The Tri-Quarter Framework: Unifying Complex Coordinates with Topological and Reflective Duality across Circles of Any Radius

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Abstract

In this paper, we introduce the *Tri-Quarter Topological Duality Theorem*, the foundation of a novel mathematical framework that unifies complex, Cartesian, and polar coordinate systems on the complex plane $\mathbb{C}$ while equipping the circle T_r of radius $r > 0$ with a new topological property. Our framework integrates a generalized coordinate system—where real and imaginary components are assigned unique phase pairs—with a structured orientation that elevates T_r to an active separator with intrinsic directional properties. We prove that T_r, as the boundary zone, exhibits topological duality with the inner zone $X_{-,r}$ ($||\vec{x}|| < r$) and outer zone $X_{+,r}$ ($||\vec{x}|| > r$), ensuring consistent separation between inner and outer radial directions across T_r with a phase pair map encoding additional information. We also introduce the *Escher Tri-Quarter Reflective Duality Theorem*, proving reflective duality across T_r via a circle inversion map that preserves phase pairs while swapping $X_{-,r}$ and $X_{+,r}$. Moreover, these phase pair assignments provide a combinatorial classification of directional structures in $\mathbb{C}$, enhancing the topological analysis. This approach offers insights into topological separation, orientation, and reflection, facilitating analysis of systems with circular symmetry, with potential applications in fields such as black hole physics, signal processing, and other areas reliant on complex domain partitioning. A case study on quadrant-based transformations demonstrates streamlined directional mappings, geometric elegance, unified classification, and computational efficiency in $\mathbb{C}$. A software tool visualizes some of these concepts, with future work aimed at exploring practical implementations.

1 Introduction

The complex plane $\mathbb{C}$, viewed as a complex 1D Riemann surface, serves as a foundational structure in mathematics and its applications, bridging algebraic, geometric, and analytic perspectives [1, 2]. In some contexts, traditional coordinate systems, such as Cartesian and polar, while powerful, treat the circle of radius $r > 0$ as a mere boundary without fully exploiting its topological role. In this work, we address this gap by introducing the Tri-Quarter Topological Duality Theorem, which establishes a generalized complex-Cartesian-polar coordinate system while applying a trichotomy—a three zone partition of $\mathbb{C}$ based on the circle of radius $r > 0$—to the four quadrants of $\mathbb{C}$ to achieve a new topological property.

This paper includes two main results. Our first main result demonstrates that the circle T_r of radius $r > 0$ is topologically dual to both the inner zone $X_{-,r}$ ($||\vec{x}|| < r$) and outer zone $X_{+,r}$ ($||\vec{x}|| > r$), with a structured orientation that ensures consistent directional separation across $\mathbb{C}$—more specifically, between the inner radial direction (approaching T_r from $X_{-,r}$) and the outer radial direction (approaching T_r from $X_{+,r}$), which is combined with a novel phase pair assignment based on well-defined quadrant and axis boundary rules. This framework unifies three coordinate representations and elevates the circle's role from a passive boundary to an active separator with intrinsic properties. By assigning discrete phase pairs to the real and imaginary components, our framework establishes a combinatorial encoding of geometric directions, unifying algebraic and geometric perspectives. Our second main result leverages a circle inversion map [3] to demonstrate reflective duality across T_r to preserve the assigned phase pairs while swapping $X_{-,r}$ and $X_{+,r}$—the

Escher Tri-Quarter Reflective Duality Theorem—inspired by the artistic symmetry of M.C. Escher [4].

The significance lies in its potential to refine analyses, computations, modeling, and design in fields that rely on complex numbers and complex domain spatial partitioning, such as offering a nuanced approach to handling precise boundary conditions, directional properties, and reflective properties across $\mathbb{C}$. For instance, potential applications may include deciphering the quasi-normal modes of black holes and enhancing the computational efficiency of signal processors. To demonstrate its practical utility, a case study in Section 7 applies the framework to quadrant-based transformations, revealing its capacity to unify classification and leverage geometric structure for efficient and consistent directional analysis across $\mathbb{C}$.

In Section 2, we define the coordinate system, followed by the structured orientation in Section 3, the topological zones in Section 4, the topological duality proof in Section 5, the reflective duality proof in Section 6, and the aforementioned case study in Section 7. In Section 8, we conclude with a discussion of its strengths, software animation tool, and potential future applications.

2 Generalized Complex-Cartesian-Polar Coordinate System

Let $X = \mathbb{C}$ be the complex plane and complex 1D Riemann surface [1, 2] with the standard topology. For a complex number $x = x_{\mathbb{R}} + ix_{\mathbb{I}} \in X$, we define the complex vector $\vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}} \in X$ as a position-vector on X with Cartesian coordinate form

$$(x_{\mathbb{R}}, x_{\mathbb{I}})_{Cartesian} = (x_{\mathbb{R}}, x_{\mathbb{I}})_C, \tag{1}$$

where $_C$ denotes Cartesian form. Vectors are denoted with arrows (e.g., $\vec{x}$) throughout this paper for consistency. Here $\vec{x}_{\mathbb{R}}$ and $\vec{x}_{\mathbb{I}}$ are axis-aligned orthogonal components that we treat interchangeably as both vectors and points in specific subsets of X. Specifically, we define

$$\begin{aligned} \vec{x}_{\mathbb{R}} &\in \mathbb{R} \times \{0\} = \{(x_{\mathbb{R}}, 0)_C \mid x_{\mathbb{R}} \in \mathbb{R}\} \\ \vec{x}_{\mathbb{I}} &\in \{0\} \times \mathbb{I} = \{(0, x_{\mathbb{I}})_C \mid x_{\mathbb{I}} \in \mathbb{R}\}, \end{aligned} \tag{2}$$

where $\mathbb{I} = i\mathbb{R}$ denotes the imaginary axis. Therefore, $\vec{x}_{\mathbb{R}} = (x_{\mathbb{R}}, 0)_C$ is a "real Euclidean 2-vector" aligned with the real axis, and $\vec{x}_{\mathbb{I}} = (0, x_{\mathbb{I}})_C$ is an "imaginary Euclidean 2-vector" aligned with the imaginary axis. Throughout this framework, this dual interpretation allows $\vec{x}_{\mathbb{R}}$ and $\vec{x}_{\mathbb{I}}$ to serve as both individual vectors and representative points within their respective sets, enabling their use interchangeably with the complex number x and its vector representation $\vec{x}$, which they form as a linear combination (emphasizing that $\vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}}$ is the direct sum of these vectors). As we will demonstrate in this paper, by subtly defining the components $\vec{x}_{\mathbb{R}}$ and $\vec{x}_{\mathbb{I}}$ to be orthogonal axis-aligned vectors—instead of just scalars—we achieve some useful properties such as orthogonality and direction.

Throughout this paper, we consider $X \setminus \{(0,0)_C\}$ to exclude the origin where phase assignments are undefined. This focuses our framework on regions that are critical to the main results.

Next, $\vec{x} \in X \setminus \{(0,0)_C\}$ is a Euclidean 2-vector with norm $||\vec{x}|| \in [0, \infty)$ and phase $\arg(\vec{x}) = \langle\vec{x}\rangle \in [0, 2\pi)$, which are respectively analogous to *magnitude* and *direction* in conventional notation with polar coordinate form

$$(||\vec{x}||, \langle\vec{x}\rangle)_{Polar} = (||\vec{x}||, \langle\vec{x}\rangle)_P, \tag{3}$$

where $_P$ denotes polar form, to give

$$\begin{aligned} x_{\mathbb{R}} &= ||\vec{x}|| \cos\langle\vec{x}\rangle \\ x_{\mathbb{I}} &= ||\vec{x}|| \sin\langle\vec{x}\rangle \end{aligned} \tag{4}$$

with Pythagorean form

$$||\vec{x}||^2 = x_{\mathbb{R}}^2 + x_{\mathbb{I}}^2. \tag{5}$$

(Note: From this point forward, for conciseness, for the phase angle notation of a given $\vec{x} \in X \setminus \{(0,0)_C\}$, we will just say $\langle\vec{x}\rangle$ instead of $\arg\vec{x}$ also.) Thus, similarly to $\vec{x}$, for the axis-aligned orthogonal vectors $\vec{x}_{\mathbb{R}}$ and $\vec{x}_{\mathbb{I}}$, we also obtain the polar coordinate form

$$\begin{gathered}(||\vec{x}_{\mathbb{R}}||,\langle\vec{x}_{\mathbb{R}}\rangle)_P \\ (||\vec{x}_{\mathbb{I}}||,\langle\vec{x}_{\mathbb{I}}\rangle)_P,\end{gathered} \tag{6}$$

such that $||\vec{x}_{\mathbb{R}}||, ||\vec{x}_{\mathbb{I}}|| \in [0,\infty)$, $\langle\vec{x}_{\mathbb{R}}\rangle \in \{0,\pi\}$, and $\langle\vec{x}_{\mathbb{I}}\rangle \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$.

Henceforth, the Pythagorean form of Equation (5) becomes

$$||\vec{x}||^2 = ||\vec{x}_{\mathbb{R}}||^2 + ||\vec{x}_{\mathbb{I}}||^2 = x_{\mathbb{R}}^2 + x_{\mathbb{I}}^2. \tag{7}$$

Thus, for all $\vec{x} \in X$ we define the 2D generalized complex-Cartesian-polar coordinate system as

$$(\vec{x}) = (\vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}}) = (x_{\mathbb{R}}, x_{\mathbb{I}})_C = (||\vec{x}||\cos\langle\vec{x}\rangle, ||\vec{x}||\sin\langle\vec{x}\rangle)_C = (||\vec{x}||, \langle\vec{x}\rangle)_P \tag{8}$$

unifying complex, Cartesian, and polar representations. This unification is further enriched by the phase pair assignments and structured orientation defined in Section 3, which provide directional properties that are critical to our topological analysis.

3 Structured Orientation

Building on the foundation of the unified coordinate representation in Equation (8), for any $\vec{x} \in X \setminus \{(0,0)_C\}$, we now assign phase pairs to establish a structured orientation across $X \setminus \{(0,0)_C\}$. We define the *quadrant phase pair assignments* as

$$\begin{aligned}
\textbf{I:}\ \ &\langle\vec{x}\rangle \in \left(0,\tfrac{\pi}{2}\right) \iff (x_{\mathbb{R}} > 0) \wedge (x_{\mathbb{I}} > 0) \iff (\langle\vec{x}_{\mathbb{R}}\rangle = 0) \wedge \left(\langle\vec{x}_{\mathbb{I}}\rangle = \tfrac{\pi}{2}\right) \iff (0,\tfrac{\pi}{2})_\phi \\
\textbf{II:}\ \ &\langle\vec{x}\rangle \in \left(\tfrac{\pi}{2},\pi\right) \iff (x_{\mathbb{R}} < 0) \wedge (x_{\mathbb{I}} > 0) \iff (\langle\vec{x}_{\mathbb{R}}\rangle = \pi) \wedge \left(\langle\vec{x}_{\mathbb{I}}\rangle = \tfrac{\pi}{2}\right) \iff (\pi,\tfrac{\pi}{2})_\phi \\
\textbf{III:}\ \ &\langle\vec{x}\rangle \in \left(\pi,\tfrac{3\pi}{2}\right) \iff (x_{\mathbb{R}} < 0) \wedge (x_{\mathbb{I}} < 0) \iff (\langle\vec{x}_{\mathbb{R}}\rangle = \pi) \wedge \left(\langle\vec{x}_{\mathbb{I}}\rangle = \tfrac{3\pi}{2}\right) \iff (\pi,\tfrac{3\pi}{2})_\phi \\
\textbf{IV:}\ \ &\langle\vec{x}\rangle \in \left(\tfrac{3\pi}{2},2\pi\right) \iff (x_{\mathbb{R}} > 0) \wedge (x_{\mathbb{I}} < 0) \iff (\langle\vec{x}_{\mathbb{R}}\rangle = 0) \wedge \left(\langle\vec{x}_{\mathbb{I}}\rangle = \tfrac{3\pi}{2}\right) \iff (0,\tfrac{3\pi}{2})_\phi,
\end{aligned} \tag{9}$$

where $(\langle\vec{x}_{\mathbb{R}}, \langle\vec{x}_{\mathbb{I}}\rangle)_\phi$ denotes the phase pair assigned by the coordinate system. Then to maintain continuity and ensure that no boundary remains undefined, we define the *axis boundary phase pair assignments* as

$$\begin{aligned}
\textbf{“East”:}\ \ &\langle\vec{x}\rangle = 0 \iff (x_{\mathbb{R}} > 0) \wedge (x_{\mathbb{I}} = 0) \iff (\langle\vec{x}_{\mathbb{R}}\rangle = 0) \wedge (\langle\vec{x}_{\mathbb{I}}\rangle = 0) \iff (0,0)_\phi \\
\textbf{“North”:}\ \ &\langle\vec{x}\rangle = \tfrac{\pi}{2} \iff (x_{\mathbb{R}} = 0) \wedge (x_{\mathbb{I}} > 0) \iff (\langle\vec{x}_{\mathbb{R}}\rangle = 0) \wedge \left(\langle\vec{x}_{\mathbb{I}}\rangle = \tfrac{\pi}{2}\right) \iff (0,\tfrac{\pi}{2})_\phi \\
\textbf{“West”:}\ \ &\langle\vec{x}\rangle = \pi \iff (x_{\mathbb{R}} < 0) \wedge (x_{\mathbb{I}} = 0) \iff (\langle\vec{x}_{\mathbb{R}}\rangle = \pi) \wedge (\langle\vec{x}_{\mathbb{I}}\rangle = 0) \iff (\pi,0)_\phi \\
\textbf{“South”:}\ \ &\langle\vec{x}\rangle = \tfrac{3\pi}{2} \iff (x_{\mathbb{R}} = 0) \wedge (x_{\mathbb{I}} < 0) \iff (\langle\vec{x}_{\mathbb{R}}\rangle = 0) \wedge \left(\langle\vec{x}_{\mathbb{I}}\rangle = \tfrac{3\pi}{2}\right) \iff (0,\tfrac{3\pi}{2})_\phi,
\end{aligned} \tag{10}$$

where each of these assigned phase pairs uniquely corresponds to each axis direction, aligning with $\langle\vec{x}\rangle$ and maintaining a structured orientation. To establish a consistent rule for handling the edge cases when $x_{\mathbb{R}} = 0$ or $x_{\mathbb{I}} = 0$, we can simply set $\langle\vec{x}_{\mathbb{R}}\rangle = 0$ or $\langle\vec{x}_{\mathbb{I}}\rangle = 0$, respectively. Thus, from Equations (9–10) we derive the following *phase assignment rules*:

- For the real component $\vec{x}_{\mathbb{R}}$:
  - If $x_{\mathbb{R}} > 0$, then set $\langle\vec{x}_{\mathbb{R}}\rangle = 0$.
  - If $x_{\mathbb{R}} = 0$, then set $\langle\vec{x}_{\mathbb{R}}\rangle = 0$.
  - If $x_{\mathbb{R}} < 0$, then set $\langle\vec{x}_{\mathbb{R}}\rangle = \pi$.
- For the imaginary component $\vec{x}_{\mathbb{I}}$:
  - If $x_{\mathbb{I}} > 0$, then set $\langle\vec{x}_{\mathbb{I}}\rangle = \frac{\pi}{2}$.
  - If $x_{\mathbb{I}} = 0$, then set $\langle\vec{x}_{\mathbb{I}}\rangle = 0$.
  - If $x_{\mathbb{I}} < 0$, then set $\langle\vec{x}_{\mathbb{I}}\rangle = \frac{3\pi}{2}$.

The phase assignment rules serve as "building blocks" for the phase pair assignments. They maintain consistency across all vectors—including the axis-bound vectors—and indicate the absence of direction in a zero-magnitude component.

In standard complex analysis, the phase (argument) of the zero vector is undefined because it has no direction. However, in this framework, we adopt the convention such that when a component is zero (e.g., $x_{\mathbb{R}} = 0$ or $x_{\mathbb{I}} = 0$), then its corresponding phase is set to 0 (e.g., $\langle \vec{x}_{\mathbb{R}} \rangle = 0$ or $\langle \vec{x}_{\mathbb{I}} \rangle = 0$). This choice simplifies the phase pair assignments, especially on the real and imaginary axes, and ensures that for all $\vec{x} \in X \setminus \{(0,0)_C\}$ there exists a well-defined phase pair $(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi$. This convention is consistently applied throughout the paper and does not affect the main results because the origin is excluded from our analysis.

These phase pair assignments establish a structured orientation that not only unifies the complex-Cartesian-polar coordinate representations but also upgrades $X \setminus \{(0,0)_C\}$ with consistent directional properties. Furthermore, in the context of algebraic combinatorics, these phase pair assignments offer a novel combinatorial classification of directional structures in $X \setminus \{(0,0)_C\}$. By mapping each vector $\vec{x} \in X \setminus \{(0,0)_C\}$ to a pair $\left(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathrm{I}} \rangle\right)_\phi$ based on the signs of its $x_{\mathbb{R}}$ and $x_{\mathbb{I}}$ components, we effectively partition $X \setminus \{(0,0)_C\}$ into a finite set of directional categories. This discrete encoding not only unifies geometric and algebraic perspectives but also invites further exploration of symmetry and transformation properties through the lens of combinatorial algebra. This system forms the foundation for the topological partitioning into three distinct zones in Section 4, where the circle's role as a separator is empowered by these orientations.

4 Topological Zones

For any $r > 0$ and $\vec{x} \in X \setminus \{(0,0)_C\}$, we use trichotomy of the norm $||\vec{x}||$ to partition X into three distinct topological zones:

(1) the inner zone $X_{-,r}$,

(2) the boundary zone T_r, and

(3) the outer zone $X_{+,r}$.

Thus, for $\vec{x} \in X \setminus \{(0,0)_C\}$ we know that precisely one of the following trichotomy conditions must be satisfied

$$\begin{aligned} ||\vec{x}|| < r &\iff \vec{x} \in X_{-,r} \\ ||\vec{x}|| = r &\iff \vec{x} \in T_r \\ ||\vec{x}|| > r &\iff \vec{x} \in X_{+,r}, \end{aligned} \tag{11}$$

where clearly $X_{-,r} \cap T_r = T_r \cap X_{+,r} = X_{-,r} \cap X_{+,r} = \emptyset$ and $X_{-,r} \cup T_r \cup X_{+,r} = X$, such that

$$\begin{aligned} X_{-,r} &= \{\vec{x} \in X : ||\vec{x}|| < r\} \\ T_r &= \{\vec{x} \in X : ||\vec{x}|| = r\} \\ X_{+,r} &= \{\vec{x} \in X : ||\vec{x}|| > r\} \end{aligned} \tag{12}$$

because the boundary zone T_r is a circle of radius r and 1-manifold, which serves as the separator between $X_{-,r}$ and $X_{+,r}$. So the four quadrants of X are partitioned into three zones via norm $||\vec{x}||$ trichotomy—a "tri-quarter" if you will.

Definition 4.1 (Tri-Quarter Topological Duality). Let $X = \mathbb{C}$ be equipped with the standard topology. For any $r > 0$, define $X_{-,r} = \{\vec{x} \in X : ||\vec{x}|| < r\}$, $T_r = \{\vec{x} \in X : ||\vec{x}|| = r\}$, and $X_{+,r} = \{\vec{x} \in X : ||\vec{x}|| > r\}$, where $X_{-,r} \cup T_r \cup X_{+,r} = X$ and $X_{-,r} \cap X_{+,r} = \emptyset$. The circular boundary zone T_r with radius r is *topologically dual* to the inner zone $X_{-,r}$ and outer zone $X_{+,r}$ if:

(1) $T_r = \partial X_{-,r}$ and $T_r = \partial \overline{X}_{+,r}$, where $\overline{X}_{+,r}$ denotes the closure of $X_{+,r}$ in X,

(2) T_r, as a 1-manifold, has a structured orientation defined by a map $\phi : T_r \to \{0, \pi\} \times \{0, \frac{\pi}{2}, \frac{3\pi}{2}\}$, assigning phase pairs $(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi$ based on quadrant and axis boundary phase assignments per Equations (9–10), and

(3) the structured orientation ϕ is constant along each ray that emanates from the origin $(0,0)_C$ in $X \setminus \{(0,0)_C\}$, enabling the norm $||\vec{x}||$ to consistently separate the inner radial direction (approaching T_r from $X_{-,r}$ as $||\vec{x}|| \to r^-$) from the outer radial direction (approaching T_r from $X_{+,r}$ as $||\vec{x}|| \to r^+$).

For the upcoming theorem and proof in Section 5, this duality captures T_r's role as a separator with an intrinsic directional structure to unify the topological and geometric properties of the partition.

5 Tri-Quarter Topological Duality

In this section, we prove the first of the two main results of this paper. The topological duality of T_r with $X_{-,r}$ and $X_{+,r}$ not only formalizes the T_r's role as a separator but also integrates the structured orientation to provide a unified framework for analyzing radial directions in the complex plane. This duality lays the groundwork for exploring additional symmetries, such as the reflective duality addressed in the subsequent section.

Theorem 5.1 (Tri-Quarter Topological Duality). Let $X = \mathbb{C}$ be equipped with the standard topology and the generalized complex-Cartesian-polar coordinate system of Equation (8), where $\vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}}$ for $\vec{x} \in X$, with $\vec{x}_{\mathbb{R}} = (x_{\mathbb{R}}, 0)_C \in \mathbb{R} \times \{0\}$, $\vec{x}_{\mathbb{I}} = (0, x_{\mathbb{I}})_C \in \{0\} \times \mathbb{I}$, and phase pairs $(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi$ assigned as:

- if $x_{\mathbb{R}} > 0$ then $\langle \vec{x}_{\mathbb{R}} \rangle = 0$, if $x_{\mathbb{R}} = 0$ then $\langle \vec{x}_{\mathbb{R}} \rangle = 0$, if $x_{\mathbb{R}} < 0$ then $\langle \vec{x}_{\mathbb{R}} \rangle = \pi$,
- if $x_{\mathbb{I}} > 0$ then $\langle \vec{x}_{\mathbb{I}} \rangle = \frac{\pi}{2}$, if $x_{\mathbb{I}} = 0$ then $\langle \vec{x}_{\mathbb{I}} \rangle = 0$, if $x_{\mathbb{I}} < 0$ then $\langle \vec{x}_{\mathbb{I}} \rangle = \frac{3\pi}{2}$,

per the phase assignment rules following Equations (9–10). For any $r > 0$, define $X_{-,r} = \{\vec{x} \in X : ||\vec{x}|| < r\}$, $T_r = \{\vec{x} \in X : ||\vec{x}|| = r\}$, $X_{+,r} = \{\vec{x} \in X : ||\vec{x}|| > r\}$. Then T_r is topologically dual to $X_{-,r}$ and $X_{+,r}$, where the structured orientation ensures consistent separation between the inner radial direction (approaching T_r from $X_{-,r}$) and the outer radial direction (approaching T_r from $X_{+,r}$) across $X \setminus \{(0,0)_C\}$ with respect to T_r.

Proof. Let $X = \mathbb{C}$ with the standard topology, and for any $r > 0$, define $X_{-,r}$, T_r, $X_{+,r}$ as above. We proceed in steps to verify each condition of Definition 4.1:

- **Step 1: Verify Partition and Boundaries on X**
  From the Jordan Curve Theorem [5, 6] we obtain the boundary zone $T_r = \{\vec{x} \in X : ||\vec{x}|| = r\}$ (a simple closed curve) that separates X into two disjoint open sets: the inner zone $X_{-,r}$ and the outer zone $X_{+,r}$, where $X \setminus T_r = X_{-,r} \cup X_{+,r}$ and $X_{-,r} \cap X_{+,r} = \emptyset$, such that:
  - $X_{-,r}$ is open, so $\partial X_{-,r} = \{\vec{x} \in X : ||\vec{x}|| = r\} = T_r$ (every neighborhood of $\vec{x} \in T_r$ intersects $X_{-,r}$ and $X_{+,r}$), and
  - $X_{+,r}$ is open, with $\overline{X}_{+,r} = \{\vec{x} \in X : ||\vec{x}|| \geq r\}$, so $\partial \overline{X}_{+,r} = T_r$ (similar reasoning).

This satisfies condition (1) of topological duality. ☑

- **Step 2: Verify Structured Orientation on T_r**
We define $\phi : T_r \to \{0, \pi\} \times \{0, \frac{\pi}{2}, \frac{3\pi}{2}\}$ by $\phi(\vec{x}) = (\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi$, where for $\vec{x} = (x_{\mathbb{R}}, x_{\mathbb{I}})_C \in T_r$ with $||\vec{x}|| = r$:

  - if $x_{\mathbb{R}} > 0$ then $\langle \vec{x}_{\mathbb{R}} \rangle = 0$, if $x_{\mathbb{R}} = 0$ then $\langle \vec{x}_{\mathbb{R}} \rangle = 0$, if $x_{\mathbb{R}} < 0$ then $\langle \vec{x}_{\mathbb{R}} \rangle = \pi$,
  - if $x_{\mathbb{I}} > 0$ then $\langle \vec{x}_{\mathbb{I}} \rangle = \frac{\pi}{2}$, if $x_{\mathbb{I}} = 0$ then $\langle \vec{x}_{\mathbb{I}} \rangle = 0$, if $x_{\mathbb{I}} < 0$ then $\langle \vec{x}_{\mathbb{I}} \rangle = \frac{3\pi}{2}$.

  Since T_r is a 1-manifold, we parametrize it as $\vec{x} = (||\vec{x}|| \cos\langle \vec{x} \rangle, ||\vec{x}|| \sin\langle \vec{x} \rangle)_C = (r \cos\langle \vec{x} \rangle, r \sin\langle \vec{x} \rangle)_C$, with $\langle \vec{x} \rangle \in [0, 2\pi)$. Then we check the structured orientation as $\langle \vec{x} \rangle \to 2\pi^-$:

  - Quadrant I ($\langle \vec{x} \rangle \in (0, \frac{\pi}{2})$): $x_{\mathbb{R}} > 0$, $x_{\mathbb{I}} > 0$, so $\phi = (0, \frac{\pi}{2})_\phi$.
  - "North" boundary at $\langle \vec{x} \rangle = \frac{\pi}{2}$: $\vec{x} = (0, r)_C$, $x_{\mathbb{R}} = 0$, $x_{\mathbb{I}} > 0$, so $\phi = (0, \frac{\pi}{2})_\phi$, consistent with Quadrant I.
  - Quadrant II ($\langle \vec{x} \rangle \in (\frac{\pi}{2}, \pi)$): $x_{\mathbb{R}} < 0$, $x_{\mathbb{I}} > 0$, so $\phi = (\pi, \frac{\pi}{2})_\phi$.
  - "West" boundary at $\langle \vec{x} \rangle = \pi$: $\vec{x} = (-r, 0)_C$, $x_{\mathbb{R}} < 0$, $x_{\mathbb{I}} = 0$, so $\phi = (\pi, 0)_\phi$.
  - Quadrant III ($\langle \vec{x} \rangle \in (\pi, \frac{3\pi}{2})$): $x_{\mathbb{R}} < 0$, $x_{\mathbb{I}} < 0$, so $\phi = (\pi, \frac{3\pi}{2})_\phi$.
  - "South" boundary at $\langle \vec{x} \rangle = \frac{3\pi}{2}$: $\vec{x} = (0, -r)_C$, $x_{\mathbb{R}} = 0$, $x_{\mathbb{I}} < 0$, so $\phi = (0, \frac{3\pi}{2})_\phi$, consistent with Quadrant IV.
  - Quadrant IV ($\langle \vec{x} \rangle \in (\frac{3\pi}{2}, 2\pi)$): $x_{\mathbb{R}} > 0$, $x_{\mathbb{I}} < 0$, so $\phi = (0, \frac{3\pi}{2})_\phi$.
  - "East" boundary at $\langle \vec{x} \rangle = 0$: $\vec{x} = (r, 0)_C$, $x_{\mathbb{R}} > 0$, $x_{\mathbb{I}} = 0$, so $\phi = (0, 0)_\phi$.

  Thus, ϕ is constant on each open arc of T_r and adjusts at axis boundaries (e.g., $\langle \vec{x} \rangle = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$). So it is piecewise continuous, which suffices for a structured orientation on a 1-manifold. This satisfies condition (2) of topological duality. ☑

- **Step 3: Verify Inner and Outer Radial Directional Separation across T_r**
For $\vec{x} \in T_r$, fix $\langle \vec{x} \rangle \in [0, 2\pi)$. We define two paths approaching T_r:

  - Inner radial path from $X_{-,r}$: $\vec{x} = (||\vec{x}|| \cos\langle \vec{x} \rangle, ||\vec{x}|| \sin\langle \vec{x} \rangle)_C$, $||\vec{x}|| \to r^-$, so $||\vec{x}|| < r$.
  - Outer radial path from $X_{+,r}$: $\vec{x} = (||\vec{x}|| \cos\langle \vec{x} \rangle, ||\vec{x}|| \sin\langle \vec{x} \rangle)_C$, $||\vec{x}|| \to r^+$, so $||\vec{x}|| > r$.

  Then we compute ϕ along both paths:

  - For $\langle \vec{x} \rangle \in (0, \frac{\pi}{2})$, we obtain $\vec{x} = (||\vec{x}|| \cos\langle \vec{x} \rangle, ||\vec{x}|| \sin\langle \vec{x} \rangle)_C$, $x_{\mathbb{R}} > 0$, $x_{\mathbb{I}} > 0$, $\phi = (0, \frac{\pi}{2})_\phi$; as $||\vec{x}|| \to r^-$, $\vec{x} \to (r \cos\langle \vec{x} \rangle, r \sin\langle \vec{x} \rangle)_C \in T_r$, where $\phi = (0, \frac{\pi}{2})_\phi$ remains constant.
  - Similarly, as $||\vec{x}|| \to r^+$, $\phi = (0, \frac{\pi}{2})_\phi$.

  The phase pair $(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi$ is constant, where its direction is distinguished by $||\vec{x}||$:

  - a "contractive" ($< r$) inner radial direction from $X_{-,r}$, and
  - an "expansive" ($> r$) outer radial direction from $X_{+,r}$.

  This holds for all $\langle \vec{x} \rangle \in [0, 2\pi)$ as it adjusts at axis boundaries (e.g., if $\langle \vec{x} \rangle = 0$ then $\phi = (0, 0)_\phi$). This structured orientation ensures consistent separation between the inner radial direction (approaching T_r from $X_{-,r}$ as $||\vec{x}|| \to r^-$) and the outer radial direction (approaching T_r from $X_{+,r}$ as $||\vec{x}|| \to r^+$) across $X \setminus \{(0, 0)_C\}$ with respect to T_r. This satisfies condition (3) of topological duality. ☑

Thus, T_r is topologically dual to $X_{-,r}$ and $X_{+,r}$. □

Now that we've established the topological duality of the circular boundary zone T_r with respect to the inner and outer zones, we now turn to a complementary symmetry that leverages this structured orientation to achieve a reflective duality across T_r for swapping between the inner and outer zones.

6 Escher Tri-Quarter Reflective Duality

In this section, we prove the second of the two main results of this paper. Building upon the structured orientation and topological duality established with the Tri-Quarter Topological Duality Theorem in the previous section, we now establish a reflective duality across the circle T_r of radius $r > 0$ in the complex plane $X = \mathbb{C}$, inspired by M.C. Escher's artistic symmetry [4]. This duality is achieved through a circle inversion map [3], which preserves the structured orientation defined by the phase pair map ϕ while swapping the inner zone $X_{-,r}$ and outer zone $X_{+,r}$.

Definition 6.1 (Circle Inversion Map). For any $r > 0$, the *circle inversion map* ι_r : $X \setminus \{(0,0)_C\} \to X \setminus \{(0,0)_C\}$ is defined as

$$\iota_r(\vec{x}) = \frac{r^2\vec{x}}{||\vec{x}||^2},$$

where $\vec{x} \in X \setminus \{(0,0)_C\}$ and $||\vec{x}||$ is the Euclidean norm [3].

Definition 6.2 (Escher Tri-Quarter Reflective Duality). Let $X = \mathbb{C}$ be equipped with the standard topology. For any $r > 0$, define $X_{-,r} = \{\vec{x} \in X : ||\vec{x}|| < r\}$, $T_r = \{\vec{x} \in X : ||\vec{x}|| = r\}$, and $X_{+,r} = \{\vec{x} \in X : ||\vec{x}|| > r\}$, where $X_{-,r} \cup T_r \cup X_{+,r} = X$ and $X_{-,r} \cap X_{+,r} = \emptyset$. Let the circular boundary zone T_r with radius r be topologically dual to both the inner zone $X_{-,r}$ and outer zone $X_{+,r}$ as in Theorem 5.1. Then T_r exhibits *reflective duality* between $X_{-,r}$ and $X_{+,r}$ if there exists a map ι_r such that:

(1) $\iota_r(\vec{x}) = \vec{x}$ for all $\vec{x} \in T_r$,

(2) $\iota_r(X_{-,r}) = X_{+,r}$ and $\iota_r(X_{+,r}) = X_{-,r}$,

(3) $\phi(\iota_r(\vec{x})) = \phi(\vec{x})$ for all $\vec{x} \in X \setminus \{(0,0)_C\}$, and

(4) ι_r is an involution, i.e., $\iota_r^{-1} = \iota_r$.

Theorem 6.3 (Escher Tri-Quarter Reflective Duality). Let $X = \mathbb{C}$ be equipped with the standard topology. For any $r > 0$, define $X_{-,r} = \{\vec{x} \in X : ||\vec{x}|| < r\}$, $T_r = \{\vec{x} \in X : ||\vec{x}|| = r\}$, and $X_{+,r} = \{\vec{x} \in X : ||\vec{x}|| > r\}$, where $X_{-,r} \cup T_r \cup X_{+,r} = X$ and $X_{-,r} \cap X_{+,r} = \emptyset$. Let T_r be topologically dual to $X_{-,r}$ and $X_{+,r}$ as in Theorem 5.1. Then the circle inversion map ι_r establishes reflective duality across T_r between $X_{-,r}$ and $X_{+,r}$ that satisfies the conditions:

(1) $\iota_r(\vec{x}) = \vec{x}$ for all $\vec{x} \in T_r$,

(2) $\iota_r(X_{-,r}) = X_{+,r}$ and $\iota_r(X_{+,r}) = X_{-,r}$,

(3) $\phi(\iota_r(\vec{x})) = \phi(\vec{x})$ for all $\vec{x} \in X \setminus \{(0,0)_C\}$, and

(4) $\iota_r^{-1} = \iota_r$.

Proof. Let $X = \mathbb{C}$ with the standard topology, and for any $r > 0$, define $X_{-,r}$, T_r, $X_{+,r}$ to satisfy Theorem 5.1 as above. Let $\iota_r : X \setminus \{(0,0)_C\} \to X \setminus \{(0,0)_C\}$ be given by Definition 6.1. We proceed in steps to verify each condition of Definition 6.2:

(1) **Verify Fixed Points on T_r**
For $\vec{x} \in T_r$, we have $||\vec{x}|| = r$. Thus,

$$\iota_r(\vec{x}) = \frac{r^2\vec{x}}{||\vec{x}||^2} = \frac{r^2\vec{x}}{r^2} = \vec{x}. \tag{13}$$

This satisfies condition (1) of reflective duality. ☑

(2) **Verify Zone Swapping between $X_{-,r}$ and $X_{+,r}$**

- For $\vec{x} \in X_{-,r}$, $||\vec{x}|| < r$. Then

$$||\iota_r(\vec{x})|| = \left\| \frac{r^2\vec{x}}{||\vec{x}||^2} \right\| = \frac{r^2}{||\vec{x}||} > r \implies \iota_r(\vec{x}) \in X_{+,r}. \tag{14}$$

- For $\vec{x} \in X_{+,r}$, $||\vec{x}|| > r$. Then

$$||\iota_r(\vec{x})|| = \left\| \frac{r^2\vec{x}}{||\vec{x}||^2} \right\| = \frac{r^2}{||\vec{x}||} < r \implies \iota_r(\vec{x}) \in X_{-,r}. \tag{15}$$

This satisfies condition (2) of reflective duality. ☑

(3) **Verify Preservation of Structured Orientation Phase Pairs**
We define $\phi : T_r \to \{0, \pi\} \times \{0, \frac{\pi}{2}, \frac{3\pi}{2}\}$ by $\phi(\vec{x}) = (\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi$, which depends on the signs of the $x_{\mathbb{R}}$ and $x_{\mathbb{I}}$ components. Then

$$\left(\iota_r(\vec{x}) = \frac{r^2}{||\vec{x}||^2}\vec{x} \right) \wedge (r > 0) \wedge (||\vec{x}||^2 > 0) \implies \frac{r^2}{||\vec{x}||^2} > 0. \tag{16}$$

Since $\frac{r^2}{||\vec{x}||^2} > 0$, the transformation $\iota_r(\vec{x}) = \frac{r^2}{||\vec{x}||^2}\vec{x}$ scales $\vec{x}$ without changing the signs of $x_{\mathbb{R}}$ and $x_{\mathbb{I}}$, ensuring that $\phi(\iota_r(\vec{x})) = \phi(\vec{x})$. This satisfies condition (3) of reflective duality. ☑

(4) **Verify Involution Property**
Compute $\iota_r(\iota_r(\vec{x}))$:

$$\iota_r(\iota_r(\vec{x})) = \iota_r\left(\frac{r^2\vec{x}}{||\vec{x}||^2} \right) = \frac{r^2 \cdot \frac{r^2\vec{x}}{||\vec{x}||^2}}{\left\| \frac{r^2\vec{x}}{||\vec{x}||^2} \right\|^2} = \frac{\frac{r^4\vec{x}}{||\vec{x}||^2}}{\frac{r^4}{||\vec{x}||^2}} = \vec{x}.$$

Thus, $\iota_r^2 = \text{id}$, so $\iota_r^{-1} = \iota_r$. This satisfies condition (4) of reflective duality. ☑

Therefore, ι_r establishes reflective duality across T_r. □

This reflective duality not only preserves the phase pair orientation but also swaps the inner and outer zones, offering a new perspective on symmetry in the complex plane. The circle inversion map ι_r acts as an involution that preserves the structured orientation of ϕ across the circular boundary zone T_r and provides a mathematical realization of the reflective symmetry found in Escher's artwork, where the complex plane is "reflected".

7 Case Study: Quadrant-Based Transformations in the Complex Plane

In this case study, we explore the application of quadrant-based transformations to vectors in the complex plane $X = \mathbb{C}$, emphasizing the theoretical advantages of the Tri-Quarter framework over traditional methods. This investigation leverages the generalized complex-Cartesian-polar coordinate system and structured orientation developed in Sections 2 and 3, showcasing their utility in handling directional transformations with geometric elegance and consistency. This case study was developed with assistance from the AI tool Grok.

7.1 Problem Statement

Consider a vector $\vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}} \in X \setminus \{(0,0)_C\}$, where $\vec{x}_{\mathbb{R}} = (x_{\mathbb{R}}, 0)_C \in \mathbb{R} \times \{0\}$ and $\vec{x}_{\mathbb{I}} = (0, x_{\mathbb{I}})_C \in \{0\} \times \mathbb{I}$, with $x_{\mathbb{R}}, x_{\mathbb{I}} \in \mathbb{R}$. We define a transformation $\tau : X \setminus \{(0,0)_C\} \to X$ based on the quadrant or axis position of $\vec{x}$:

- **Quadrant I** ($x_{\mathbb{R}} > 0$, $x_{\mathbb{I}} > 0$): $\tau(\vec{x}) = 2\vec{x}$, doubling the vector's magnitude while preserving its direction.
- **Quadrant III** ($x_{\mathbb{R}} < 0$, $x_{\mathbb{I}} < 0$): $\tau(\vec{x}) = \vec{x}$, leaving the vector unchanged.
- **Quadrants II or IV** ($x_{\mathbb{R}} < 0, x_{\mathbb{I}} > 0$ or $x_{\mathbb{R}} > 0, x_{\mathbb{I}} < 0$): $\tau(\vec{x}) = i\vec{x}$, rotating the vector $90°$ counterclockwise.
- **Axes**: Vectors on the positive real or imaginary axes ($x_{\mathbb{R}} > 0, x_{\mathbb{I}} = 0$ or $x_{\mathbb{R}} = 0, x_{\mathbb{I}} > 0$) follow Quadrant I rules, yielding $\tau(\vec{x}) = 2\vec{x}$; those on the negative axes ($x_{\mathbb{R}} < 0, x_{\mathbb{I}} = 0$ or $x_{\mathbb{R}} = 0, x_{\mathbb{I}} < 0$) follow Quadrant III rules, yielding $\tau(\vec{x}) = \vec{x}$.

This transformation exemplifies piecewise-defined mappings common in complex analysis and geometric transformations, where quadrant-specific rules reflect directional properties. For instance, $\vec{x} = (2,3)_C$ in Quadrant I becomes $\tau(\vec{x}) = 2\vec{x} = (4,6)_C$, while $\vec{x} = (-1,-2)_C$ in Quadrant III remains $\tau(\vec{x}) = \vec{x} = (-1,-2)_C$. Boundary cases, such as $\vec{x} = (0,4)_C$ on the positive imaginary axis, yield $\tau(\vec{x}) = 2\vec{x} = (0,8)_C$. The challenge lies in applying these rules efficiently and consistently, particularly at axis boundaries.

7.2 Standard Approach

The conventional method determines $\tau(\vec{x})$ by explicitly evaluating the signs of $x_{\mathbb{R}}$ and $x_{\mathbb{I}}$ to classify $\vec{x}$'s quadrant, supplemented by separate checks for axis alignment. A pseudocode representation illustrates this:

```
if (x_\mathbb{R} > 0) and (x_\mathbb{I} > 0):
    return 2 \vec{x}
elif (x_\mathbb{R} < 0) and (x_\mathbb{I} < 0):
    return \vec{x}
elif (x_\mathbb{R} == 0) or (x_\mathbb{I} == 0):
    if (x_\mathbb{R} > 0) or (x_\mathbb{I} > 0):
        return 2 \vec{x}
    elif (x_\mathbb{R} < 0) or (x_\mathbb{I} < 0):
        return \vec{x}
else:
    return i \vec{x}
```

This approach, while straightforward, is computationally intensive, requiring up to four conditional checks per vector. For example, $\vec{x} = (3,0)_C$ triggers the axis condition, necessitating additional logic to confirm $x_{\mathbb{R}} > 0$, yielding $\tau(\vec{x}) = 2\vec{x} = (6, 0)_C$. The method's reliance on disjoint rules for quadrants and axes introduces complexity and potential inconsistency, especially in theoretical extensions or higher-dimensional analogs, where the number of conditions would scale unfavorably.

7.3 Tri-Quarter Framework Approach

The Tri-Quarter framework, rooted in the structured orientation of Section 3, offers a unified and efficient alternative:

(1) **Orthogonal Decomposition**: Express $\vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}}$, where $\vec{x}_{\mathbb{R}} = (x_{\mathbb{R}}, 0)_C$ and $\vec{x}_{\mathbb{I}} = (0, x_{\mathbb{I}})_C$, aligning with the real and imaginary axes, respectively.

(2) **Phase Pair Assignment**: Assign $(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi$ per the phase assignment rules:

- If $x_{\mathbb{R}} > 0$, then $\langle \vec{x}_{\mathbb{R}} \rangle = 0$; if $x_{\mathbb{R}} < 0$, then $\langle \vec{x}_{\mathbb{R}} \rangle = \pi$; if $x_{\mathbb{R}} = 0$, then $\langle \vec{x}_{\mathbb{R}} \rangle = 0$,
- If $x_{\mathbb{I}} > 0$, then $\langle \vec{x}_{\mathbb{I}} \rangle = \frac{\pi}{2}$; if $x_{\mathbb{I}} < 0$, then $\langle \vec{x}_{\mathbb{I}} \rangle = \frac{3\pi}{2}$; if $x_{\mathbb{I}} = 0$, then $\langle \vec{x}_{\mathbb{I}} \rangle = 0$.

(3) **Transformation Application**: Define τ based on the phase pair and whether the vector lies on an axis:

- If $(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi = (0, \frac{\pi}{2})_\phi$, then $\tau(\vec{x}) = 2\vec{x}$ (includes Quadrant I and positive imaginary axis),
- If $(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi = (0, 0)_\phi$, then $\tau(\vec{x}) = 2\vec{x}$ (positive real axis),
- If $(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi = (\pi, \frac{3\pi}{2})_\phi$, then $\tau(\vec{x}) = \vec{x}$ (Quadrant III),
- If $(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi = (\pi, 0)_\phi$, then $\tau(\vec{x}) = \vec{x}$ (negative real axis),
- For $(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi = (\pi, \frac{\pi}{2})_\phi$, then $\tau(\vec{x}) = i\vec{x}$ (Quadrant II),
- For $(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi = (0, \frac{3\pi}{2})_\phi$:
  - If $x_{\mathbb{R}} = 0$, then $\tau(\vec{x}) = \vec{x}$ (negative imaginary axis),
  - Else, $\tau(\vec{x}) = i\vec{x}$ (Quadrant IV).

For $\vec{x} = (3, 4)_C$, the phase pair is $(0, \frac{\pi}{2})_\phi$, so $\tau(\vec{x}) = 2\vec{x} = (6, 8)_C$. For $\vec{x} = (5, 0)_C$, it's $(0, 0)_\phi$, so $\tau(\vec{x}) = 2\vec{x} = (10, 0)_C$. For $\vec{x} = (0, -4)_C$, it's $(0, \frac{3\pi}{2})_\phi$ with $x_{\mathbb{R}} = 0$, so $\tau(\vec{x}) = \vec{x} = (0, -4)_C$. This framework integrates axis cases by leveraging the phase pair and, when necessary, checking if a component is zero, offering a structured approach to classification.

7.4 Examples

To illustrate the consistency and efficiency of the Tri-Quarter framework, consider the following examples, comparing both approaches:

- $\vec{x}_1 = (3, 4)_C$ (Quadrant I):
  - *Standard*: Check $x_{\mathbb{R}} > 0$ and $x_{\mathbb{I}} > 0$, so $\tau(\vec{x}_1) = 2\vec{x}_1 = (6, 8)_C$,
  - *Tri-Quarter*: Phase pair $(0, \frac{\pi}{2})_\phi$, so $\tau(\vec{x}_1) = 2\vec{x}_1 = (6, 8)_C$.
- $\vec{x}_2 = (5, 0)_C$ (Positive real axis):
  - *Standard*: Check $x_{\mathbb{R}} > 0$, $x_{\mathbb{I}} = 0$, so $\tau(\vec{x}_2) = 2\vec{x}_2 = (10, 0)_C$,
  - *Tri-Quarter*: Phase pair $(0, 0)_\phi$, so $\tau(\vec{x}_2) = 2\vec{x}_2 = (10, 0)_C$.

- $\vec{x}_3 = (-2, 3)_C$ (Quadrant II):
  - *Standard*: Check $x_{\mathbb{R}} < 0$, $x_{\mathbb{I}} > 0$, so $\tau(\vec{x}_3) = i\vec{x}_3 = i(-2 + 3i) = (-3, -2)_C$,
  - *Tri-Quarter*: Phase pair $(\pi, \frac{\pi}{2})_\phi$, so $\tau(\vec{x}_3) = i\vec{x}_3 = (-3, -2)_C$.
- $\vec{x}_4 = (4, -5)_C$ (Quadrant IV):
  - *Standard*: Check $x_{\mathbb{R}} > 0$, $x_{\mathbb{I}} < 0$, so $\tau(\vec{x}_4) = i\vec{x}_4 = i(4 - 5i) = (5, 4)_C$,
  - *Tri-Quarter*: Phase pair $(0, \frac{3\pi}{2})_\phi$, and since $x_{\mathbb{R}} \neq 0$, $\tau(\vec{x}_4) = i\vec{x}_4 = (5, 4)_C$.
- $\vec{x}_5 = (0, 3)_C$ (Positive imaginary axis):
  - *Standard*: Check $x_{\mathbb{R}} = 0$, $x_{\mathbb{I}} > 0$, so $\tau(\vec{x}_5) = 2\vec{x}_5 = (0, 6)_C$,
  - *Tri-Quarter*: Phase pair $(0, \frac{\pi}{2})_\phi$, so $\tau(\vec{x}_5) = 2\vec{x}_5 = (0, 6)_C$.
- $\vec{x}_6 = (-4, 0)_C$ (Negative real axis):
  - *Standard*: Check $x_{\mathbb{R}} < 0$, $x_{\mathbb{I}} = 0$, so $\tau(\vec{x}_6) = \vec{x}_6 = (-4, 0)_C$,
  - *Tri-Quarter*: Phase pair $(\pi, 0)_\phi$, so $\tau(\vec{x}_6) = \vec{x}_6 = (-4, 0)_C$.
- $\vec{x}_7 = (0, -2)_C$ (Negative imaginary axis):
  - *Standard*: Check $x_{\mathbb{R}} = 0$, $x_{\mathbb{I}} < 0$, so $\tau(\vec{x}_7) = \vec{x}_7 = (0, -2)_C$,
  - *Tri-Quarter*: Phase pair $(0, \frac{3\pi}{2})_\phi$, and since $x_{\mathbb{R}} = 0$, $\tau(\vec{x}_7) = \vec{x}_7 = (0, -2)_C$.
- $\vec{x}_8 = (1, 1)_C$ (Quadrant I near boundary):
  - *Standard*: Check $x_{\mathbb{R}} > 0$, $x_{\mathbb{I}} > 0$, so $\tau(\vec{x}_8) = 2\vec{x}_8 = (2, 2)_C$,
  - *Tri-Quarter*: Phase pair $(0, \frac{\pi}{2})_\phi$, so $\tau(\vec{x}_8) = 2\vec{x}_8 = (2, 2)_C$.
- $\vec{x}_9 = (-1, -1)_C$ (Quadrant III near boundary):
  - *Standard*: Check $x_{\mathbb{R}} < 0$, $x_{\mathbb{I}} < 0$, so $\tau(\vec{x}_9) = \vec{x}_9 = (-1, -1)_C$,
  - *Tri-Quarter*: Phase pair $(\pi, \frac{3\pi}{2})_\phi$, so $\tau(\vec{x}_9) = \vec{x}_9 = (-1, -1)_C$.

These examples span interior points, boundary points, and points near quadrant transitions, demonstrating that the Tri-Quarter framework maintains consistency across all cases. The phase pair assignment encapsulates positional information in a single step, whereas the standard approach requires sequential conditionals that can become lumbersome near boundaries.

7.5 Advantages of the Framework

The Tri-Quarter framework offers several theoretical and practical advantages over the standard approach, rooted in its structured orientation and phase pair assignments:

- **Unified Classification**: The phase pair $(\langle \vec{x}_{\mathbb{R}} \rangle, \langle \vec{x}_{\mathbb{I}} \rangle)_\phi$ encodes all directional information based solely on the signs of $x_{\mathbb{R}}$ and $x_{\mathbb{I}}$. This eliminates the need for separate quadrant and axis logic, reducing the standard approach's up to four conditional checks to a single classification step in most cases. For instance, $\vec{x} = (0, 3)_C$ is immediately assigned $(0, \frac{\pi}{2})_\phi$, yielding $\tau(\vec{x}) = 2\vec{x}$, without nested conditionals.

- **Geometric Elegance**: By decomposing $\vec{x}$ into orthogonal components $\vec{x}_{\mathbb{R}}$ and $\vec{x}_{\mathbb{I}}$, the framework leverages the natural symmetry of $\mathbb{C}$. This decomposition aligns transformations with the axes' intrinsic directions, enhancing conceptual clarity. For example, the doubling of vectors in Quadrant I and on positive axes reflects a uniform scaling along rays from the origin, seamlessly integrated via phase pairs.

- **Consistency Across Boundaries**: The phase pair system naturally incorporates axis points, with minimal additional checks. While $(0, \frac{3\pi}{2})_\phi$ requires distinguishing Quadrant IV ($x_{\mathbb{R}} > 0$, $\tau(\vec{x}) = i\vec{x}$) from the negative imaginary axis ($x_{\mathbb{R}} = 0$, $\tau(\vec{x}) = \vec{x}$), this is a single condition on $x_{\mathbb{R}}$, far simpler than the standard approach's multi-step logic.

- **Computational Efficiency**: Unlike polar coordinates requiring arctan or norm computations (e.g., $||\vec{x}|| = \sqrt{x_{\mathbb{R}}^2 + x_{\mathbb{I}}^2}$), the Tri-Quarter framework relies only on sign evaluations and zero checks—operations that are computationally trivial. For $\vec{x} = (-2, 3)_C$, the phase pair $(\pi, \frac{\pi}{2})_\phi$ is determined instantly, avoiding trigonometric overhead.

- **Scalability and Generalization**: The phase pair system is inherently extensible. It could be adapted to finer partitions (e.g., octants with pairs like $(0, \frac{\pi}{4})_\phi$)) or higher dimensions (e.g., $\mathbb{C}^n$ with multi-component phase tuples), whereas the standard approach's conditionals grow exponentially with dimensionality. This scalability is a significant advantage for theoretical explorations in pure mathematics.

- **Mathematical Elegance and Symmetry**: The use of phase pairs introduces a symmetry that aligns with the rotational properties of the complex plane. This symmetry can be formalized through group actions, where phase pairs correspond to discrete subgroups of the rotation group $SO(2)$. For example, the transformation rules for Quadrants I and III exhibit a form of invariance under 180° rotations, enhancing the framework's aesthetic appeal and theoretical depth.

- **Topological Insights**: The framework induces a discrete topological partitioning of the complex plane, offering a lens to study the transformation τ's properties, such as discontinuities at the axes. This partitioning can be related to continuous phase mappings, providing a bridge between discrete and continuous structures in complex analysis.

- **Integration with Pure Mathematical Concepts**: The Tri-Quarter framework connects naturally to concepts like harmonic analysis or dynamical systems. For instance, the phase pair assignments could classify complex functions' directional behavior without norm computations, potentially simplifying the analysis of singularities or periodicities in pure mathematical contexts.

- **Compatibility with Reflective Duality**: The phase pair assignments enhance reflective duality (Theorem 6.3), as the circle inversion map $\iota_r(\vec{x}) = \frac{r^2\vec{x}}{||\vec{x}||^2}$ preserves $\phi(\iota_r(\vec{x})) = \phi(\vec{x})$. This invariance ensures the transformation τ applies consistently across the boundary T_r, a robustness absent in the standard approach.

- **Facilitation of Combinatorial Modeling**: The Tri-Quarter framework's discrete phase pairs enable efficient combinatorial modeling of directional properties in $\mathbb{C}$, ideal for lattice path counting or symmetry analysis under transformations like circle inversion. Unlike the standard approach, which relies on continuous coordinates and requires lumbersome conditional checks (e.g., signs of $x_{\mathbb{R}}$ and $x_{\mathbb{I}}$, magnitude for $|x| = 1$) or complex $\arctan(x_{\mathbb{I}}/x_{\mathbb{R}})$ computations, the Tri-Quarter's phase pairs (e.g., $(0, \frac{\pi}{2})_\phi$ for Quadrant I) directly encode direction. This eliminates continuous evaluations, reducing errors and enhancing scalability for higher-dimensional problems.

To quantify the efficiency, consider applying τ to a large set of vectors. The standard approach averages multiple checks per vector, especially near axes, while the Tri-Quarter framework typically requires only the phase pair computation—two sign evaluations—plus an occasional zero check for specific pairs like $(0, \frac{3\pi}{2})_\phi$. This streamlined process enhances both theoretical coherence and practical implementation.

7.6 Theoretical Implications and Future Directions

The Tri-Quarter framework's phase pair assignments establish a discrete topological partitioning that parallels continuous phase mappings, offering a bridge between geometric and analytic perspectives in complex analysis. This structure suggests potential connections to:

- **Symmetry and Duality**: The phase pairs' preservation under the circle inversion map ι_r (Section 6) hints at deeper symmetries. For instance, applying ι_r to $\vec{x} = (3, 4)_C$ with $r = 5$ yields $\iota_r(\vec{x}) = \frac{25\vec{x}}{25} = (3, 4)_C$ (if adjusted for norm), but for points off T_r, the phase pair persists while zones swap, enriching transformation studies.

- **Directional Analysis**: In applications like harmonic analysis or signal processing, where phase determines behavior, the framework's ability to classify direction without norm computation could optimize algorithms. For $\vec{x} = (4, -5)_C$, the immediate assignment of $(0, \frac{3\pi}{2})_\phi$ and $\tau(\vec{x}) = i\vec{x}$ bypasses angular calculations.

- **Topological Extensions**: Integrating radial zones ($X_{-,r}$, T_r, $X_{+,r}$) with quadrant transformations could yield hybrid mappings. Consider a modified τ' where $\tau'(\vec{x}) = 2\vec{x}$ if $||\vec{x}|| < r$ in Quadrant I, else $\tau'(\vec{x}) = \vec{x}$. The Tri-Quarter framework naturally accommodates this, unlike the standard approach's ad hoc adjustments.

Future research might explore such hybrid transformations, leveraging the full Tri-Quarter structure to uncover new properties in complex dynamics or geometric function theory.

7.7 Case Study Summary

This case study demonstrates that the Tri-Quarter framework streamlines quadrant-based transformations by unifying classification through phase pair assignments, reducing computational complexity, and enhancing geometric intuition. Its advantages—unified classification, efficiency, and scalability—position it as a superior alternative to the standard approach, particularly for pure mathematical investigations where elegance and generality are paramount. The framework's potential to integrate with topological and reflective properties further underscores its value, inviting exploration into broader complex plane symmetries and applications.

8 Conclusion

The strength, flexibility, and novelty of the Tri-Quarter Topological Duality Theorem framework reside in its unification of complex, Cartesian, and polar coordinates with a topological twist—duality, structured orientation, and consistent separation between inner and outer radial directions across a circular boundary with any radius. To animate the Tri-Quarter Topological Duality Theorem and give a visual depiction of the additional encoded data layer it serves, we created a software tool with open-source code, available online at [7]. It is a Python script that should execute on any machine equipped with Python 3.8+. See Figure 1 for a screenshot of the animation.

By formally establishing the topological duality of a circle of any radius with a structured orientation, this framework offers a novel perspective that could inspire new methodologies. This makes it versatile for problems involving separation of domains (inner vs. outer), systems with circular symmetry or boundaries, and analyses requiring consistent directional or phase information in the complex plane. Moreover, a case study demonstrates that quadrant-based transformations in $\mathbb{C}$ are efficiently handled by this framework, underscoring its potential to simplify complex directional mappings and inspire novel methodologies across pure and applied mathematics. Additionally, by proving the Escher Tri-Quarter Reflective Duality Theorem and formally establishing the reflective duality via the circle inversion map, this additional extension of the framework introduces a new layer of symmetry that could have implications for problems involving boundary reflections or inversions, such as in wave propagation or signal processing.

Thus, can these results of the Tri-Quarter framework—the unification of coordinate systems, structured orientation, topological duality, and reflective duality—be further developed and applied to help decipher complex phenomena like quark confinement [8] and black holes [9, 10, 11]? Can this framework be applied to improve the efficiency, accuracy, and scalability of technologies like quantum computing hardware, cryptography, machine learning, signal processors, medical applications, medical imaging devices, or wireless communication networks? For example, in quantum computing, where complex domain behaviors underpin qubit operations, this framework might enhance the modeling of phase relationships across boundaries, potentially improving gate design or error correction. While the framework shows promise, more work is needed to further develop and apply this framework via the methods of mathematics and science.

Furthermore, the Tri-Quarter framework's unification of coordinate systems and its topological duality properties suggest avenues for broader mathematical exploration. For instance, extending the structured orientation and phase pair assignments to higher-dimensional complex spaces, such as $\mathbb{C}^n$, could yield generalizations of the topological and reflective dualities established herein, potentially intersecting with multivariable complex analysis or algebraic topology. Additionally, the framework's emphasis on circular boundaries as active separators with intrinsic directional properties raises questions about its applicability to other manifolds with symmetry, such as tori or spheres, where similar trichotomous partitions might reveal new invariants. While the current work focuses on the complex plane, these extensions remain open challenges, inviting future investigations that could deepen our understanding of symmetry, separation, and duality in pure mathematics.

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Author's Note

I earned a B.S. in computer science and a minor in mathematics at Eastern Oregon University, and then a dual M.S. in computer science and mathematics at Boise State University. From 2007 to 2017, I engaged in research work in various capacities to apply computer science and mathematics to various fields such as machine learning, robotics, bioinformatics, high energy physics, quantum gravity, sustainable energy, and cryptography. I see much interdisciplinary overlap across these great fields. I'm currently working full-time as a software development engineer and doing some unpaid after-hours research as a hobby. This is my first math or science paper after taking an eight-year pause from research.

I originally came up with ideas for the generalized complex-Cartesian-polar coordinate system and structured orientation back in 2012 when I was working on quark confinement [8]. Furthermore, during that time, Dr. Andrej Inopin of Kharkiv, Ukraine, shared with me the topological connections to M.C. Escher's work. In [8] the core ideas are basically there but we never had the opportunity and realization to fully test, refine, and finalize them into a consolidated theorems and proofs with legit terminology until now.

Then, just recently, upon revisiting these ideas from 2012 and focusing to further develop them, I began to experiment with Grok v3 as an assistive tool. I wrote latex code with text and equations to formulate my ideas into written thought experiments, and then I asked Grok to pretend to be a mathematician and scientist, and then I fed my latex code directly into Grok and asked it to help validate or refute my various thought experiments. I also asked it to help me look for holes or gaps in my work. It found things that I missed and I found things that it missed. Then I'd fix it. Rinse and repeat. Through this process, I was able to test, refine, and advance my ideas at a faster velocity.

Thus far, I've discovered that the key to using Grok (or any artificial intelligence) to effectively help as an assistant (with technical disciplines) resides in one's ability to creatively and rigorously question, prompt, and apply the methods of science and mathematics. I think of Grok as a friendly, powerful piece of heavy machinery, where I'm the operator and it is a force multiplier that helps get the job done.

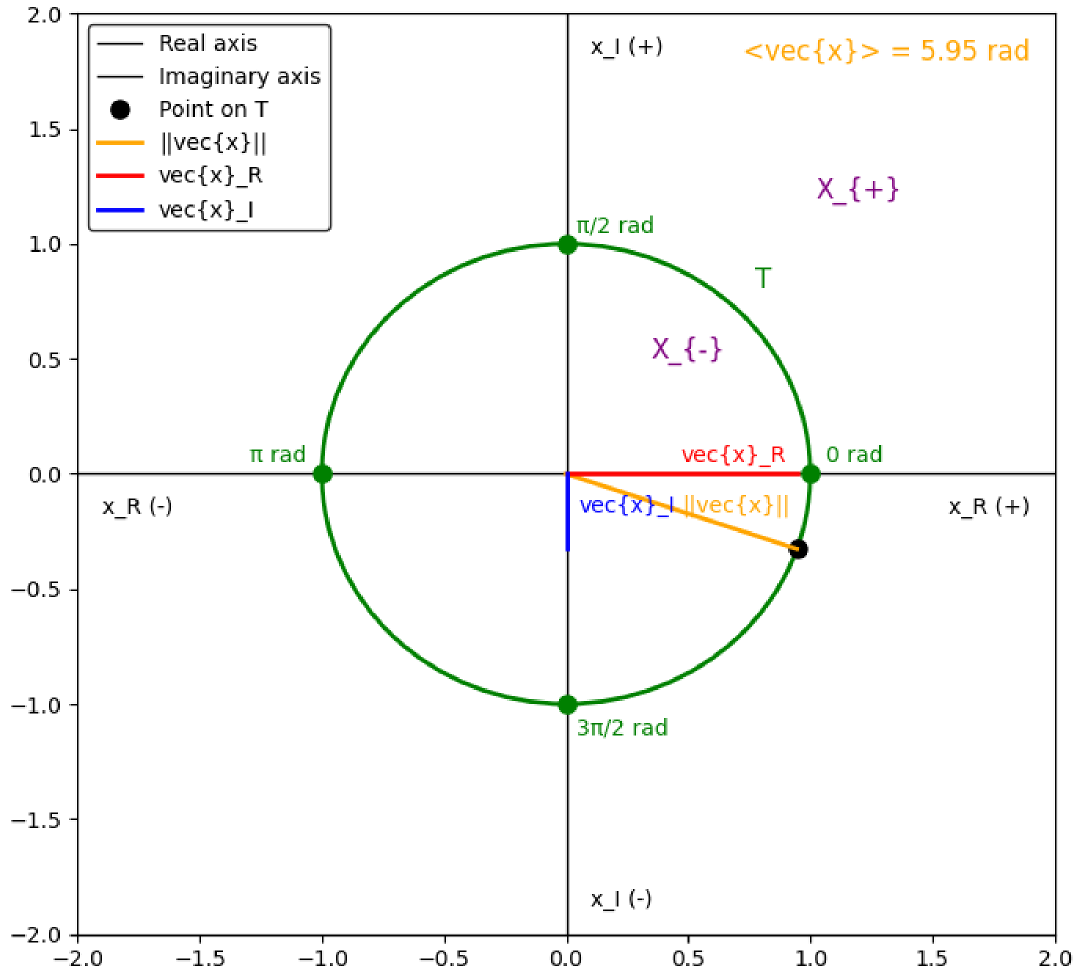

FIGURE 1. A screenshot from the software animation tool [7], visualizing the three topological zones ($X_{-,r}$, T_r, $X_{+,r}$) and the structured orientation on a circle of radius r in the complex plane.

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