The Overlooked Geometric Conflict Between Special Relativity and Sommerfeld's 1909 Spherical Model

Aswan Korula

March 10, 2025

Abstract

Special Relativity (SR) assumes that velocity vectors, displacement vectors and direction vectors coincide, since they are projected onto a flat analytical space (Minkowski space) to interpret inertial relative motion. In contrast, Arnold Sommerfeld's 1909 spherical model interprets velocity composition on a spherical surface, where displacement and constant direction vectors are fundamentally distinct. This paper demonstrates that because Einstein and Sommerfeld use different surfaces to project their interpretation of the Michelson-Morley interferometry, their predictions for relativistic effects diverge. Specifically, while Einstein's model necessitates Lorentz contraction and time dilation, Sommerfeld's spherical formulation predicts no such distortions. Notably, compatibility between SR and Sommerfeld's spherical model exists only in the special case of two right-angled triangles. By extension, any hyperbolic model that claims compatibility with SR must also be restricted to this special case. This work highlights a fundamental geometric conflict between Special Relativity and Sommerfeld's alternative formulation, warranting further examination of the geometric nature of inertial motion.

Keywords— Michelson-Morley, special relativity, Sommerfeld, hyperbolic geometry

1 Introduction

Special relativity [1], formulated by Albert Einstein in 1905, fundamentally altered our understanding of space, time, and motion. It provided a framework where the speed of light remains constant in all inertial frames, leading to counter-intuitive but experimentally verified concepts like time dilation and length contraction. A contemporary work by Sommerfeld, *On the Composition of Velocities in the Theory of Relativity* [2] was published just four years after Einstein. In his short paper Sommerfeld examines the problem of relative velocity composition from the perspective of spherical trigonometry.

Restricting himself at first to two congruent right spherical triangles (shown later to be equivalent to fig. 1 $\triangle ABQ$ and $\triangle CBQ$), Sommerfeld arrives at the spherical equivalent of Einstein's addition theorem [3] for velocities. Proceeding further, they invoke the cosine rule of spherical trigonometry to present a general solution to all triangles of the form $AB'C$ (refer fig. 1). Sommerfeld summarizes, "For the composition of velocities in the theory of relativity, not the formulas of the plane, but the formulas of the spherical trigonometry (with imaginary sides) are valid. By this remark the complicated transformation calculus becomes dispensable, and can be replaced by a lucid construction on a sphere" [2]. Sommerfeld's work leads Varicak to coin the term *rapidity* in 1910 thereby extending Sommerfeld's derived compatibility with special relativity into the hyperbolic geometry of Lobachevsky and Bolyai [4].

The aim of this paper is to conduct an in-depth theoretical re-visitation of the paradigm shifting Michelson-Morley (MM) experiment. We will interpret the MM geometry using the spherical model and compare Sommerfeld's predictions with Einstein's paradigm. This investigation into inertial motion will show that Einstein and Sommerfeld differ on the properties of the vector that each chooses to manipulate in their analysis of the MM geometry. We will show that Sommerfeld's interpretation of inertial motion is a geometry of shortest distance paths and is distinct from Einstein's geometry of inertial motion employing velocity vectors on a flat surface. This fundamental disagreement on the properties of inertial motion leads to conflict between Sommerfeld's solution and the predictions of special relativity.

2 Euclidean Geometry

On a flat surface [1], we draw any angle θ at origin Q bounded by two equal length line segments $QB = QB' = h$. We join points B and B' to points A and C such that the line segment AC is perpendicular to QB and centred at Q. Fig. 1 illustrates.

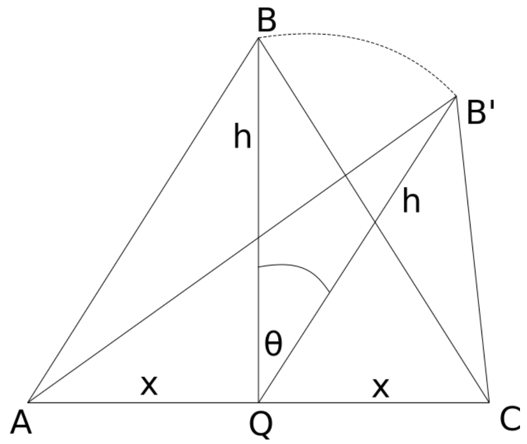

Figure 1: Triangles ABC and $AB'C$ rendered on a flat surface.

From fig. 1, we posit the following:

1. If $x > 0$, physical measurements will verify the theoretical statement $AB + BC \neq AB' + B'C$ remains true for all $\theta \neq 0, \pi, 2\pi...$
2. Independent of x and keeping h constant, if line segment QB' is rotated over π radians about point Q then curve BB' will take the form of a circle of radius h about point Q.
3. If $x > 0$, physical measurements will verify the theoretical statement $\angle AB'Q \neq \angle QB'C$ remains true over all $\theta \neq 0, \pi/2, \pi...$

3 A Template of the MM Experiment

Now we turn to theoretical aspects of relativistic optical interferometry to demonstrate that the geometry and sequence of events within an MM interferometer always templates to that of fig. 1.

3.1 Frames of Reference

Consider two imaginary euclidean reference frames that are in relative motion with respect to each other. Let us arbitrarily assume one of these frames is at rest and the other moves with some velocity v with respect to the rest frame. Accordingly we refer to fig. 1 and declare,

1. A rest frame I_0 centered at point Q.
2. A moving frame I_1 that translates from point A to point C with some velocity v relative to rest frame I_0.

3.2 Geometry and Sequence of Events

Now let us consider the structure of an MM interferometer [5](see fig. 2). By fixing $\angle B_1'QB_2' = \pi/2$, line segments QB_1' and QB_2' form the arms of the interferometer. Mirrors B_1 and B_2 are aligned perpendicular to their respective arms. The apparatus may be rotated about its source and consequently each arm subtends its own angle θ_i measured from a perpendicular to line segment AC. Let us affix moving frame I_1 to the source of the interferometer. Now let us imagine this interferometer moving through space under inertial rules such that,

1. v remains constant ($AQ = QC$).
2. The interferometer orientation (θ_i) with respect to line segment AC remains constant.

Reference frame I_1 (affixed to the source) translates with constant velocity v from point A to point C. From the perspective of the rest frame I_0, a discrete event cycle begins with the source at point A marking the simultaneous emission of a pair of photons (wavelength=λ). As the entire apparatus moves with some constant ($AQ = QC$) velocity v relative to origin Q along line segment AC, the photons are emitted at point A, reflect from mirrors B_1 and B_2 to finally arrive simultaneously (in phase with each other) at point C. This geometry and sequence of events remains true over all possible orientations θ of an MM interferometer [6] and over all $0 \le v < c$ where c represents the velocity of light in free space [7].

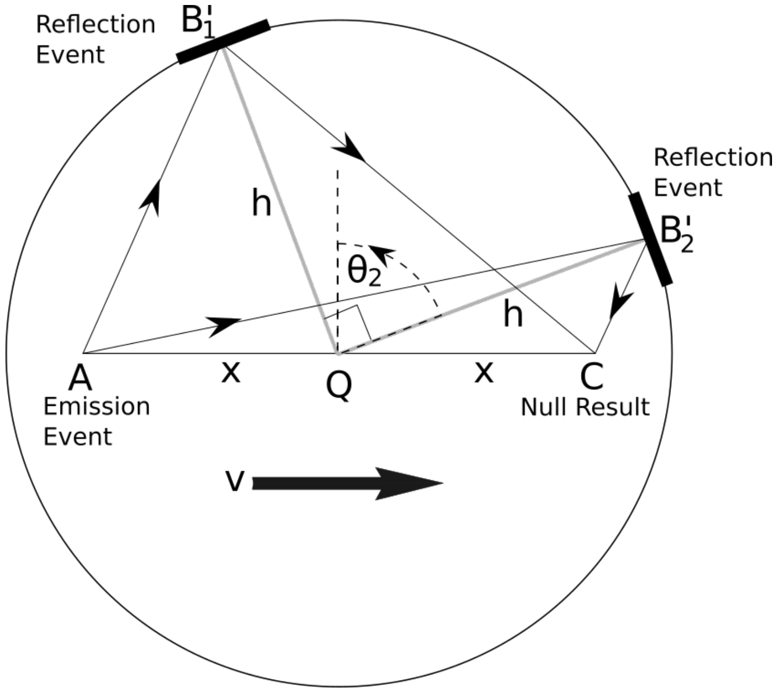

Figure 2: Geometry of the Michelson-Morley experiment depicting the general case $v \neq 0$ and $\theta_i \neq 0, \pi/2, \pi....$ Point Q is chosen as the origin. Only the events within the interferometer that are relevant to relativistic discussion are shown. Independent of the orientation of the interferometer, rest frame I_0 will find triangle $AB_i'C$ is a generalisation of triangle $AB'C$ in fig. 1. Identical to fig.1, physical measurements of the geometry of events will confirm that $AB_i'+B_i'C \neq AB_j'+B_j'C$ for all $\sin\theta_i \neq \sin\theta_j$ (inequality in path lengths) and $\angle AB'Q \neq \angle QB'C$ (inequality in angles of incidence and reflection) for all $\theta_i \neq 0, \pi/2, \pi...$ By setting $v = 0$ ($x = 0$), the figure represents the observational perspective of moving frame I_1. By setting $v > 0$ ($x > 0$), the figure represents the observational perspective of rest frame I_0. It is evident from fig. 1 that curve BB' will take the form of a stationary circle of radius h about point Q independent of θ_i (i.e. orientation) and v (i.e. frame of reference).

4 MM Geometry in Sommerfeld's Spherical Model

4.1 Inertial Motion in Euclidean Space

It may be seen from fig. 1, that when events are drawn on an unbounded flat surface, the velocity vectors v and the displacement vectors x are colinear and both vectors are compatible with the properties of inertial motion i.e. nil acceleration. In flat space, velocity vectors are compatible with:

1. Constant displacement over time along a path of shortest distance
2. Constant sense of direction over time

Figure 1 is the drawing basis [1] for Einstein's special theory of relativity.

4.2 Inertial Motion on Spherical Surfaces

In his consideration of the problem, Sommerfeld [2] instead chooses to project the MM geometry onto a spherical surface. In this interpretation of space, constant direction vectors and constant displacement vectors *are not* equivalent. Constant displacement vectors (shortest distance paths between two points) are drawn as arcs of great circles on the sphere and constant direction vectors are drawn as *surface* tangents (i.e. instantaneous arcs of small circles, known to navigators as rhumb lines [8]) to these great circles. This crucial difference in the spatial properties of direction and displacement vectors arises from the shape of the sphere and the geometry of the lines of meridian that drawn on its surface. To measure a sense of direction, the meridians on a sphere are analogous to parallel lines on a flat surface. At any point along a shortest distance path, an instantaneous sense of direction is measured by the angle bounded by the local meridian and the *surface* tangent to the great circle path at that point. When extended indefinitely on a sphere, a path of constant displacement will return to its origin whereas a path of constant direction will eventually spiral into the nearest pole [8] forming a non-inertial track and requiring some form of constantly changing acceleration input directed toward the pole to maintain this track. In contrast to Einstein's flat geometry, the notions of a constant sense of direction and a shortest distance path are both conceptually and mathematically distinct from each other when projected onto a bounded spherical surface [9] and only the displacement vectors retain conformity with inertial motion i.e. nil acceleration.

Sommerfeld recognises this distinction between great circle displacement (shortest distance path) and tangential velocity (constant direction) vectors and advises that if our discussion is restricted to inertial conditions and projected onto a spherical surface, the constant direction vectors should be ignored since they represent non-inertial tracks. Sommerfeld thus interprets the MM problem using great circle arcs rather than their tangents, stating, "it apparently better corresponds to the meaning of the theory of relativity, to calculate and (by consideration of the reality relations) to construct by rotation angles, **instead of only using its tangents, the velocities**" [2]. Sommerfeld's invocation of great circle trigonometry (i.e. the cosine rule) confirms that from a mathematical perspective, the spherical solution is a model of shortest distance vectors not constant direction vectors.

4.2.1 Agreement With Einstein's Addition Theorem

In his 1909 paper, Sommerfeld demonstrates agreement between the spherical model and Einstein's 1905 special relativity by arriving at a common velocity addition formula [2][1]. But the derivation of Sommerfeld's addition formula originates in two congruent right spherical triangles (see $\triangle ABQ$ and $\triangle CBQ$ in fig. 1). It is only in this special case i.e. a pair of congruent right spherical triangles that Einstein (velocity vector) and Sommerfeld (displacement vector) will agree with each other. This is because if two great circles intersect at a right angle then one forms the local line of meridian and the other forms the local equator. In this special configuration, constant direction vectors and constant displacement vectors are equivalent leading to agreement between Einstein and Sommerfeld. This conceptual and theoretical agreement between flat space and spherical space does not extend to the general case i.e. triangles of the form $AB'C$ (refer fig. 1) where constant direction vectors take on the non-inertial properties argued above.

4.3 Spherical Trigonometry

With these considerations in mind, let rest frame I_0 project fig. 1 onto the surface of an imaginary sphere of arbitrary radius R such that the shortest distance path between any two points are described by great circles on the sphere [9]. Thus the magnitude of physical distances $x, h, AB', B'C$ in fig. 1 are measured analytically as rotations in radians subtended at the centre of this sphere. The angles depicted in fig. 1 are measured on the surface of the sphere and curve BB' takes the form of a small circle on the surface of this sphere having radius h radians and centred at point Q. Since Sommerfeld has already provided the cosine rule as a solution to all triangles of the form $AB'C$, let us generalise further and invoke instead the sine rule of spherical trigonometry to see where it leads us.

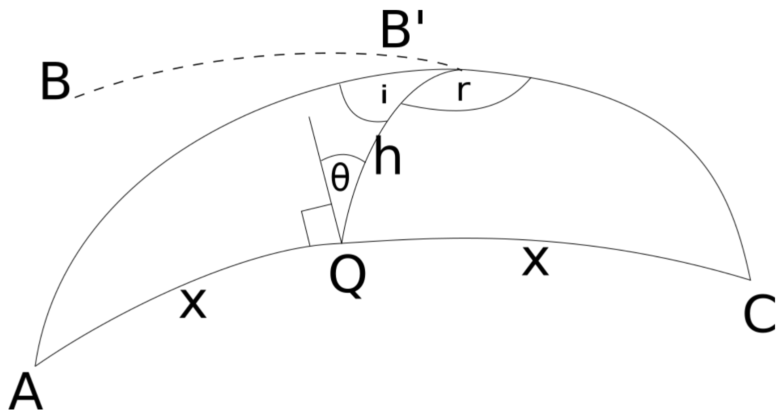

Figure 3: Spherical Trigonometry. Angles $h, x, AB', B'C$ are measured analytically at the centre of an imaginary sphere. Angles i, r, A, C, θ are measured analytically on the surface of the sphere.

4.4 Analysis of Spherical Model

From fig. 3 and the rule of sines for spherical triangles [10], rest frame I_0 finds in $\triangle AB'Q$:

$$\frac{\sin AB'}{\sin(\pi/2+\theta)} = \frac{\sin h}{\sin A} = \frac{\sin x}{\sin i} \tag{1}$$

where $i = \angle AB'Q$. Similarly for $\triangle CB'Q$:

$$\frac{\sin CB'}{\sin(\pi/2-\theta)} = \frac{\sin h}{\sin C} = \frac{\sin x}{\sin r} \tag{2}$$

where $r = \angle CB'Q$.

From equations, 1 and 2 rest frame I_0 finds in all spherical triangles of the form $AB'C$:

$$\frac{\sin(AB')}{\sin(CB')} = 1 \tag{3}$$

From eq. 3, rest frame I_0 finds AB' and CB' are supplementary angles of each other i.e. always summing to π radians. Referring now to fig. 2, eq. 3 guarantees that by interpreting the MM null result geometry with this analytical approach, rest frame I_0 is assured the theoretical statement $AB'_i + B'_iC = AB'_j + B'_jC$ remains true independent of x, h, θ. Thus the paradox of unequal path lengths presented by physical measurements of fig. 1 vanishes independent of frame of reference x_i or orientation of the interferometer θ_i and significantly, the MM geometry is retained *verbatim*.

Thus rest frame I_0 finds that Sommerfeld's model, when applied to the MM problem predicts no distortions in space and time and retains the circularity of curve BB'. These predictions are in sharp contrast to Einstein's special relativity which mandates distortions in space and time as a function of the velocity vector v,

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \tag{4}$$

Einstein's solution is restricted to the domain $0 \leq v < c$ whereas Sommerfeld's solution is valid over all $0 \leq x/h < \infty$.

5 Conclusion

Sommerfeld's title "*On the **Composition of Velocities** in the Theory of Relativity*" may inadvertently lead the reader to an interpretation that differs from the work's mathematical content i.e. a method of composition of shortest distance paths by projecting them onto a sphere. Mathematically, employing great circle trigonometry ultimately leads to conflict with special relativity's employment of velocity vectors except in the special case of two right congruent triangles. This conflict between Sommerfeld and Einstein arises from the non-inertial properties of constant direction vectors when these are drawn on a spherical surface. By extension, this fundamental incompatibility with special relativity must also infect other models that conjure the imaginary counter-image of spherical geometry i.e. hyperbolic geometry such as Varicak cited above.

6 Statements and Declarations

The author has no competing interests to declare that are relevant to the content of this article. There are no data associated with this article.

References

[1] A Einstein. On the electrodynamics of moving bodies. *Annalen der physik*, 17(10):891–921, 1905.

[2] A. Sommerfeld. On the composition of velocities in the theory of relativity. *Relativtheorie, Verh. der DPG*, 21:577–582, 1909.

[3] A. Einstein. *Relativity The Special and General Theory*, page 33. 1920.

[4] V Varićak. On the relativistic interpretation of so-called hyperbolic geometry. *Annual Report of the German Mathematical Association*, 19:211–248, 1910.

[5] C.R. Burnett, J.G. Hirschberg, and J.E. Mack. Diffraction and interference. In *Handbook of Physics*, pages 6–91. McGraw Hill Book Company Inc., 1958.

[6] Ch Eisele, A Yu Nevsky, and S Schiller. Laboratory test of the isotropy of light propagation at the 10- 17 level. *Physical Review Letters*, 103(9):090401, 2009.

[7] J. Walker, D. Halliday, and R. Resnick. *Principles of Physics, 10 Ed., International Student Version*, page 393. Wiley India Pvt Ltd, 2016.

[8] The Sea Lords of the Admiralty. *Admiralty Manual of Navigation Volume 2, Revised*, page 1. Her Majesty's Stationary Office, London, 1960.

[9] The Sea Lords of the Admiralty. *Admiralty Manual of Navigation Volume 2, Revised*, page 283. Her Majesty's Stationary Office, London, 1960.

[10] I Todhunter. *Spherical Trigonometry For the Use of Colleges and Schools, 5th edition*, page 21. MacMillan and Co., 1886.