Disprove of a Proof of Euler's Formula

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Abstract

In 1991, Gilbert Strang published a proof of Euler's formula using polar coordinates in Calculus, Wellesley-Cambridge, p. 389. In the following we show that this alleged proof is not valid.

Comments

The so-called proof uses the fact that all complex numbers can be expressed in polar coordinates. It is assumed that $e^{ix} = r(\cos\theta + i\sin\theta)$ at least for some r and θ.

No assumptions are made for x, r and θ, especially for the relationship between x on one side and r and θ on the other side

In the following, the derivative of x for the assumed equation will be calculated. The derivative of e^{ix} with respect to x is ie^{ix}. Calculating the derivative with respect to x for $r(\cos\theta + \iota\sin\theta)$ yields the following equation:

$ie^{ix} = (dr/dx)(\cos\theta + i\sin\theta) + (d\theta/dx)\, ir(\cos\theta + i\sin\theta)$

Using the assumption for e^{ix} as given above, this yields the equation:

$ie^{ix} = (dr/dx)(\cos\theta + i\sin\theta) + (d\theta/dx)\, i\, e^{ix.}$

Making the right and the left side of the equation equal can be achieved by setting:

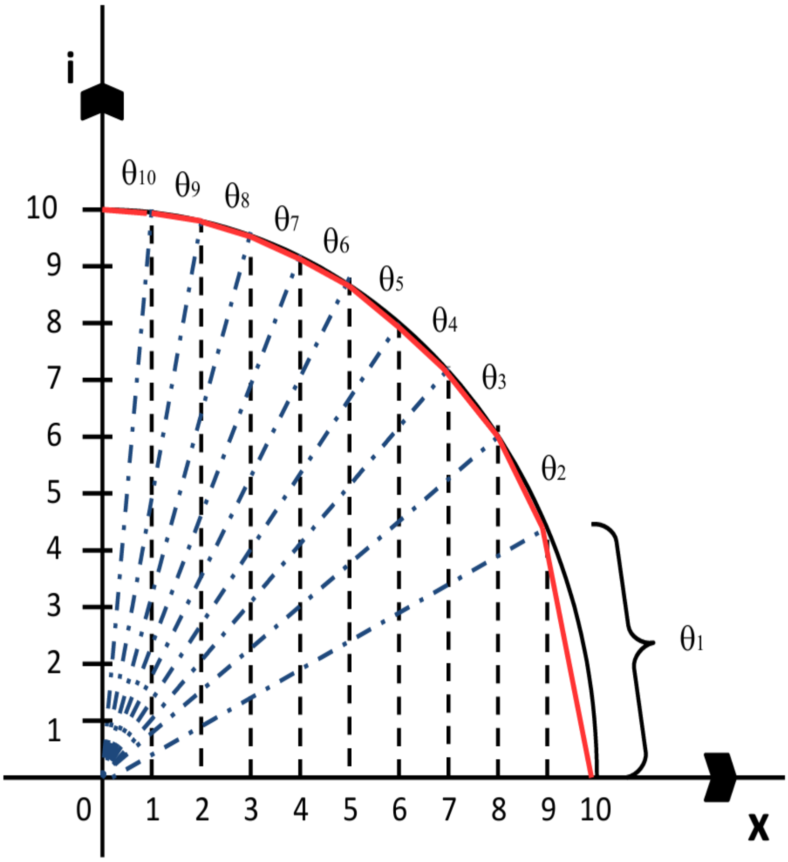
$(dr/dx) = 0$ and $(d\theta/dx) = 1$.

Integrating the latter assumption yields a linear relationship between x and θ, which can be written as:

$\theta = x + C$, with C being a constant.

While the former assumption holds true, the latter assumption is obviously incorrect, as demonstrated in the following by figure 1.

Figure 1



In figure 1, each unit Δx on the x-axis is associated with an angle θ in arc measure on the arc of the circle with radius r and its corresponding secant (shown in red).

From figure 1 it becomes obvious, that the secants and, thus, the angles are not in a linear relationship with x, since each secant has different length. Even if a constant C is introduced into the secant between $x = 9$ and $x = 10$, only the secants between $x = 8$ and $x = 9$ as well as between $x = 9$ and $x = 10$ can be made of equal length. The rest of the secants still remains different. Accordingly, it is obvious that a linear relationship between x and θ does not exist, and therefore, the alleged proof of Euler's formula in polar coordinates does not hold.