Gravity and black holes

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Abstract

The question of how Newton's inverse-square law of gravity relates to general relativity (GR) is discussed in this paper. In GR, gravity is considered as a consequence of space and time curvature, whereas Newton's law is restricted to a flat space. Logically, then, Newtonian gravity must relate solely to the time curvature contribution in GR. Instances where Newton's law does not describe phenomena correctly, such as the perihelion rotation of the planet Mercury and the bending of starlight, are therefore attributable to spatial curvature. The GR solution for a static point mass, calculated on this basis for correspondence with Newton's law, is entirely regular and agrees with all the usual predictions of GR except the one leading to an event horizon. This suggests that the currently accepted model of a static black-hole, although mathematically possible, is non-physical. Not only is there no horizon in spacetime, but gravitational attraction between two masses does not diverge to infinity as they approach each other. This means there is no singularity at the origin of coordinates where physical laws would break down, and relative speeds do not exceed the speed of light.

1 Introduction

The current status quo amongst scientists is that black holes are ubiquitous in the universe. Matter collapses in on itself due to gravitational attraction, and an astronomical object, such as a very compact star, becomes a black hole with its mass contained within a so-called event horizon. Spacetime is then curved so much that nothing can escape from within that event horizon, not even light. Furthermore, at the centre of the black hole is a singularity where the laws of physics break down. Since the year 2016, observational evidence for such objects has been claimed both from gravitational wave signals, thought to emanate from the coalescence of spinning black holes [1], as well as observations of images from galactic centres, where there is thought to be a supermassive black hole [2]. The purpose of this paper is to question some of this.

2 Background

Black holes were given a compelling theoretical foundation a century ago through application of Albert Einstein's theory of general relativity GR [3] to the problem of finding the gravitational field due to the curvature of spacetime near a static point mass (Schwarzschild, 1916 [4]). A metric line element in a 4D spacetime with spherical spatial symmetry around a point mass may be written in its most general form as:

$$d\tilde{s}^2 = c^2 dt'^2 = A(r)\, c^2 dt^2 - B(r)\, dr^2 - C(r)(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (1)$$

where (r, θ, ϕ) are spherical polar coordinates and t is time, $d\tilde{s}$ is a spacetime increment, c the speed of light, dt' an increment of proper time, dt an increment of coordinate time, and dr an increment of radial coordinate distance. A, B and C are radially dependent metric coefficients describing the curvature of time, radial and angular metric coefficients, respectively. If the spacetime were flat, A and B would be unity and C equal to r^2. In a curved space, the radial coordinate r is the radial distance of a point from the coordinate origin viewed from an infinitely long distance from the mass causing curvature, or if the distance were measured using a hypothetically rigid or non-deformable ruler.

The calculus of variations is applied to the metric in a standard way to obtain the geodesic equations, from which the Christoffel curvature coefficients are obtained and used to find the Ricci tensor components. These are then all set to zero to satisfy Einstein's field equations of

GR for the vacuum outside the point mass. This procedure leads to two independent equations for the three variables, A, B and C, which means they cannot be solved explicitly, but only deliver a relationship between the three quantities in terms of each other.

To overcome this problem, the metric is "simplified" by writing it as:

$$d\tilde{s}^2 = c^2 dt'^2 = A(\tilde{r})\, c^2 dt^2 - B(\tilde{r})\, d\tilde{r}^2 - \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (2)$$

where the angular curvature $C(r)$ has been replaced by $\tilde{r}^2$. Thus, instead of the solid angle being curved in a radial direction, the space-time geometry fixes it to be the same as in a flat space, and this is then equivalent to introducing a new radial coordinate that I have denoted in Equation 2 as $\tilde{r}$, in order to distinguish it from r in Equation 1.

The advantage of this substitution is that Einstein's field equations of GR can then be used to find A and B explicitly in terms of $\tilde{r}$, giving the well-known result:

$$A(\tilde{r}) = \frac{1}{B(\tilde{r})} = 1 - \frac{\alpha}{\tilde{r}} \qquad (3)$$

where α is a constant of integration.

The idea of a black hole now becomes apparent, since $A(\tilde{r})$ changes sign if $\tilde{r}$ passes through α, while the reciprocal function $B(\tilde{r})$ becomes infinite, as well as changing sign at $\tilde{r} = \alpha$. A hypothetical sphere of radius α is now imagined called the event horizon, centred at $\tilde{r} = 0$, where it is tacitly assumed the point mass is situated. The radius of this sphere is referred to as the Schwarzschild radius or gravitational radius, and the coordinates used in Equation 2 $(t, \tilde{r}, \theta, \phi)$ are commonly called Schwarzschild coordinates. However, it is important to realise that $\tilde{r}$ is not the same as r in the general metric of Equation 1, and therefore the coordinate distance $\tilde{r}$ in the solution of Equation 3 is not an exact measure of the "undeformed ruler" or true distance r from the mass at the origin (see, e.g. Rindler, [5]).

In his original solution, Schwarzschild [4] did realise that the metric coefficients can change sign to become negative, but he defined an auxiliary radial coordinate that prevented this happening, as he regarded a coordinate discontinuity of this nature as non-physical. Schwarzschild therefore did not predict black holes himself, even though current popular science often suggests he did. Einstein, too, did not believe black holes existed and thought the solution was a mathematical quirk. Soon after Schwarzschild tragically died at an early age, both Droste [6] and Weyl [7] published a different variant of the solution, essentially in which they avoided the discontinuity by limiting the range of $\tilde{r}$ to

$\infty > \tilde{r} > \alpha$. Subsequently, Hilbert [8] extended Droste and Weyl's solution into the region $\tilde{r} < \alpha$, where the functions $A(\tilde{r})$ and $B(\tilde{r})$ become negative. Hilbert's argument for extending the solution was essentially that GR is conceived as a generally covariant theory, meaning that a change of coordinates should not alter the physics of the situation, and so now, the black-hole solution introduced by Hilbert has come to be accepted as correct by the scientific community.

3 Correspondence with Newton's classical law

GR is essentially an abstract geometrical description of how space and time are curved by stress-energy. In order to obtain correspondence between GR and real physical quantities, it is customary to make use of Newton's law of gravity. This is conventionally achieved in the region of gravity where Newton's law holds and where curvature is considered to be small. I shall adopt a different approach here from most textbooks, one which I think is most illustrative.

Consider a test particle falling towards the point mass along a radial direction, starting a long way away with zero velocity. The geodesic equation in r (which I have not specifically derived here) obtained from the metric in Equation 1 is found to be given by:

$$\ddot{r} + \frac{A'}{2B}c^2\dot{t}^2 + \frac{B'}{2B}\dot{r}^2 - \frac{C'}{2B}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = 0$$

From this, the following expression for the radial equation of motion is obtained (setting $\dot{\theta} = \dot{\phi} = 0$):

$$\ddot{r} = -\frac{A'}{2B}c^2\dot{t}^2 - \frac{B'}{2B}\dot{r}^2 \tag{4}$$

where $\ddot{r} = d^2r/dt'^2$ is the proper acceleration, $\dot{r} = dr/dt'$ is the proper velocity, $\dot{t} = dt/dt'$, $A' = dA/dr$ and $B' = dB/dr$. Now writing the metric (Equation 1) in the form:

$$\dot{t}^2 = \frac{1 + B\dot{r}^2}{Ac^2} \tag{5}$$

where again $\dot{\theta} = \dot{\phi} = 0$, the equation of motion may be rewritten (eliminating $\dot{t}$) as:

$$\ddot{r} = -\frac{1}{2}\frac{A'}{AB}c^2 - \frac{1}{2}\left(\frac{A'}{A} + \frac{B'}{B}\right)\dot{r}^2 \tag{6}$$

The same mathematical form for the equation of motion is derived irrespective of which radial coordinate is used (r or $\tilde{r}$), since C does not appear in it, so the metric in Equation 2 gives us an equivalent equation to Equation 6 in terms of $\tilde{r}$. In any case, we shall consider the situation when gravity is weak, i.e. we are well away from any extreme effects that occur near a point mass, such that any difference between $\tilde{r}$ and r can be regarded as negligible. Now apply the GR solution in Equation 3 ($A = 1/B = 1 - \alpha/r$ with $\tilde{r} = r$) to Equation 6 and we have

$$\ddot{r} = -\frac{1}{2}A'c^2 = -\frac{1}{2}\frac{\alpha c^2}{r^2} \tag{7}$$

substituting $A' = \alpha/r^2$. This neatly appears to agree with the inverse-square form of Newton's gravitational law, which may be expressed as:

$$a = -\frac{GM}{r^2} \tag{8}$$

where a is the classical or absolute acceleration, G Newton's gravitational constant, M the mass causing the acceleration, and where we identify the proper quantities in GR with the corresponding classical quantities in Newton's law. This correspondence gives $\alpha = 2GM/c^2$. In doing so, this appears to confirm there is an event horizon at a radius α, since A and B appear to change sign at $r = \alpha$. However, that would be a flawed conclusion, since the analysis is only valid when gravity is weak and $r >> \alpha$. It is fundamentally incorrect to extrapolate this behavior to values of r of the order of α, where $\tilde{r}$ may differ significantly from r.

The feature that characterizes Newtonian behaviour is the strict inverse-square dependence of the acceleration on distance, i.e. it relates strictly to a flat or Euclidean space with the absence of spatial curvature. Thus, to obtain the correct correspondence between Newton's law and the equation of motion from the calculus of variations the metric coefficient for spatial curvature B must be set to unity in Equation 6, which gives:

$$\ddot{r} = -\frac{A'}{2A}\left(c^2 + \dot{r}^2\right) \qquad [B = 1] \tag{9}$$

This expression now describes Newtonian free-fall motion in the four-dimensional spacetime geometry of GR.

Solving this differential equation by using the asymptotic condition, $A = 1$ for $\dot{r} = 0$, we obtain

$$A = \left(1 + \frac{\dot{r}^2}{c^2}\right)^{-1} \tag{10}$$

and if we now set $\dot{r}$ equal to the Newtonian velocity of free-fall $\sqrt{2GM/r}$, we obtain

$$A(r) = \left(1 + \frac{\alpha}{r}\right)^{-1} \tag{11}$$

which gives $A \rightarrow 1$ for $r \rightarrow \infty$ and $A \rightarrow 0$ for $r \rightarrow 0$, where $\alpha = 2GM/c^2$ is the same quantity as before.

This expression for A is regular for all values of r and can be regarded as resulting from positing that Newtonian gravity is determined purely by the curvature of *time* in a Lorentzian four-dimensional space-time. The corollary to this is that deviations from Newtonian gravity, such as the anomalous perihelion rotation of the planet Mercury and the bending of starlight past the Sun, i.e. gravitational lensing effects generally, are caused by the additional effect of spatial curvature.

4 Back to *GR*

The solution $A = (1 + \alpha/r)^{-1}$ with $B = 1$ does not satisfy Einstein's *GR* field equations for the vacuum exactly, since it was derived to represent the conditions under which Newton's law applies, viz. with no space curvature. However, there is no reason to suppose that this time-curvature part of the solution does not remain valid under all circumstances, and then assume that deviations from Newton's law are due entirely to spatial curvature.

Comparing time curvature from the *GR* solution with that deduced from Newton's law, one obtains (with $A(\tilde{r}) = A(r)$):

$$1 - \frac{\alpha}{\tilde{r}} = \left(1 + \frac{\alpha}{r}\right)^{-1} \tag{12}$$

Rearranging this gives the following linear relationship between $\tilde{r}$ and r:

$$\tilde{r} = r + \alpha \tag{13}$$

This now solves the issue of how the Schwarzschild radial coordinate $\tilde{r}$ is related to the true radial coordinate r.

In addition, the spatial curvature functions are related by

$$B(r)dr^2 = B(\tilde{r})d\tilde{r}^2$$

which gives

$$B(r) = \left(1 - \frac{\alpha}{\tilde{r}}\right)^{-1} \left(\frac{d\tilde{r}}{dr}\right)^2 = 1 + \frac{\alpha}{r}$$

$A(r)$ and $B(r)$ thus remain regular for all values of the true radial coordinate distance, $\infty > r > 0$. $A(\tilde{r})$ and $B(\tilde{r})$ change sign at $\tilde{r} = \alpha$, but this does not represent a discontinuity in spacetime, since from Equation 13 the range of $\tilde{r}$ is restricted to $\infty > \tilde{r} > \alpha$. Thus, the extension made by Hilbert to $\tilde{r} < \alpha$ is invalid and the concept of a static black hole with an event horizon is false.

5 New law of gravitational attraction

Substituting the proposed GR solution $A = 1/B = (1 + \alpha/r)^{-1}$ into the equation of motion gives for the free-fall acceleration:

$$\ddot{r} = -\frac{1}{2}\,c^2 A' = -\frac{1}{2}\,\frac{c^2\alpha}{(r+\alpha)^2} \tag{14}$$

This expression shows classical Newtonian behaviour for $r \gg \alpha$ but deviates from inverse-square law behaviour for r of the order of α. For $r \to 0$ it reaches a constant value of $c^4/4GM$, rather than increasing to ∞. This means that the attractive force between two masses does not increase to infinity as masses approach each other closely. Qualitatively, the time curvature produces an ever-increasing attractive force, but as spatial curvature becomes important at small r, this produces a repulsive force that counteracts the attraction. There is therefore no singularity at the origin of coordinates where the laws of physics would break down.

The proper velocity of free-fall may be written:

$$\frac{\dot{r}^2}{c^2} = 1 - A$$

which gives

$$\dot{r} = c\sqrt{\frac{\alpha}{r+\alpha}} \tag{15}$$

This means that for $r \to 0$ the proper velocity $\dot{r}$ goes to the speed of light. On the other hand, Hilbert's solution predicts

$$\dot{r} = c\sqrt{\frac{\alpha}{r}} \tag{16}$$

which implies $\dot{r} \to c$ for $r \to \alpha$, and $\dot{r} \to \infty$ for $r \to 0$. In other words, whereas Hilbert's mathematical solution permits superluminal velocities, the regular solution presented here limits velocities to the speed of light.

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