On Area Element in Polar and Volume element in Spherical Coordinates

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Abstract

A simple and elementary derivation for the formula for the area element in polar coordinates, and the volume element in spherical coordinates is given.

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1 Introduction

To convert a double integral from Cartesian coordinates to polar coordinates, we have to convert the area element $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ to polar coordinates. For triple integral, the volume element $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ has to be converted to spherical coordinates. Most derivations of the formula, both in two dimensions [1, 2, 4, 5, 7, 8, 10] and in three dimensions [2, 3, 6, 9, 11, 12, 13], are either based on geometry or Jacobian (or tensor product).

This note first gives a self-contained and simple derivation for the area element without using geometry or Jacobian. Then, it is shown that the technique can also be extended for the volume element in spherical coordinates.

The conversion, in two dimensions, is also required, for example, for computing the integral:

$$\int_0^\infty e^{-x^2}\mathrm{d}x$$

1.1 Preliminaries

If P is a point in the plane with coordinates (x_P, y_P), then the distance of P from the origin, say r, will be the length of the segment OP, and $r^2 = (OP)^2 = x_P^2 + y_P^2$. Let us choose $x = r\cos\theta$, then as $\cos^2\theta + \sin^2\theta = 1$,

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y can be taken as $y = r\sin\theta$.

Coordinates (r, θ) are called polar coordinates.

Thus, to summarise, if polar coordinates are (r, θ), then Cartesian coordinates are

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

If we are using some other Cartesian coordinate system with the same origin, in which the coordinates of P are (x'_P, y'_P) then also $(OP)^2 = x'^2_P + y'^2_P$.

Let us assume that the second coordinate frame is at an angle φ with respect to the first. Then $\theta' = \theta + \varphi$, and $x' = r\cos(\theta + \varphi)$ and $y' = r\sin(\theta + \varphi)$.

Thus, $x' = r\cos(\theta + \varphi) = r\cos\theta\cos\varphi - r\sin\theta\sin\varphi = x\cos\varphi - y\sin\varphi$. Here we used $x = r\cos\theta$, and $y = r\sin\theta$.

Similarly, $y' = r\sin(\theta + \varphi) = r\sin\theta\cos\varphi + r\cos\theta\sin\varphi = y\cos\varphi + x\sin\varphi = x\sin\varphi + y\cos\varphi$.

Or in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix $R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ is the rotation matrix.

2 Area Element

Differentiating

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

We get

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}r\cos\theta - r\sin\theta\mathrm{d}\theta \\ \mathrm{d}y &= \mathrm{d}r\sin\theta + r\cos\theta\mathrm{d}\theta \end{aligned}$$

In matrix form, the equation can be written as

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\theta \end{pmatrix}$$

Or equivalently,

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ r\mathrm{d}\theta \end{pmatrix}$$

As the square matrix is a 2-D rotation matrix, and as areas do not change under rotation the area element $\mathrm{d}A = (\mathrm{d}r)(r\mathrm{d}\theta)$.

3 Volume Element

In 3-dimensions, if r if the distance of point $P = (x, y, z)$ from origin, then $r^2 = x^2 + y^2 + z^2$. If $z = r\cos\theta$ then $x^2 + y^2 = r^2\sin^2\theta$. As in 2-d, $x = (r\sin\theta)\cos\varphi$ and $y = (r\sin\theta)\sin\varphi$. Thus,

$$\begin{aligned}
\mathrm{d}x &= \sin\varphi\cos\theta\mathrm{d}r + r\cos\varphi\cos\theta\mathrm{d}\varphi - r\sin\varphi\sin\theta\mathrm{d}\theta \\
\mathrm{d}y &= \cos\varphi\cos\theta\mathrm{d}r + r\cos\varphi\sin\theta\mathrm{d}\varphi + r\sin\varphi\cos\theta\mathrm{d}\theta \\
\mathrm{d}z &= \cos\varphi\mathrm{d}r - r\sin\varphi\mathrm{d}\varphi
\end{aligned}$$

In matrix form,

$$\begin{aligned}
\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}x \end{pmatrix} &= \begin{pmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\varphi \\ \mathrm{d}\theta \end{pmatrix} \\
&= \begin{pmatrix} \sin\varphi\cos\theta & \cos\varphi\cos\theta & -\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi\sin\theta & \cos\theta \\ \cos\varphi & -\sin\varphi & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ r\mathrm{d}\varphi \\ r\sin\varphi\mathrm{d}\theta \end{pmatrix} \\
&= \begin{pmatrix} 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ r\mathrm{d}\varphi \\ r\sin\varphi\mathrm{d}\theta \end{pmatrix} \\
&= \begin{pmatrix} 0 & R(\theta) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} R(\varphi) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ r\mathrm{d}\varphi \\ r\sin\varphi\mathrm{d}\theta \end{pmatrix}
\end{aligned}$$

Here, $R(\theta)$ and $R(\varphi)$ are 2-dimensional rotation matrices. As rotation does not change area or volume, the volume element $\mathrm{d}V = r^2\sin\varphi\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r$ (corresponding to $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ in Cartesian coordinates).

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