The Tri-Quarter Theorem: Unifying Complex Coordinates with Topological Duality on the Unit Circle

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Abstract

In this paper, we introduce the Tri-Quarter Theorem, a new mathematical framework with a generalized complex-Cartesian-polar coordinate system on the complex plane $\mathbb{C}$ and a new topological property. We discuss the coordinate system's construction and prove that the boundary zone T, defined as the unit circle, separates the inner zone X_- ($||\vec{x}|| < 1$) and outer zone X_+ ($||\vec{x}|| > 1$), where T exhibits topological duality with X_- and X_+ through a novel phase pair assignment and structured orientation that ensures consistent directional separation across $\mathbb{C}$. This result unifies complex, Cartesian, and polar representations while offering new insights into topological separation and orientation. This framework provides a unified perspective on complex coordinates and a tool for analyzing systems with circular symmetry or boundary-dependent behaviors, with potential applications in fields such as black hole physics, signal processing, and other areas reliant on complex domain partitioning. We also created a software tool to animate the theorem. More work is needed to further develop and apply this framework via the methods of science and mathematics.

1 Introduction

The complex plane $\mathbb{C}$, viewed as a complex 1D Riemann surface, serves as a foundational structure in mathematics and its applications, bridging algebraic, geometric, and analytic perspectives [1, 2]. In some contexts, traditional coordinate systems, such as Cartesian and polar forms, while powerful, treat the unit circle as a mere boundary without fully exploiting its topological role. In this work, we address this gap by introducing the Tri-Quarter Theorem, which establishes a generalized complex-Cartesian-polar coordinate system while applying a trichotomy—a three zone partition of $\mathbb{C}$ based on the unit circle—to the four quadrants of $\mathbb{C}$ to achieve a new topological property.

Our main result demonstrates that the unit circle T is topologically dual to both the inner zone X_- ($||\vec{x}|| < 1$) and outer zone X_+ ($||\vec{x}|| > 1$), with a structured orientation (through a novel phase pair assignment) that ensures consistent directional separation across $\mathbb{C}$. This framework unifies three coordinate representations and elevates the unit circle's role from a passive boundary to an active separator with intrinsic properties.

The significance lies in its potential to refine analyses, computations, modeling, and design in fields that rely on complex numbers and complex domain spatial partitioning, such as offering a nuanced approach to handling precise boundary conditions and directional properties across $\mathbb{C}$. For instance, potential applications may include deciphering the quasi-normal modes of black holes and enhancing the computational efficiency of signal processors.

In Section 2, we define the coordinate system, followed by the structured orientation in Section 3, the topological zones in Section 4, and the theorem's proof in Section 5. In Section 6, we conclude with a discussion of its origin, development, strengths, software animation tool, and potential future applications.

2 Generalized Complex-Cartesian-Polar Coordinate System

Let $X = \mathbb{C}$ be the complex plane and complex 1D Riemann surface [1, 2] with the standard topology. For a complex number $x = x_{\mathbb{R}} + ix_{\mathbb{I}} \in X$, we define the complex vector $\vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}} \in X$ as a position-point and position-vector on X with Cartesian coordinate form

$$(x_{\mathbb{R}}, x_{\mathbb{I}})_{Cartesian} = (x_{\mathbb{R}}, x_{\mathbb{I}})_C, \tag{1}$$

where $_C$ denotes Cartesian form. Vectors are denoted with arrows (e.g., $\vec{x}$) throughout this paper for consistency. Here $\vec{x}_{\mathbb{R}}$ and $\vec{x}_{\mathbb{I}}$ are axis-aligned orthogonal components that we treat interchangeably as both vectors and points in specific subsets of X. Specifically, we define

$$\begin{aligned} \vec{x}_{\mathbb{R}} &\in \mathbb{R} \times \{0\} = \{(x_{\mathbb{R}}, 0)_C \mid x_{\mathbb{R}} \in \mathbb{R}\} \\ \vec{x}_{\mathbb{I}} &\in \{0\} \times \mathbb{I} = \{(0, x_{\mathbb{I}})_C \mid x_{\mathbb{I}} \in \mathbb{R}\}, \end{aligned} \tag{2}$$

where $\mathbb{I} = i\mathbb{R}$ denotes the imaginary axis. Therefore, $\vec{x}_{\mathbb{R}} = (x_{\mathbb{R}}, 0)_C$ is a "real Euclidean 2-vector" aligned with the real axis, and $\vec{x}_{\mathbb{I}} = (0, x_{\mathbb{I}})_C$ is an "imaginary Euclidean 2-vector" aligned with the imaginary axis. Throughout this framework, this dual interpretation allows $\vec{x}_{\mathbb{R}}$ and $\vec{x}_{\mathbb{I}}$ to serve as both individual vectors and representative points within their respective sets, enabling their use interchangeably with the complex number x and its vector representation $\vec{x}$, which they form as a linear combination (emphasizing that $\vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}}$ is the direct sum of these vectors). As we will demonstrate in this paper, by subtly defining the components $\vec{x}_{\mathbb{R}}$ and $\vec{x}_{\mathbb{I}}$ to be orthogonal axis-aligned vectors—instead of just scalars—we achieve some useful properties.

Throughout this paper, we consider $X \setminus \{(0, 0)_C\}$ to exclude the origin where phase assignments are undefined. This focuses our framework on regions that are critical to the main results.

Next, $\vec{x} \in X \setminus \{(0, 0)_C\}$ is a Euclidean 2-vector with norm $||\vec{x}|| \in [0, \infty)$ and phase $\arg(\vec{x}) = \langle\vec{x}\rangle \in [0, 2\pi)$, which are respectively analogous to *magnitude* and *direction* in conventional notation with polar coordinate form

$$(||\vec{x}||, \langle\vec{x}\rangle)_{Polar} = (||\vec{x}||, \langle\vec{x}\rangle)_P, \tag{3}$$

where $_P$ denotes polar form, to give

$$\begin{aligned} x_{\mathbb{R}} &= ||\vec{x}|| \cos\langle\vec{x}\rangle \\ x_{\mathbb{I}} &= ||\vec{x}|| \sin\langle\vec{x}\rangle \end{aligned} \tag{4}$$

with Pythagorean form

$$||\vec{x}||^2 = x_{\mathbb{R}}^2 + x_{\mathbb{I}}^2. \tag{5}$$

(Note: From this point forward, for the sake of conciseness, for the phase angle notation of a given $\vec{x} \in X \setminus \{(0, 0)_C\}$, we will just say $\langle\vec{x}\rangle$ instead of $\arg \vec{x}$ also.) Thus, similarly to $\vec{x}$, for the axis-aligned orthogonal vectors $\vec{x}_{\mathbb{R}}$ and $\vec{x}_{\mathbb{I}}$, we also obtain the polar coordinate form

$$\begin{aligned} &(||\vec{x}_{\mathbb{R}}||, \langle\vec{x}_{\mathbb{R}}\rangle)_P \\ &(||\vec{x}_{\mathbb{I}}||, \langle\vec{x}_{\mathbb{I}}\rangle)_P, \end{aligned} \tag{6}$$

such that $||\vec{x}_{\mathbb{R}}||, ||\vec{x}_{\mathbb{I}}|| \in [0, \infty)$, $\langle\vec{x}_{\mathbb{R}}\rangle \in \{0, \pi\}$, and $\langle\vec{x}_{\mathbb{I}}\rangle \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$.

Henceforth, the Pythagorean form of Equation (5) becomes

$$||\vec{x}||^2 = ||\vec{x}_{\mathbb{R}}||^2 + ||\vec{x}_{\mathbb{I}}||^2 = x_{\mathbb{R}}^2 + x_{\mathbb{I}}^2. \tag{7}$$

Thus, $\forall \vec{x} \in X$ we define the 2D generalized complex-Cartesian-polar coordinate system as

$$(\vec{x}) = (\vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}}) = (x_{\mathbb{R}}, x_{\mathbb{I}})_C = (||\vec{x}|| \cos\langle\vec{x}\rangle, ||\vec{x}|| \sin\langle\vec{x}\rangle)_C = (||\vec{x}||, \langle\vec{x}\rangle)_P \tag{8}$$

unifying complex, Cartesian, and polar representations. This unification is further enriched by the phase pair assignments and structured orientation defined in Section 3, which provide directional properties that are critical to our topological analysis.

3 Structured Orientation

Building on the foundation of the unified coordinate representation in Equation (8), for any $\vec{x} \in X \setminus \{(0,0)_C\}$, we now assign phase pairs to establish a structured orientation across $X \setminus \{(0,0)_C\}$. We define the *quadrant phase pair assignments* as

$$\begin{aligned}
&\text{I:} && \langle\vec{x}\rangle \in \left(0, \tfrac{\pi}{2}\right) \Longleftrightarrow (x_{\mathbb{R}} > 0) \wedge (x_{\mathbb{I}} > 0) \Longleftrightarrow (\langle\vec{x}_{\mathbb{R}}\rangle = 0) \wedge \left(\langle\vec{x}_{\mathbb{I}}\rangle = \tfrac{\pi}{2}\right) \Longleftrightarrow (0, \tfrac{\pi}{2})_\phi \\
&\text{II:} && \langle\vec{x}\rangle \in \left(\tfrac{\pi}{2}, \pi\right) \Longleftrightarrow (x_{\mathbb{R}} < 0) \wedge (x_{\mathbb{I}} > 0) \Longleftrightarrow (\langle\vec{x}_{\mathbb{R}}\rangle = \pi) \wedge \left(\langle\vec{x}_{\mathbb{I}}\rangle = \tfrac{\pi}{2}\right) \Longleftrightarrow (\pi, \tfrac{\pi}{2})_\phi \\
&\text{III:} && \langle\vec{x}\rangle \in \left(\pi, \tfrac{3\pi}{2}\right) \Longleftrightarrow (x_{\mathbb{R}} < 0) \wedge (x_{\mathbb{I}} < 0) \Longleftrightarrow (\langle\vec{x}_{\mathbb{R}}\rangle = \pi) \wedge \left(\langle\vec{x}_{\mathbb{I}}\rangle = \tfrac{3\pi}{2}\right) \Longleftrightarrow (\pi, \tfrac{3\pi}{2})_\phi \\
&\text{IV:} && \langle\vec{x}\rangle \in \left(\tfrac{3\pi}{2}, 2\pi\right) \Longleftrightarrow (x_{\mathbb{R}} > 0) \wedge (x_{\mathbb{I}} < 0) \Longleftrightarrow (\langle\vec{x}_{\mathbb{R}}\rangle = 0) \wedge \left(\langle\vec{x}_{\mathbb{I}}\rangle = \tfrac{3\pi}{2}\right) \Longleftrightarrow (0, \tfrac{3\pi}{2})_\phi,
\end{aligned} \tag{9}$$

where $(\langle\vec{x}_{\mathbb{R}}, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi$ denotes the phase pair assigned by the coordinate system. Then to maintain continuity and ensure that no boundary remains undefined, we define the *axis boundary phase pair assignments* as

$$\begin{aligned}
&\text{"East":} && \langle\vec{x}\rangle = 0 \Longleftrightarrow (x_{\mathbb{R}} > 0) \wedge (x_{\mathbb{I}} = 0) \Longleftrightarrow (\langle\vec{x}_{\mathbb{R}}\rangle = 0) \wedge (\langle\vec{x}_{\mathbb{I}}\rangle = 0) \Longleftrightarrow (0, 0)_\phi \\
&\text{"North":} && \langle\vec{x}\rangle = \tfrac{\pi}{2} \Longleftrightarrow (x_{\mathbb{R}} = 0) \wedge (x_{\mathbb{I}} > 0) \Longleftrightarrow (\langle\vec{x}_{\mathbb{R}}\rangle = 0) \wedge \left(\langle\vec{x}_{\mathbb{I}}\rangle = \tfrac{\pi}{2}\right) \Longleftrightarrow (0, \tfrac{\pi}{2})_\phi \\
&\text{"West":} && \langle\vec{x}\rangle = \pi \Longleftrightarrow (x_{\mathbb{R}} < 0) \wedge (x_{\mathbb{I}} = 0) \Longleftrightarrow (\langle\vec{x}_{\mathbb{R}}\rangle = \pi) \wedge (\langle\vec{x}_{\mathbb{I}}\rangle = 0) \Longleftrightarrow (\pi, 0)_\phi \\
&\text{"South":} && \langle\vec{x}\rangle = \tfrac{3\pi}{2} \Longleftrightarrow (x_{\mathbb{R}} = 0) \wedge (x_{\mathbb{I}} < 0) \Longleftrightarrow (\langle\vec{x}_{\mathbb{R}}\rangle = 0) \wedge \left(\langle\vec{x}_{\mathbb{I}}\rangle = \tfrac{3\pi}{2}\right) \Longleftrightarrow (0, \tfrac{3\pi}{2})_\phi,
\end{aligned} \tag{10}$$

where each of these assigned phase pairs uniquely corresponds to each axis direction, aligning with $\langle\vec{x}\rangle$ and maintaining a structured orientation. To establish a consistent rule for handling the edge cases when $x_{\mathbb{R}} = 0$ or $x_{\mathbb{I}} = 0$, we can simply set $\langle\vec{x}_{\mathbb{R}}\rangle = 0$ or $\langle\vec{x}_{\mathbb{I}}\rangle = 0$, respectively. Thus, we derive the following *phase assignment* definition rules:

- For the real component $\vec{x}_{\mathbb{R}}$:
  - If $x_{\mathbb{R}} > 0$, then set $\langle\vec{x}_{\mathbb{R}}\rangle = 0$.
  - If $x_{\mathbb{R}} = 0$, then set $\langle\vec{x}_{\mathbb{R}}\rangle = 0$.
  - If $x_{\mathbb{R}} < 0$, then set $\langle\vec{x}_{\mathbb{R}}\rangle = \pi$.
- For the imaginary component $\vec{x}_{\mathbb{I}}$:
  - If $x_{\mathbb{I}} > 0$, then set $\langle\vec{x}_{\mathbb{I}}\rangle = \frac{\pi}{2}$.
  - If $x_{\mathbb{I}} = 0$, then set $\langle\vec{x}_{\mathbb{I}}\rangle = 0$.
  - If $x_{\mathbb{I}} < 0$, then set $\langle\vec{x}_{\mathbb{I}}\rangle = \frac{3\pi}{2}$.

The phase assignments establish a convention that aligns with the axis boundary phase pair assignments of Equation (10), indicates the absence of direction in a zero-magnitude component, and maintains consistency across all points, including the axis-bound points.

These phase pair assignments and phase assignments establish a structured orientation that not only unifies the complex-Cartesian-polar coordinate representations but also upgrades $X \setminus \{(0,0)_C\}$ with consistent directional properties. This system forms the foundation for the topological partitioning into three distinct zones in Section 4, where the unit circle's role as a separator is empowered by these orientations.

4 Topological Zones

We use trichotomy to partition X into three distinct topological zones:

(1) the inner zone X_-,

(2) the boundary zone T, and

(3) the outer zone X_+.

$\forall \vec{x} \in X$ we know that precisely one of the following trichotomy conditions must be satisfied

$$\begin{aligned} ||\vec{x}|| < 1 &\iff \vec{x} \in X_- \\ ||\vec{x}|| = 1 &\iff \vec{x} \in T \\ ||\vec{x}|| > 1 &\iff \vec{x} \in X_+, \end{aligned} \tag{11}$$

where clearly $X_- \cap T = T \cap X_+ = X_- \cap X_+ = \emptyset$ and $X_- \cup T \cup X_+ = X$, such that

$$\begin{aligned} X_- &= \{\vec{x} \in X : ||\vec{x}|| < 1\} \\ T &= \{\vec{x} \in X : ||\vec{x}|| = 1\} \\ X_+ &= \{\vec{x} \in X : ||\vec{x}|| > 1\} \end{aligned} \tag{12}$$

because the boundary zone T is a unit circle, which serves as the multiplicative group of all non-zero complex unit vectors separating X_- and X_+. So the four quadrants of X are partitioned into three zones—a "tri-quartering" if you will.

Definition 4.1 (Topological Duality). In the context of a topological space X partitioned by a boundary zone T into disjoint open sets X_- and X_+ such that $X \setminus T = X_- \cup X_+$ and $X_- \cap X_+ = \emptyset$, we say that T is *topologically dual* to X_- and X_+ if:

(1) $T = \partial X_-$ is the topological boundary of X_-, and $T = \partial \overline{X}_+$ is the topological boundary of the closure of X_+, where $\overline{X}_+$ denotes the closure of X_+ in X, and

(2) T is equipped with a structured orientation, defined by a consistent assignment of directional properties (e.g., phase pairs $(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi$) that distinguishes the approach from X_- and X_+ while maintaining separation across X.

For the upcoming theorem and proof in Section 5, this duality captures T's role as a separator with an intrinsic directional structure to unify the topological and geometric properties of the partition.

5 Tri-Quarter Theorem

Here we prove the main result of this paper.

Theorem (Tri-Quarter). In the generalized complex-Cartesian-polar coordinate system, the boundary zone T is topologically dual to both the inner zone X_- and outer zone X_+, with its structured orientation ensuring consistent directional separation across $X = \mathbb{C}$.

Proof. Let $X = \mathbb{C}$ be the complex 1D Riemann surface equipped with the 2D generalized complex-Cartesian-polar coordinate system given by Equation (8). Let $X_-, T, X_+ \subset X$ be given by Equations (11–12) so the boundary zone T separates X into the two disjoint open sets X_- and X_+ from the Jordan Curve Theorem [3, 4], where

$$X \setminus T = X_- \cup X_+ \text{ and } X_- \cap T = T \cap X_+ = X_- \cap X_+ = \emptyset. \tag{13}$$

The system assigns phase pairs per Equations (9–10).

First, we wish to show *structured orientation* on T. Now $\forall \vec{x} \in T$ we have $||\vec{x}|| = 1$ so Equation (8) gives

$$\begin{aligned}\vec{x} &= (\vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}}) \\ &= (x_{\mathbb{R}}, x_{\mathbb{I}})_C \\ &= (||\vec{x}|| \cos\langle\vec{x}\rangle, ||\vec{x}|| \sin\langle\vec{x}\rangle)_C \\ &= (1 \cdot \cos\langle\vec{x}\rangle, 1 \cdot \sin\langle\vec{x}\rangle)_C \\ &= (\cos\langle\vec{x}\rangle, \sin\langle\vec{x}\rangle)_C \\ &= (||\vec{x}||, \langle\vec{x}\rangle)_P \\ &= (1, \langle\vec{x}\rangle)_P,\end{aligned} \tag{14}$$

where as per Equations (9–10) the system assigns the following structured orientation:

- **Quadrant Phase Pairs**
  - **I:** $\langle\vec{x}\rangle \in (0, \frac{\pi}{2}) \implies (\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi = (0, \frac{\pi}{2})_\phi$: all points in this arc of T have a positive real direction and a positive imaginary direction (e.g., $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})_C$).
  - **II:** $\langle\vec{x}\rangle \in (\frac{\pi}{2}, \pi) \implies (\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi = (\pi, \frac{\pi}{2})_\phi$: all points in this arc of T have a negative real direction and a positive imaginary direction (e.g., $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})_C$).
  - **III:** $\langle\vec{x}\rangle \in (\pi, \frac{3\pi}{2}) \implies (\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi = (\pi, \frac{3\pi}{2})_\phi$: all points in this arc of T have a negative real direction and a negative imaginary direction (e.g., $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})_C$).
  - **IV:** $\langle\vec{x}\rangle \in (\frac{3\pi}{2}, 2\pi) \implies (\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi = (0, \frac{3\pi}{2})_\phi$: all points in this arc of T have a positive real direction and a negative imaginary direction (e.g., $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})_C$).
- **Axis Boundary Phase Pairs**
  - **"East":** $\langle\vec{x}\rangle = 0 \implies (\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi = (0, 0)_\phi$: the single point $(1, 0)_C$ on this position of T has a positive real direction and a zero imaginary direction.
  - **"North":** $\langle\vec{x}\rangle = \frac{\pi}{2} \implies (\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi = (0, \frac{\pi}{2})_\phi$: the single point $(0, 1)_C$ on this position of T has a zero real direction and a positive imaginary direction.
  - **"West":** $\langle\vec{x}\rangle = \pi \implies (\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi = (\pi, 0)_\phi$: the single point $(-1, 0)_C$ on this position of T has a negative real direction and a zero imaginary direction.
  - **"South":** $\langle\vec{x}\rangle = \frac{3\pi}{2} \implies (\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi = (0, \frac{3\pi}{2})_\phi$: the single point $(0, -1)_C$ on this position of T has a zero real direction and a negative imaginary direction.

Thus, we can take any point on T to obtain a fixed phase pair; this structured orientation on T gives it an intrinsic directional structure with phase pairs encoding real and imaginary directions (positive, negative, or zero) across all $\langle\vec{x}\rangle \in [0, 2\pi)$.

Having established T's structured orientation, we now demonstrate how this orientation distinguishes the approach from X_- and X_+. Since T separates X_- and X_+, we have $\partial X_- = T = \partial\overline{X}_+$, where ∂X_- is the boundary of the inner zone X_- and $\partial\overline{X}_+$ is the boundary of the closure of the outer zone X_+ with $\overline{X}_+ = \{\vec{x} \in X : ||\vec{x}|| \geq 1\}$. Now $\forall \vec{x} \in T$, the assigned phase pair $(\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi$ identifies a consistent directional label based on $\langle\vec{x}\rangle$. Specifically, the approach direction is distinguished by combining phase pairs with $||\vec{x}||$'s limiting behavior as it approaches the unit radius of T, which can be from below in X_- as $||\vec{x}|| \to 1^-$ or from above in X_+ as $||\vec{x}|| \to 1^+$. So we consider the following two cases:

- **Case 1: Inward Orientation from X_-.** As the boundary of the inner zone X_-, T is approached radially by vectors in X_- along rays defined by $\langle\vec{x}\rangle$. So take $\vec{x} \in X_-$. Then $||\vec{x}|| < 1$. Now fix $\langle\vec{x}\rangle \in (0, \frac{\pi}{2})$ as an example, with the structured orientation of Equations (9–10) ensuring that the system's phase pair assignment remains constant across all $\langle\vec{x}\rangle \in [0, 2\pi)$ as $\vec{x} \to T$ from within X_-. Then $\vec{x} = (||\vec{x}|| \cos\langle\vec{x}\rangle, ||\vec{x}|| \sin\langle\vec{x}\rangle)_C = (x_{\mathbb{R}}, x_{\mathbb{I}})_C$ where $x_{\mathbb{R}} > 0$ and $x_{\mathbb{I}} > 0$. So the system assigns the phase pair $(\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi = (0, \frac{\pi}{2})_\phi$. As the inward approach from within X_- is identified by $||\vec{x}|| \to 1^-$, we have $\vec{x} \to (\cos\langle\vec{x}\rangle, \sin\langle\vec{x}\rangle)_C \in T$, where the system's continuously assigned phase pair $(\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi = (0, \frac{\pi}{2})_\phi$ remains fixed, serving as a consistent reference for the inward radial approach from X_- to T. This encodes a *contractive orientation* as the increasing magnitude $||\vec{x}|| \to 1^-$ scales toward T's unit radius from below. This is a structured orientation with a consistent directional encoding.

- **Case 2: Outward Orientation to X_+.** As the boundary of the outer zone X_+, T is approached radially by vectors in X_+ along rays defined by $\langle\vec{x}\rangle$. So take $\vec{x} \in X_+$. Then $||\vec{x}|| > 1$. Now (again) fix $\langle\vec{x}\rangle \in (0, \frac{\pi}{2})$ as an example, with the structured orientation of Equations (9–10) ensuring that the system's phase pair assignment remains constant across all $\langle\vec{x}\rangle \in [0, 2\pi)$ as $\vec{x} \to T$ from X_+. Then $\vec{x} = (||\vec{x}|| \cos\langle\vec{x}\rangle, ||\vec{x}|| \sin\langle\vec{x}\rangle)_C = (x_{\mathbb{R}}, x_{\mathbb{I}})_C$ where $x_{\mathbb{R}} > 0$ and $x_{\mathbb{I}} > 0$. So the system assigns the phase pair $(\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi = (0, \frac{\pi}{2})_\phi$. As the outward approach from X_+ is identified by $||\vec{x}|| \to 1^+$, we have $\vec{x} \to (\cos\langle\vec{x}\rangle, \sin\langle\vec{x}\rangle)_C \in T$, where the system's continuously assigned phase pair $(\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi = (0, \frac{\pi}{2})_\phi$ remains fixed, serving as a consistent reference for the outward radial approach from X_+ to T. This encodes an *expansive orientation* as the decreasing magnitude $||\vec{x}|| \to 1^+$ scales toward T's unit radius from above. This is a structured orientation with a consistent directional encoding.

Therefore, $\forall \vec{x} \in T$, the assigned phase pair $(\langle\vec{x}_{\mathbb{R}}\rangle, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi$ dualizes the approach: inward to X_- and outward to X_+. The structured orientation on T distinguishes the approach from X_- and X_+ by maintaining consistent phase pairs that encode distinct directional behaviors (contractive from X_-, expansive from X_+) across all $\langle\vec{x}\rangle \in [0, 2\pi)$.

Thus, T is topologically dual to X_- and X_+, with its structured orientation ensuring consistent directional separation across X. □

6 Conclusion

This Tri-Quarter Theorem mathematical framework's strength resides in its unification of complex, Cartesian, and polar coordinates with a topological twist—duality and orientation across a boundary. This makes it versatile for:

- Problems involving separation of domains (inner vs. outer).
- Systems with circular symmetry or boundaries.
- Analyses requiring consistent directional or phase information in the complex plane.

To animate the Tri-Quarter Theorem and give a visual depiction of the additional data layer it serves, we created a software tool with open source code that is available online at [5]. It is a Python script that should execute on any machine equipped with Python 3.8+. So feel free to check that out. See Figure 1 for a screenshot of the animation.

My hypothesis is that this mathematical framework may be further developed and applied to help decipher great beasts like quark confinement [6] and black holes [7, 8, 9]. It could inspire or enhance technologies that rely on complex numbers, boundary behaviors, or spatial partitioning in engineering, computing, and applied sciences. Can it be applied to improve things like quantum computing hardware, cryptography, machine learning, digital signal processors, medical imaging devices, or wireless communication networks? Perhaps someday we will find out.

By formally establishing the topological duality of the unit circle with a structured orientation, the Tri-Quarter Theorem offers a novel perspective that could inspire new methodologies. For example, in quantum computing, where complex domain behaviors underpin qubit operations, this framework might enhance the modeling of phase relationships across boundaries, potentially improving gate design or error correction. Indeed, more work is needed to further develop and apply this framework via the methods of science and mathematics.

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Author's Note

I originally came up with the ideas for the generalized complex-Cartesian-polar coordinate system, structured orientation, and consistent directional separation back in 2012 when I was working on quark confinement [6]. In that work, the core ideas are basically there but I never had the opportunity and realization to fully test, refine, and finalize them into a single consolidated theorem and proof with legit terminology until now.

Then, just recently, upon revisiting these ideas from 2012 and focusing to further develop them, I began to experiment with Grok v3 as an assistive tool. I wrote latex code with text and equations to formulate my ideas into written thought experiments, and then I asked Grok to pretend to be a mathematician and scientist, and then I fed my latex code directly into Grok and asked it to help validate or refute my various thought experiments. I also asked it to help me look for holes or gaps in my work. It found things that I missed and I found things that it missed. Fix it. Rinse and repeat. Through this process, I was able to test, refine, and advance my ideas at a faster velocity.

Thus far, I've discovered that the key to using Grok (or any artificial intelligence) to effectively help as an assistant—with technical disciplines like mathematics, engineering, and science—resides in one's technical ability to creatively question, prompt, and apply the methods of the mathematics and science. I think of Grok as a friendly, powerful piece of heavy machinery, where I'm the operator and it serves as a force multiplier that helps get the job done.

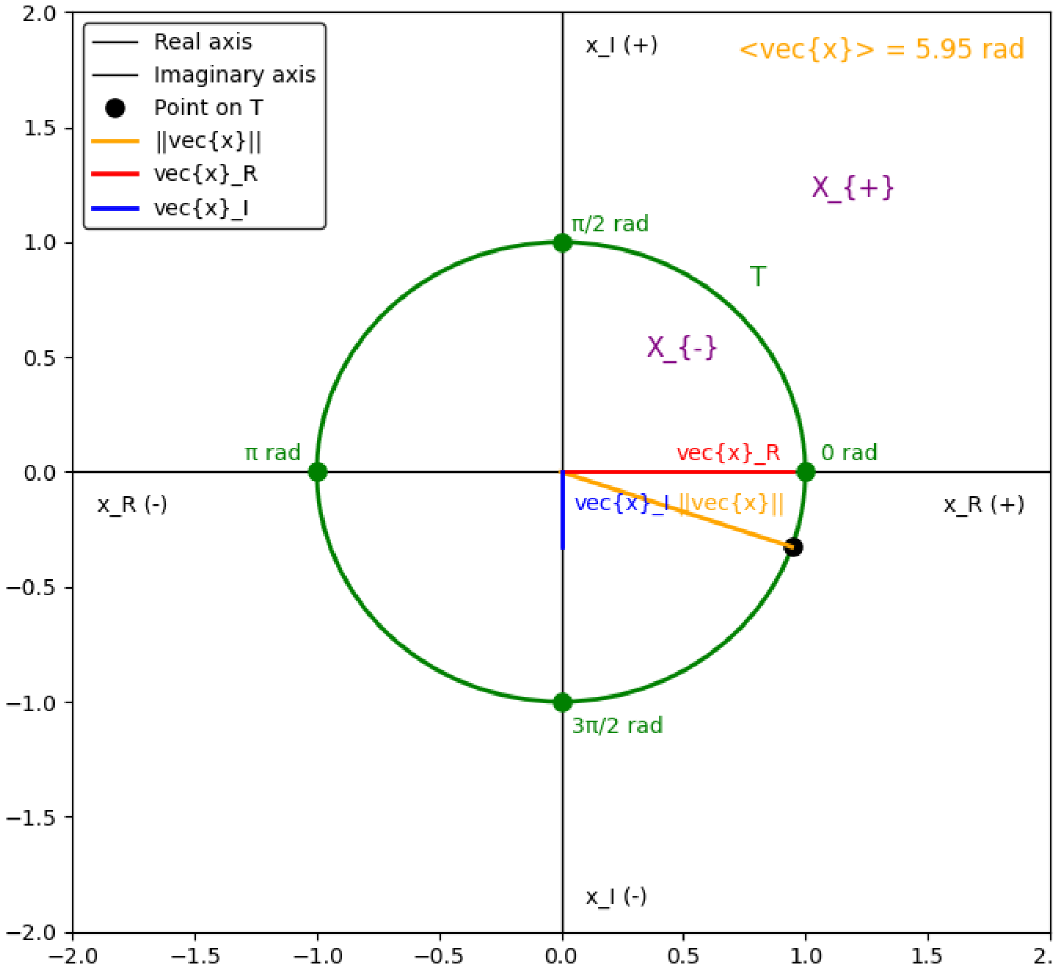

FIGURE 1. A screenshot from our software animation tool [5] illustrating the three topological zones and the structured orientation on the unit circle.

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