On the Feigenbaum Attractor and Feynman Diagrams

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Abstract

It was recently conjectured that the Standard Model of particle physics resides on a *bifurcation diagram* generated by the recursive scaling of Higgs coupling. This sequel explores the relationship between the bifurcation diagram and the Path Integral (PI) formalism of Quantum Field Theory (QFT). The long-term goal is to base the *Feynman diagrams* on the properties of the *Feigenbaum attractor* of either quadratic or cubic maps.

Key words: Bifurcation diagram, Feigenbaum universality, Standard Model, Path Integral formalism, Dimensional Fluctuations, Feynman Diagrams.

1. Introduction

It is well known that *low-dimensional maps* (such as quadratic, Hénon or cubic maps) serve as prime models of nonlinear science and chaos theory. Moreover, when tied in with the *logistic equation*, quadratic maps illustrate key concepts of chaotic behavior, including period-doubling bifurcations, universality, Lyapunov stability, strange attractors and sensitive dependence on initial conditions.

Recent research confirms that low-dimensional maps are also highly relevant to *complex dynamics*, in general, and the *nonintegrable regime* of QFT and particle physics, in particular:

1) Far above the electroweak scale, reaction-diffusion processes involving dimensional fluctuations lead to quadratic maps and the complex Ginzburg-Landau equation [2 - 3].

2) The Standard Model unfolds under recursive bifurcations of the cubic map, the latter being derived from the Renormalization Group (RG) flow

of the Higgs coupling. Both particle and Dark Matter condensates act as fixed points of the bifurcation diagram [1].

3) Large systems of evolution equations can be shown to reduce to either quadratic or cubic forms following the *center manifold theory* [4].

This exploration is a sequel to [1], where bifurcations start with the formation of a Higgs condensate and ends up with the formation of a top-antitop ($t\bar{t}$) condensate. The long-term goal of this work is to base the *Feynman diagrams* on the properties of the *Feigenbaum attractor* of either quadratic or cubic maps.

The report is organized as follows: working assumptions and conventions used throughout are covered in the next section. Section 3 outlines the remarkable (yet largely underappreciated) analogy between the self-similarity of fractal structures and Feynman diagrams; Building on section 3, section 4 elaborates on the topic of random walks on the Feigenbaum attractor. The PI formulation of the Feigenbaum attractor in terms of field theory forms the subject of section 5. A summary is included in the last section. Designed as an introductory/pedagogical study, the paper is presented in an accessible format and is open for independent scrutiny and unbiased analysis.

2. Assumptions and conventions:

A1) With reference to [1], fields undergoing bifurcations are denoted as x, the time-scale for field evolution τ is the continuous analog of the iteration index n, and the one-parameter of either quadratic or cubic map is denoted by r.

A2) The quadratic and cubic maps studied herein are one-parameter unimodal (or single peak) maps, written as

$$x_{n+1} = f_{r_1}(x_n) = r_1 x_n (1 - x_n) \tag{1}$$

$$x_{n+1} = f_{r_2}(x_n) = r_2 x_n (1 - x_n^2) \tag{2}$$

A3) There are *two representative time scales* entering the derivation below, namely,

a) an RG scale for field evolution $\tau = \log(\mu/\mu_0)$, where μ is the observation scale.

b) an RG scale for the flow of couplings denoted by l.

A4) The "Feigenbaum fixed point" and the "Feigenbaum attractor" are respectively defined as follows:

a) The ***Feigenbaum fixed point*** is a mathematical concept of Renormalization theory, describing the universal scaling of period-doubling cascades (Appendix B).

b) The ***Feigenbaum attractor*** is a fractal structure of actual dynamical systems at the transition to fully developed chaos. In quadratic or cubic maps, the attractor develops in proximity to the accumulation point

$r_{1,2} = r_\infty$, is self-similar and has a fractal dimension greater than 1 (Appendix A).

3. Self-similarity of fractal structures and Feynman diagrams

It is well known that *Feynman diagrams* are graphical representations of interactions in QFT. Particle interactions are pictured using *vertices, lines,* and *loops*, where each loop corresponds to quantum corrections in perturbation theory. The expansion in Feynman diagrams follows a recursive structure—higher-order terms in the perturbation series consider additional interaction vertices and loops. This procedure leads to *self-similar patterns* of ever-increasing complexity.

Although both Feynman diagrams and fractal structures exhibit self-similarity, they arise in different contexts within physics and mathematics. The connection between them can be explored through their shared recursive/iteration attributes. The analogy between Feynman diagrams and fractals emerges in how QFT accounts for the contribution of radiative corrections. In particular,

a) the *RG flow* describes how physical parameters of the theory (such as masses and coupling constants) change with the observation scale. This scaling behavior often exhibits self-similarity, as physics at one scale nearly replicates that at another.

b) The inclusion of additional loops and sub-diagrams in Feynman diagrams resembles the *iterative growth* of fractals. Typical examples include diagrams in quantum electrodynamics (QED) or quantum chromodynamics (QCD), which can be broken down into sub-diagrams replicating the whole. Emitted gluons in parton showers from high-energy collisions exhibit a branching geometry akin to fractal-like structures.

A gallery of representative Feynman diagrams is illustrated in Fig. 1 below.

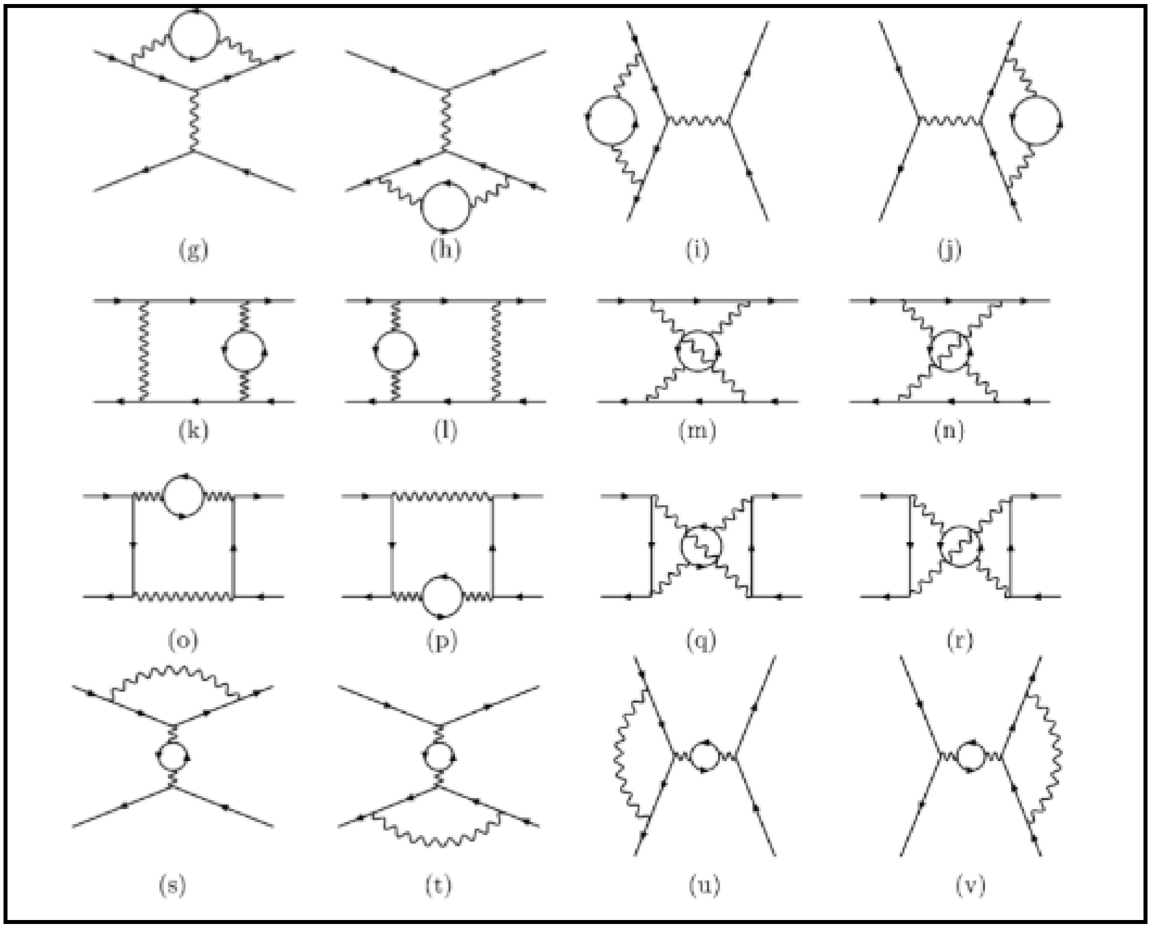


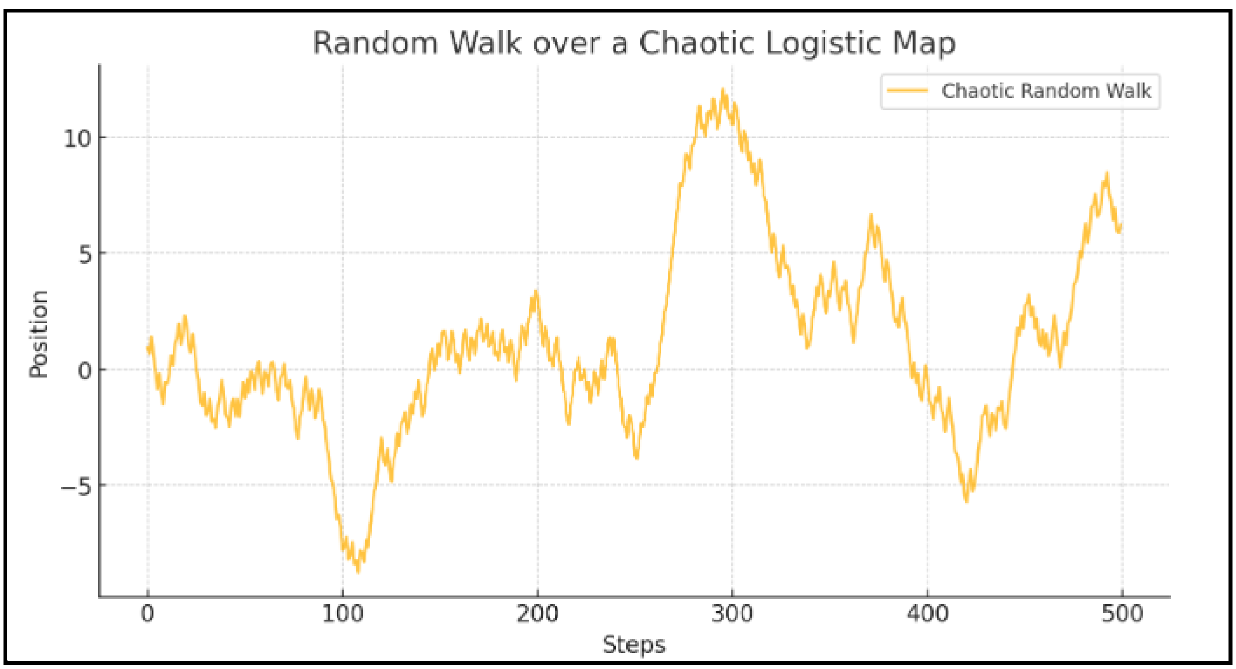
Fig. 1 Feynman diagrams containing vertices, lines and loops.

4. Random walks (RW) on the Feigenbaum attractor

By definition, a *random walk* (RW) consists of a sequence of stochastic steps, often used to model *Brownian motion* and *diffusion processes*. The probability of reaching a certain point is given by summing up all possible paths leading to that point.

RW's provide an intuitive foundation for understanding Path Integrals (PI) in statistical physics, quantum mechanics, and QFT. The PI approach in quantum theory extends the idea of summing over random walks to the quantum domain, where paths interfere according to the phase e^{iS} rather than being weighted by classical probabilities.

The couple of graphs shown below illustrate the plots of RW's over the quadratic and cubic maps displayed as "position" versus number of "stochastic steps".



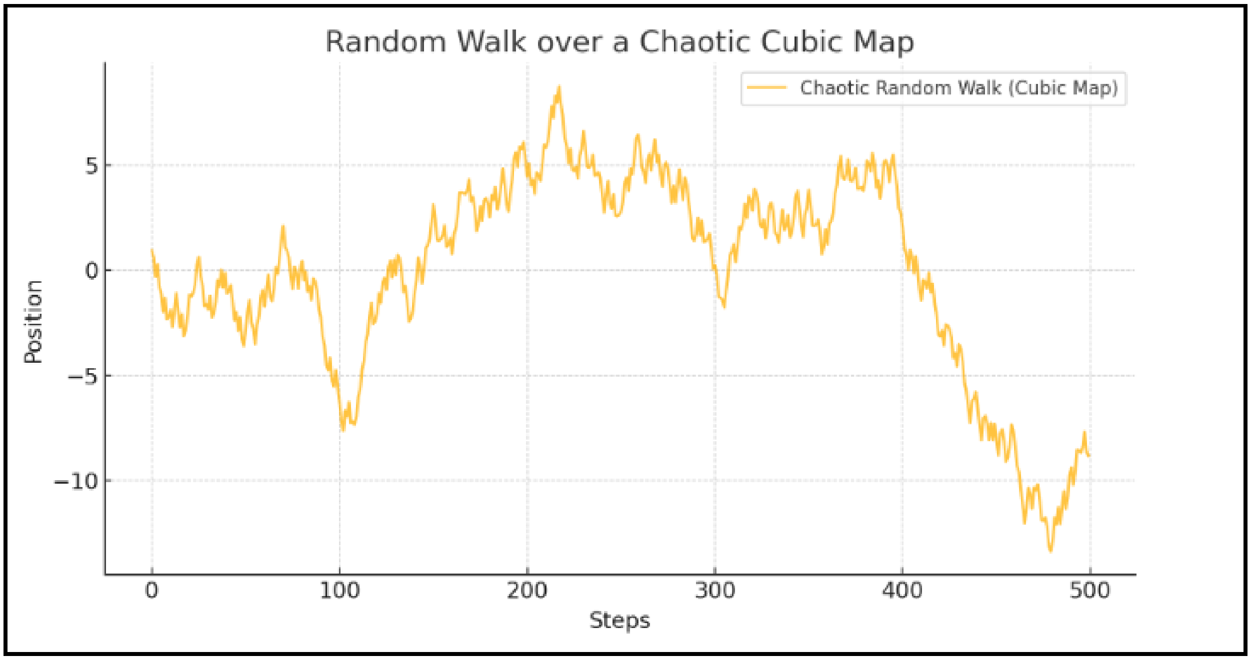


Fig. 2 below is the visualization of RW in the bifurcation diagram of the quadratic map. Here are the main features of this RW:

a) The *black points* represent the bifurcation diagram, showing the steady-state behavior of x as r varies.

b) The *red curve* traces the RW path in r - space.

c) The *blue points* highlight the specific steps of the RW.

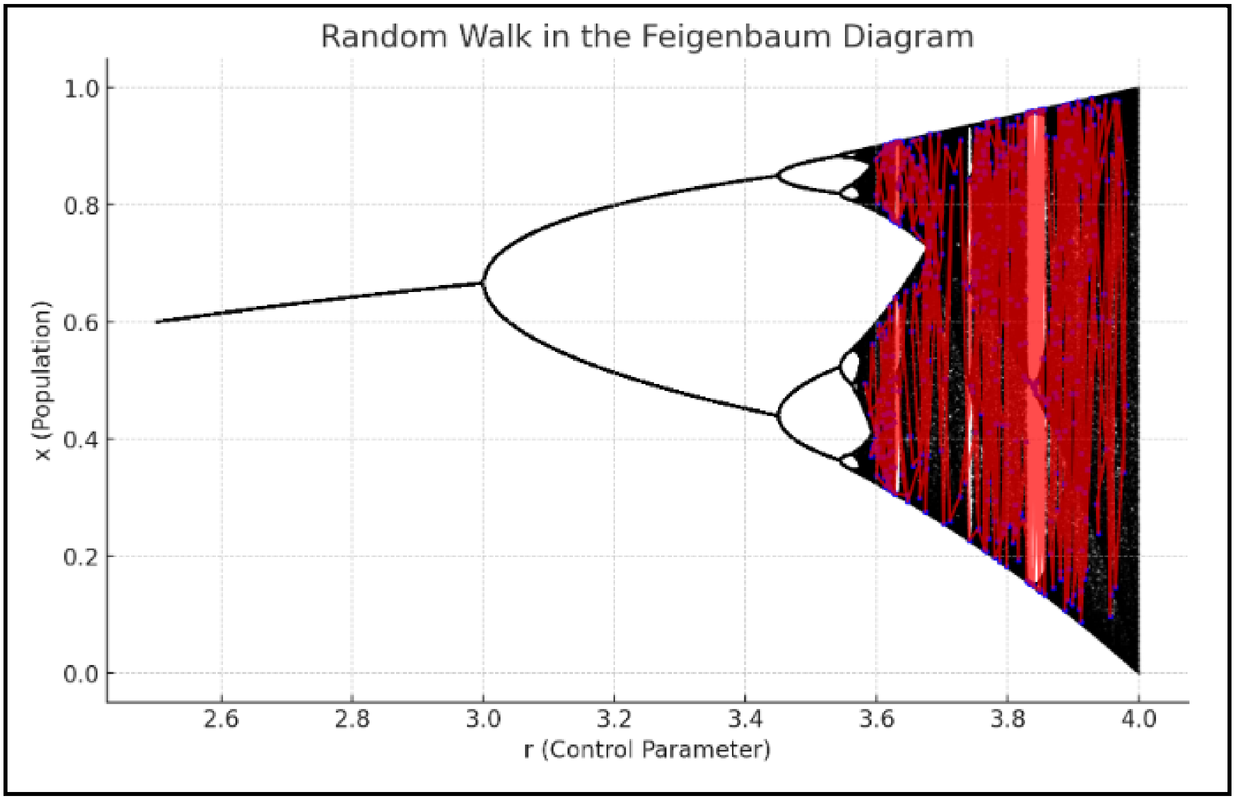


Fig. 2: Visualization of random walks on the bifurcation diagram

The walk "wanders" through *periodic regions* (where the behavior is *stable*) and *chaotic regions* (where small changes in r cause large variations in x). A similar mixture of regular and chaotic phase space orbits shows up in the approach to Hamiltonian chaos of nonintegrable dynamical systems.

Following [1], the border between Cantor Dust/Dark Matter condensate and the 3 neutrino condensates of the Standard Model is a marker of *fully developed chaos*, with few or no traces of periodic behavior.

5. Field-Theoretic Account of the Feigenbaum Attractor

The PI approach to the Feigenbaum attractor may be built upon the analogy between *summation over paths* in QFT and summation over *histories of chaotic maps* in classical chaos theory. The basis for this analogy is that the PI approach to chaos describes classical fluctuations in chaotic dynamics, in the same way quantum fluctuations are accounted for in the PI integrals of QFT. Note that the underlying principle at work here is that all nonlinear dynamical systems reducible to unimodal low-dimensional maps follow a *universal route to chaos,* regardless of their format and initial conditions [5 - 26].

In line with these observations, the goal of this section is twofold, namely,

a) to cast the scaling behavior of the map near the Feigenbaum point in terms of an effective action**,**

b) to analyze the corresponding renormalization group (RG) equations.

To this end, we introduce an *auxiliary field* $\psi(x,\tau)$ denoting perturbations around a trajectory x_* located near the Feigenbaum attractor

$$\psi(n) = x_n - x_* \tag{3}$$

The evolution equation may be then approximated as

$$\psi_{n+1} = g(\psi_n) \tag{4}$$

where the so-called Feigenbaum-Cvitanovic function $g(x)$ satisfies the scaling equation (Appendix B)

$$g(x) = -\alpha g(g(x/\alpha)) \tag{5}$$

Here, α is the second Feigenbaum universal constant ($\alpha \approx 2.5029$ for the quadratic map and $\alpha' \approx 2.3378$ for the cubic map). Per Appendix B, the main point here is that, under repeated renormalization operations consisting of iteration and rescaling, various unimodal maps *converge to the same function* - denoted as $g^*(x)$ - regardless of the form taken by the original function.

We next introduce a Lagrangian density in the continuous time limit of (4)

$$L = \frac{1}{2}(\partial_\tau \psi)^2 - V_{eff}(\psi) \tag{6}$$

in which the *effective potential* follows from the Feigenbaum renormalization flow and is given by [ ]

$$V_{eff}(\psi) = \frac{1}{2}\lambda\psi^2 + \frac{g_3}{3!}\psi^3 + \frac{g_4}{4!}\psi^4 + ... \text{ (quadratic map)} \tag{7a}$$

$$V_{eff}(\psi) = \frac{1}{2}\lambda\psi^2 + \frac{g_3}{3!}\psi^3 + \frac{g_5}{5!}\psi^5 + ... \text{(cubic map)} \tag{7b}$$

Here, by (11), λ is related to the first Feigenbaum constant δ ($\delta \approx 4.669$ for the quadratic map and $\delta' \approx 8.721$ for the cubic map), and $g_{3,4,5}$ are interaction coefficients. The corresponding *Euclidean action* governing the fluctuations about the Feigenbaum attractor is

$$S[\psi]=\int d\tau dx[\frac{1}{2}(\partial_\tau\psi)^2+V_{eff}(\psi)] \tag{8}$$

Again, by analogy with QFT, the probability amplitude of the transition between two perturbation states ψ_i and ψ_f over a time interval $T=\tau_f-\tau_i$ is given by the PI,

$$Z=\int d\psi\exp(-S[\psi]) \tag{9}$$

To perform a RG analysis in the Wilsonian sense, we first integrate out the fast modes and rescale the remaining degrees of freedom. The resulting RG equations define the *scaling behavior of fluctuations* near the Feigenbaum attractor and assume the form,

Quadratic map:

$$\frac{d\lambda}{dl}=2\lambda-\alpha g_3^2+O(g_4) \tag{10a}$$

$$\frac{dg_3}{dl}=(1-\delta)g_3-\alpha g_4 g_3+O(g_5) \tag{10b}$$

$$\frac{dg_4}{dl} = 2(1-\delta)g_4 + O(g_5) \tag{10c}$$

Here, the Feigenbaum constant δ is the rate of flow for λ at the Feigenbaum fixed point

$$\delta = \left.\frac{d\lambda}{dl}\right|_{FP} \tag{11}$$

Cubic map:

$$\frac{d\lambda}{dl} = 2\lambda - \alpha' g_5^2 \tag{12a}$$

$$\frac{dg_3}{dl} = (1-\delta')g_3 - \alpha' g_5 g_3 \tag{12b}$$

$$\frac{dg_5}{dl} = 2(1-\delta')g_5 \tag{12c}$$

Fig. 3 shows the RG flow for the Feigenbaum attractor of the quadratic map. The quadratic coupling $\lambda(l)$ grows under the RG flow, indicating that the system drifts away from criticality. The cubic and quartic couplings $g_3(l)$

and $g_4(l)$ follow a decay trend, characteristic of irrelevant couplings in RG theory.

Unlike the quadratic map, analysis shows that there is a nontrivial fixed point of the RG flow for the cubic map given by

$$\lambda^* \propto \frac{\alpha' g_5^{*2}}{2}; \;\; g_3^* = g_5^* = 0 \tag{13}$$

As the RG flow for the cubic map converges to (13), the parameter λ tends to stabilize, which indicates the transition to self-similarity characteristic for critical behavior. As a result, the cubic map follows a *different universality scaling* from the quadratic map. This result is consistent with the observation that cubic maps often exhibit *richer dynamics*, including period-doubling cascades, chaotic attractors, and multi-stability (coexistence of multiple attractors). These findings point out that the bifurcation diagram of the cubic map covered in [1] is likely to unveil many surprising details on the underlying physics of Feynman diagrams.

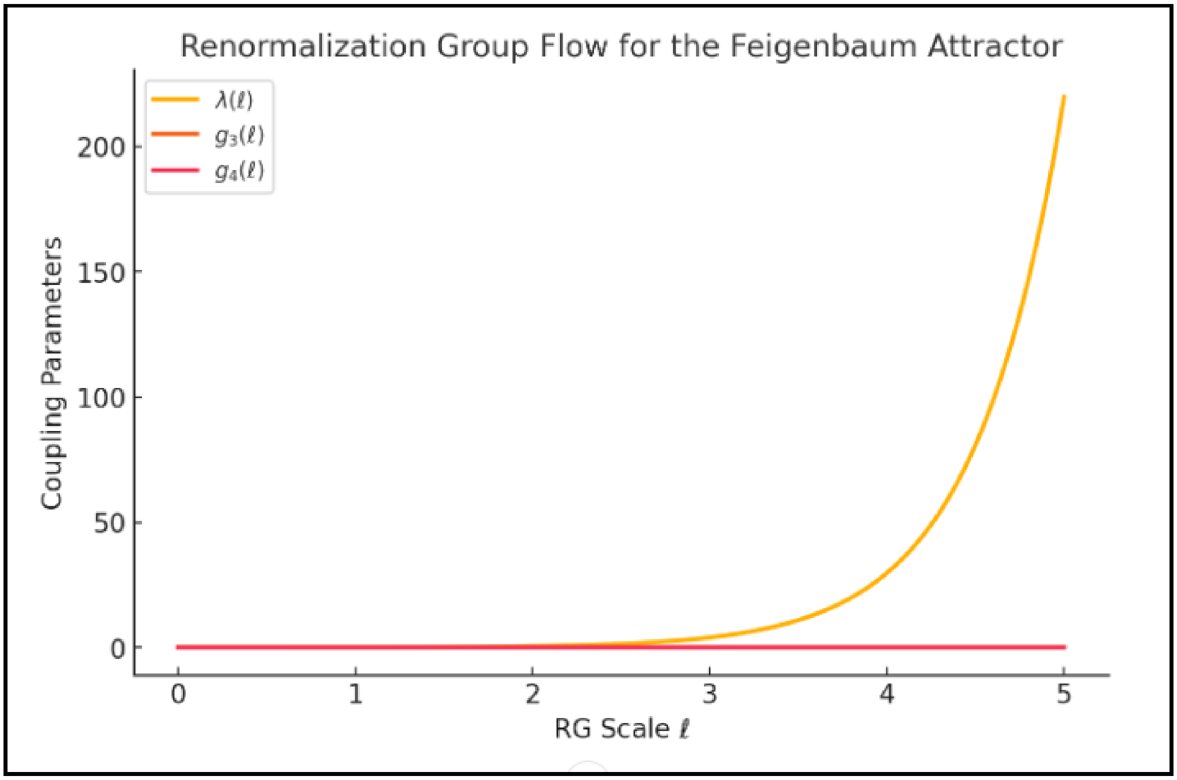


Fig. 3 RG plot of coupling parameters for the quadratic map

6. Summary

Our exploration provides a baseline for studying the link between the *Feigenbaum path to chaos* and renormalization in nonlinear dynamics, on the one hand, and *Feynman diagrams*, on the other. We believe that, in connecting QFT techniques with nonlinear maps, this modest contribution sheds unforeseen light on the common mathematical foundation of statistical physics and dynamical systems.

APPENDIX A: Feigenbaum Attractor of Low-Dimensional Unimodal Maps

The *Feigenbaum attractor* reflects the onset of chaos after an infinite sequence of period-doubling bifurcations. It is a *strange attractor* with a non-integer fractal dimension. It exhibits self-similarity and universal properties across different chaotic systems. At the Feigenbaum attractor, the system exhibits an aperiodic but nested structure of points, where zooming in shows a self-repeating pattern.

APPENDIX B: Feigenbaum Fixed Point and the Renormalization Approach to Period Doubling Bifurcations

Consider a family of one-parameter unimodal maps (1) or (2). As parameters $r_{1,2}$ ramp up, the system undergoes bifurcations in which stable periodic points double their period ($2^k, k = 1, 2, 3...$), eventually leading to fully developed chaos at r_∞. The *renormalization approach* to period-doubling bifurcations considers a transformation R whose action on the families of maps (1) or (2) is described by

$$R[f](x) = \alpha f(f(x/\alpha)) \tag{B1}$$

with α a is the second Feigenbaum constant. The *fixed-point function* represents the limit of (B1) and is defined as

$$g^*(x) = \alpha g^*(g^*(x/\alpha)) \tag{B2}$$

The meaning of (B2) is that, under repeated application of the renormalization operator R, various unimodal functions *converge to the same outcome* $g^*(x)$, regardless of the original form of the function.

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