<0$, since the function domain of Eq.(1) is $\operatorname{Re}(s)>1$. In the region $0\leq\operatorname{Re}(s)\leq 1$, the Zeta function is still not defined. It does not like what Riemann thought that the extended Zeta function holds for all regions except at point $\operatorname{Re}(s)=1$.

Therefore, some people later proposed an extension method [4], in the region $0\leq\operatorname{Re}(s)<1$, the Zeta function is written as:

$$\zeta''(s)=\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s} \qquad 0\leq\operatorname{Re}(s)<1 \qquad (19)$$

In this way, it is considered that the Riemann Zeta function of Eq.(1) is extended to the entire complex space except at the point $\operatorname{Re}(s)=1$, and Eqs.(1), (17) and (19) constitute the complete definition of Riemann Zeta function.

The problem is that if Eq.(19) is considered to be the extended Zeta function, then the discussion of Riemann conjecture should be based on it. Because Riemann conjecture indicated that the all zeros of Zeta function fall on the point $\operatorname{Re}(s)=1/2$, i.e., on the domain of the Zeta function $\zeta''(s)$. However, this is not the case in practice. Starting from Riemann, Eq.(17) has been used to discuss the zero point of Zeta function, and no one used Eq.(19). Therefore, no matter whether the definition of Eq.(19) is reasonable or not, it has nothing to do with Riemann's 1859 paper and does not belong to the discussing scope of Riemann's conjecture so it is not discussed in this paper further.

3. The complex continuation of complementary formula of Gama function

3.1 The calculation of complementary formula of Gama function

Suppose that a is a real non-integer number with $0 < a < 1$, it can be proved to exist the following formula [5]

$$\int_0^\infty \frac{x^{a-1}}{x+1}dx = \frac{\pi}{\sin a\pi} \qquad 0 < a < 1 \tag{20}$$

Though the formula is defined in the real field, it needs to use the Residue theorem of complex function to prove. Let $s = a + ib$, the complex continuation of Eq.(20) is considered to directly let $a \to s$ in Eq.(20) at present and get

$$\int_0^\infty \frac{x^{s-1}}{x+1}dx = \frac{\pi}{\sin s\pi} \qquad 0 < \mathrm{Re}(s) < 1 \tag{21}$$

Correspondingly, the complex continuation of Eq.(7) is considered becoming follow form

$$\Gamma(a)\Gamma(1-a) = \int_0^\infty \frac{x^{a-1}}{x+1}dx = \frac{\pi}{\sin a\pi} \qquad 0 < a < 1 \tag{22}$$

According to Eq.(21), the complex extension of complementary formula of Gama function is

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty \frac{x^{s-1}}{x+1}dx = \frac{\pi}{\sin s\pi} \qquad 0 < \mathrm{Re}(s) < 1 \tag{23}$$

It should be noted that Eq.(21) is obtained by analogy and is not strictly calculating result. We prove below that Eq.(21) is not true and therefore Eq.(23) is not true.

3. 2 The proof that Eq.(21) does not hold

We write the left side of Eq.(21) as

$$\int \frac{x^{s-1}}{x+1}dx = \int_0^\infty \frac{x^{a+ib-1}}{x+1}dx = \int_0^\infty \frac{x^{a-1}e^{ib\ln x}}{x+1}dx$$

$$= \int_0^\infty \frac{x^{a-1}}{x+1}\cos(b\ln x)dx + i\int_0^\infty \frac{x^{a-1}}{x+1}\sin(b\ln x)dx \tag{24}$$

According to the Euler formula, we have

$$e^{is} = e^{ia-b} = e^{-b}\left(\cos a + i\sin a\right) \tag{25}$$

$$\sin(a+ib) = \frac{e^{i(a+ib)} - e^{-i(a+ib)}}{2i} = \frac{e^{-b}(\cos a + i\sin a) - e^{b}(\cos a - i\sin a)}{2i}$$

$$= \frac{-(e^{b} - e^{-b})\cos a + i(e^{b} + e^{-b})\sin a}{2i} \tag{26}$$

$$\frac{\pi}{\sin(a+ib)\pi} = \frac{i2\pi}{-(e^{b\pi} - e^{-b\pi})\cos a\pi + i(e^{b\pi} + e^{-b\pi})\sin a\pi}$$

$$= \frac{2\pi(e^{b\pi} + e^{-b\pi})\sin a\pi}{e^{2b\pi} + e^{-2b\pi} + 2(\sin^2 a\pi - \cos^2 a\pi)} - i\frac{2\pi(e^{b\pi} - e^{-b\pi})\cos a\pi}{e^{2b\pi} + e^{-2b\pi} + 2(\sin^2 a\pi - \cos^2 a\pi)} \quad (27)$$

If Eq.(22) holds, by comparing the real parts and the imaginary parts of Eq.(24) and (27), we have

$$\int_0^\infty \frac{x^{a-1}}{x+1}\cos(b\ln x)dx = \frac{2\pi(e^{b\pi} + e^{-b\pi})\sin a\pi}{e^{2b\pi} + e^{-2b\pi} + 2(\sin^2 a\pi - \cos^2 a\pi)} \quad (28)$$

$$\int_0^\infty \frac{x^{a-1}}{x+1}\sin(b\ln x)dx = \frac{-2\pi(e^{b\pi} + e^{-b\pi})\cos a\pi}{e^{2b\pi} + e^{-2b\pi} + 2(\sin^2 a\pi - \cos^2 a\pi)} \quad (29)$$

Let $b = 0$ in Eqs.(28) and (29), we get

$$\int_0^\infty \frac{x^{a-1}}{x+1}dx = \frac{4\pi\sin a\pi}{2(1 + \sin^2 a\pi - \cos^2 a\pi)} = \frac{\pi}{\sin a\pi} \quad (30)$$

$$0 = \frac{-4\pi\cos a\pi}{2(1 + \sin^2 a\pi - \cos^2 a\pi)} = -\frac{\pi\cos a\pi}{\sin^2 a\pi} \quad (31)$$

Eq.(30) is completely the same as Eq.(21), but Eq.(31) can not hold when $a \neq (2n+1)/2$. Where is wrong? Let's analyze it below.

3. 3 The calculation of Eq.(31)

At present, the Residue theorem is used to calculate Eq.(20). Let's repeat this calculation at first. Suppose that a is a real non-integer, we consider the integral of complex function below with $0 \le \arg z \le 2\pi$ [5]

$$T(a) = \int_C z^{a-1}Q(z)dz \quad (32)$$

Here $Q(z)$ is a single value and analytic function everywhere except at several isolated singularities. There are no singularities on the positive real axis. When $|z| \to 0$ and $|z| \to \infty$, $|z^a Q(z)|$ tends to zero consistently.

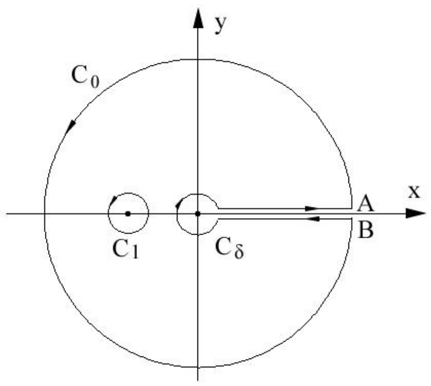

Fig.1 The contour of residue calculation

The integral contour C of residue calculation is shown in Fig.1. It stars off from the point

$z = x = \delta$ on the above positive real axis, goes along the positive real axis and arrive at point A with $x = R$. Then goes along a big circle C_0 with radius and comes back to point B on the down positive real axis. Then goes along the negative direction of real axis and arrives at point $x = \delta$. At last, goes around the small circle C_δ and reaches the starting point.

Therefore, the integral of Eq.(31) can be written as

$$\int_C z^{a-1}Q(z)dz = \int_\delta^R z^{a-1}Q(z)dz + \int_{C_0} z^{a-1}Q(z)dz = \int_R^\delta z^{a-1}Q(z)dz + \int_{C_R} z^{a-1}Q(z)dz$$

$$= (1-e^{i2a\pi})\int_\delta^R z^{a-1}Q(z)dz + \int_{C_0} z^{a-1}Q(z)dz + \int_{C_\delta} z^{a-1}Q(z)dz \quad (33)$$

By using the Residue theorem, we have

$$\int_C z^{a-1}Q(z)dz = i2\pi\sum res\{z^{a-1}Q(z)\} \quad (34)$$

For Eq.(21), we have $Q(z) = 1/(1+z)$, Eq.(34) becomes

$$\int_0^\infty \frac{x^{a-1}}{1+x}dx \to \int_C \frac{z^{a-1}}{1+z}dz = i2\pi\sum res\left\{\frac{z^{a-1}}{1+z}\right\} \quad (35)$$

According to the calculation premise of Eq.(32), under the condition $0 < a < 1$, when $|z| = R \to \infty$, due to $a < 1$, we have

$$|z^a Q(z)| = \left|\frac{z^a}{1+z}\right| \to 0 \quad (36)$$

When $|z| = R \to 0$, due to $a > 0$, we have

$$|z^a Q(z)| = \left|\frac{z^a}{1+z}\right| \to 0 \quad (37)$$

Therefore, the integral can be guaranteed to converge on both the upper and lower limits, that is

$$\int_{C_0} z^{a-1}Q(z)dz \to 0 \qquad \text{and} \qquad \int_{C_\delta} z^{a-1}Q(z)dz \to 0 \quad (38)$$

Substituting the result above in Eq.(33), we get

$$\int_\delta^R z^{a-1}Q(z)dz \to \frac{1}{1-e^{i2a\pi}}\int_C z^{a-1}Q(z)dz = \frac{i2\pi}{1-e^{i2a\pi}}\sum res\{z^{a-1}Q(z)\} \quad (39)$$

The function $Q(z) = 1/(1+z)$ has an unique singularity $z = -1 = e^{i\pi}$ on the real axis. The residue is

$$\sum res\{z^{a-1}Q(z)\} = res\left\{\frac{z^{a-1}}{(1+z)'}\right\}\Bigg|_{z=e^{i\pi}} = z^{a-1}\Big|_{z=e^{i\pi}} = e^{i(a-1)\pi} \quad (40)$$

Substituting it in Eq.(35), we get the current result with

$$\int_0^\infty \frac{x^{a-1}}{1+x}dx = \frac{i2\pi e^{i(a-1)\pi}}{1-e^{i2a\pi}} = \frac{i2\pi e^{-i\pi}}{e^{-ia\pi} - e^{i2a\pi}} = \frac{-i2\pi}{e^{-ia\pi} - e^{ia\pi}} = \frac{\pi}{\sin a\pi} \quad (41)$$

However, if let $a \to s = a + ib$, as proved in Section 3.1, it is impossible to let $\sin a\pi \to \sin s\pi$ on the right side of Eq.(39). It needs to be calculated again.

3.4 The correct calculation of Eq.(21)

Eq. (21) is recalculated below to obtain theorem 1.

Theorem1：The calculation result of Eq.(21) is

$$\int_0^\infty \frac{x^{s-1}}{1+x} dx = \frac{\pi e^{b\pi}}{\sin a\pi} \qquad 0 < \mathrm{Re}(s) < 1 \tag{42}$$

Proof: For the same contour integral, according to Eq.(24), let

$$Q_1(b,z) = \cos(b\ln z)/(1+z) \qquad Q_2(b,z) = \sin(b\ln z)/(1+z) \tag{43}$$

$$T_1(a,b) = \int_C z^{a-1} Q_1(b,z)dz \qquad T_2(a,b) = \int_C z^{a-1} Q_2(b,z)dz \tag{44}$$

When $|z| = \delta \to 0$ and $|z| = R \to \infty$, the functions $\cos(b\ln z)$ and $\sin(b\ln z)$ are uncertain, but we $|\cos(b\ln z)| \le 1$ and $|\cos(b\ln z)| \le 1$. When $|z| = R \to \infty$, due to $a < 1$, we have

$$\left| z^a Q_1(z) \right| = \left| z^a \frac{\cos b\ln z}{1+z} \right| \to 0 \tag{45}$$

When $|z| = R \to 0$, due to $a > 0$, we have

$$\left| z^a Q_1(z) \right| = \left| z^a \frac{\sin(b\ln z)}{1+z} \right| \to 0 \tag{46}$$

So we can use the residue theorem to calculate. For $T_1(a,b)$, similar to Eq.(39), the calculating result is

$$\frac{1}{i2\pi} \int_C z^{a-1} Q_1(b,z)dz = \sum res\left\{ z^{a-1} Q_1(z) \right\} = z^{a-1} \cos(b\ln z) \Big|_{z=e^{i\pi}}$$

$$= e^{i(a-1)\pi} \cos(b\ln e^{i\pi}) = e^{i(a-1)\pi} \cos(ib\pi) = e^{i(a-1)\pi} \frac{e^{b\pi} + e^{-b\pi}}{2} \tag{47}$$

For $T_2(a,b)$, the calculating result is

$$\frac{1}{i2\pi} \int_C z^{a-1} Q_2(b,z)dz = \sum res\left\{ z^{a-1} Q_2(z) \right\} = z^{a-1} \sin(b\ln z) \Big|_{z=e^{i\pi}}$$

$$= e^{i(a-1)\pi} \sin(b\ln e^{i\pi}) = e^{i(a-1)\pi} \sin(ib\pi) = e^{i(a-1)\pi} \frac{e^{-b\pi} - e^{b\pi}}{2i} \tag{48}$$

According to Eq.(39) and by considering the relation $e^{-i\pi} = -1$, the last result is

$$\int_0^\infty \frac{x^{a-1}}{1+x} \cos(b\ln x)dx = \frac{i2\pi}{1 - e^{i2a\pi}} e^{i(a-1)\pi} \frac{e^{b\pi} + e^{-b\pi}}{2}$$

$$= \frac{-i2\pi}{e^{-ia\pi} - e^{ia\pi}} \frac{e^{b\pi} + e^{-b\pi}}{2} = \frac{\pi}{\sin a\pi} \frac{e^{b\pi} + e^{-b\pi}}{2} \tag{49}$$

$$\int_0^\infty \frac{x^{a-1}}{1+x} \sin(b \ln x) dx = \frac{i2\pi}{1 - e^{i2a\pi}} e^{i(a-1)\pi} \frac{e^{b\pi} - e^{-b\pi}}{2i}$$

$$= \frac{-i2\pi}{e^{-ia\pi} - e^{ia\pi}} \frac{e^{b\pi} - e^{-b\pi}}{2i} = \frac{\pi}{\sin a\pi} \frac{e^{b\pi} - e^{-b\pi}}{2i} \tag{50}$$

When $b = 0$, Eq.(50) is consistent with Eq.(30) and Eq.(50) is equal to zero. The contradiction shown in Eq.(31) does not exist again. Eq.(24) becomes

$$\int_0^\infty \frac{x^{s-1}}{x+1} dx = \int_0^\infty \frac{x^{a+ib-1}}{x+1} dx = \frac{\pi}{\sin a\pi} \left[\frac{e^{b\pi} + e^{-b\pi}}{2} + i \frac{e^{b\pi} - e^{-b\pi}}{2i} \right] = \frac{\pi e^{b\pi}}{\sin a\pi} \tag{51}$$

The result of integral is a real number, rather than a complex number. The complex continuation formula (23) should be changed in the form of Eq.(42).

It should be noted that Eq. (42) holds only under the condition $0 < a < 1$. Beyond this condition, Eq. (42) is still invalid.

3. 5 The correct calculation of real Gama function's complementary formula

Based on Theorem 1, Theorem 2 can be obtained below.

Theorem 2: The complex extension of real Gama function's complementary formula should be revised as

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty \frac{x^{s-1}}{x+1} dx = \frac{\pi e^{b\pi}}{\sin a\pi} \qquad 0 < \text{Re}(s) < 1 \tag{52}$$

Proof: The definition of complex Gama function is

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \qquad \text{Re}(s) > 0 \tag{53}$$

If $a \leq 0$, the function is infinite and meaningless. Similarly, we have

$$\Gamma(1-s) = \int_0^\infty e^{-u} u^{1-s-1} du = \int_0^\infty e^{-u} u^{-s} du \qquad \text{Re}(s) < 1 \tag{54}$$

So in the region $0 < \text{Re}(s) < 1$, $\Gamma(s)$ and $\Gamma(1-s)$ are absolutely convergence and they can be combined into a double integral [5]

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty e^{-t} t^{s-1} dt \int_0^\infty e^{-u} u^{-s} du = \int_0^\infty \int_0^\infty e^{-(t+u)} (t/u)^s t^{-1} dt du \tag{55}$$

Let $x = t/u$, $y = t + u$, the variation range of x and y can be from 0 to infinite, the corresponding Jacobian determinant is:

$$\left| \frac{\partial(t,u)}{\partial(x,y)} \right| = \left| \frac{\partial(x,y)}{\partial(t,u)} \right|^{-1} = \frac{y}{(1+x)^2} \tag{56}$$

Substituting it in Eq.(55), we get

$$\Gamma(s)\Gamma(1-s)=\int_0^\infty\int_0^\infty e^{-y}x^s\frac{1+x}{xy}\frac{y}{(1+x)^2}dxdy$$

$$=\int_0^\infty e^{-y}dy\int_0^\infty\frac{x^{s-1}}{1+x}dx=\int_0^\infty\frac{x^{s-1}}{1+x}dx \qquad 0<\mathrm{Re}(s)<1 \tag{57}$$

So according to Theorem 1, Eq.(52) of Theorem 2 is proved.

It should be emphasized that the domain of Eq.(57) is $0<\mathrm{Re}(s)<1$. Beyond this range, the contour integral diverges and the result is invalid because Eqs.(45) and (46) do not hold. Some literature and textbooks said that since the left and right sides of formula (21) are analytic functions on the whole plane except the points $z=n=0,\ \pm1,\ \pm2,\ \dots$, Eq.(23) is valid on the plane except for these points [4]. However, this is not possible. Outside the range $0<\mathrm{Re}(s)<1$, Eq.(23) does not exist.

3.6 The influence on the problem of Riemann hypothesis

Eq.(23) was used for Riemann to deduce the Riemann Zeta function quation, so the result of Eq.(52) will have an impact on the Riemann hypothesis problem. On the basis of Eq.(6), Riemann defines a new function $\xi(s)$ to let [3, 4]

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{58}$$

It is proved to have following symmetry a

$$\xi(s)=\xi(1-s) \tag{59}$$

Due to $\zeta(s)$ and $\xi(s)$ have the same zeros, the practical calculations on the zeros of Zeta function are based on Eq.(58). According to Eq.(7), $\zeta(s)=0$ when $s=\pm2k$（$k=0,\ 1,\ 2,\ 3,\cdots$）. For the cases with $s=-2k$, the zeros are called as the trivial zeros of the Zeta function. For the cases with $s=+2k$, the current theory do not discuss them.

By considering Eq.(52), the Riemann Zeta function equation becomes

$$\zeta(s)=(2\pi)^{s-1}e^{\pi b}\sin\pi a\Gamma(1-s)\zeta(1-s) \qquad 0<\mathrm{Re}(s)<1 \tag{60}$$

The symmetry of Eq.(59) does not exist again, i.e., $\xi(s)\neq\xi(1-s)$. More serious problem is that Eq.(60) demands $0<\mathrm{Re}(s)<1$ which is the condition that the calculation of residue calculation should satisfied. However, according to the definition of Eq.(1), Eq.(60) is invalid at any point on the complex plane. So any discussion on Eq.(60) is meaningless, so we do not discuss the zero of Eq.(60) any more in this paper.

4 Conclusion

The author pointed out in Reference 1 that Riemann used a summation formula to derive the integral form of the Zeta function in his paper in 1859. This formula is applicable when $x>0$. If $x=0$, the formula does not make sense at this point. However, Riemann's calculation involves $x=0$ at the lower limit of the integral, resulting in divergence of integral form of Zeta function [1].

In this paper, it is proved that on the real number axis of the complex plane, when the left side of the

Riemann Zeta function equation is finite, the right side may be infinite, and vice versa. The Riemann Zeta function equation holds only at a point $\mathrm{Re}(s)=1/2$, where the Zeta function is infinite, not zero.The Riemann Zeta function equation exists serious inconsistencies [1].

In addition, when Riemann derived the Zeta function equation, he also missed a term, which is not equal to zero in the region $\mathrm{Re}(s)<1$, but is divergent, resulting in the Zeta function equation invalid. Therefore, the Riemann Zeta function equation does not hold, and the Riemann hypothesis is meaningless [1].

The calculations of the zero of Riemann Zeta function's zeros are also discussed. It is pointed out that the existing calculation uses an approximate method, the analytic property of complex variable function is destroyed, and the Cauchy-Riemann equation cannot be satisfied. Although a large number of non-trivial zeros are found on the line $\mathrm{Re}(s)=1/2$ of the complex plane, none of them are real zeros of the Zeta function.

The authors proved in paper 2 that even if the Riemann Zeta function equation is considered valid, the Riemann hypothesis is also invalid. The author uses a standard method to completely separate the real and imaginary parts of the Riemann Zeta function equation, and proves that the real and imaginary parts cannot equal to zeros at the same time [2]. So the Zeta function equation has no non-trivial zero, and the Riemann hypothesis solved completely.

In this paper, the existing problem of the complex extension of Riemann Zeta function is discussed. The equation is rewritten in the form of Eq.(7), which is proved to be a formula describing the relationship between the original Zeta function $\zeta(s)$ and the new function $\zeta'(s)=\zeta(1-s)$. However, the domains of two sides of the equation are different and incompatible, so the Riemann Zeta function equation is invalid. It is not true as Riemann thought, that the equations of the Zeta function hold for all points except the point $\mathrm{Re}(s)=1$ on the complex plane.

Finally, it is proved that the complex extension formula of the existing Gama function complementary formula is wrong, and the correct formula is given by strict calculation. In order to make the contour integral of the residue calculation finite, the formula needs to satisfy the conditions. But the Zeta function itself requires $\mathrm{Re}(s)>1$, both are mutually exclusive. Therefore, the Gama function equation derived by Riemann is invalid at any point of the complex plane, and it is meaningless to discuss it.

References

1. Mei Xiaochun, Inconsistencies of Riemann Zeta function Equations, Mathematics Letters, 2019; 13-22, 5 (2) : Doi: 10.11648 / j.m l. 20190502.11.
2. Mei Xiaochun, A standard method to prove that the Riemann Zeta function equation has no non-trivial Zeros, Advances in Pure Mathematics, 2020, 10, 86-99, https://www.scirp.org/journal/apm
3. Riemann G. F. B., (1859), Uber die Anzabl der Primahlem unter einer gegebenen Grosse, Monatsberichte der Berliner Akademine, 2, 671- 680.
4. Bent E. Petersen, Riemann Zeta Function, https://pan.baidu.com/s/1geQsZxL.
5. Guo Dunren, Methods of Mathematical Physics, People's Education Press, 1965, p. 15,129.
6. Liang Kunmiao, Function Theory of Complex Variables, People's Education Press, 1979, p. 49.