THICK SEQUENCES

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Abstract

We call an integer sequence *thick* if the quotients formed from its terms are dense in the set of real numbers. To find thick sequences, we consider the geometric, Fibonacci, power, and prime sequences. We show that the sequence of primes is thick provided that a conjecture **DC-2** holds. **DC-2** says that certain pairs of linear Dirichlet conditions have infinitely many solutions. It is a weak form of Dickson's conjecture, which states that a finite system of linear Dirichlet conditions has infinitely many solutions and generalizes Dirichlet's well-known result on primes in arithmetic progressions. Also, we obtain partial results for the general thickness problem for an arbitrary sequence and look at heuristic evidence for the validity of **DC-2.** We conclude with a short list of problems for further research.

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1. INTRODUCTION

We are used to approximating real numbers with rational numbers. This can always be done because the set $\mathbb{Q}$ of rational numbers is dense in the set $\mathbb{R}$ of real numbers. Recall that a set A is dense in $\mathbb{R}$ if for any real number r, any open interval containing r also contains an element of A. Using progressively shorter intervals, we can surely find a sequence of terms from A that converges to r, hence approximate r with terms from A to as high an accuracy as we please.

For example, the ubiquitous irrational number π can be approximated by taking successive terms in the sequence of its decimal expansion: 3, 3.1, 3.14, 3.141, 3.1415, 3.14159, ... One of the earliest and simplest results on this subject was obtained by Dirichlet in 1842: if r is an irrational number, then there are infinitely many rational numbers p/q ($q > 0$) such that $|r - p/q| < 1/q^2$. That is, p/q approximates r to within the squared reciprocal $1/q^2$ of the denominator q. We can think of such p/q as an 'efficient' approximation of r in the sense that its accuracy is within the squared reciprocal of its integer denominator, so that larger denominators give better accuracy. One proof of Dirichlet's result uses the well-known pigeon-hole principle. A later result by Hurwitz replaces the $1/q^2$ in Dirichlet's result by $1/(\sqrt{5}\, q^2)$, with the constant $1/\sqrt{5}$ being the best possible. The proofs of these theorems can be found in standard books on number theory, such as [NT].

The rational numbers are quotients of the sequence of integers. We do not need the entire sequence of integers to form quotients that can approximate real numbers. For instance, it would suffice to form quotients from the set of even integers, or from the set of integers whose absolute values are greater than, say, 10^{12}. Are we able to approximate with even sparser sequences?

For simplicity, *we stipulate that henceforth all numbers in this paper are positive*, although our discussion can easily be extended to include negative numbers and zero. Let s be a sequence of natural numbers. We define the set of *sequence quotients* of s to be the set of all quotients p/q where p and q are terms of s. Without loss of generality, we can assume that p/q is in lowest terms. We call a sequence *thick* if the set of its sequence quotients is dense in the set $\mathbb{R}$ of (positive) real numbers. For example, the sequences of natural numbers, even natural numbers, and natural numbers $> 10^{12}$ are thick.

This paper is organized as follows. We proceed from simple, concrete examples to general results. Starting with geometric-type sequences, we find that these are, for the most part, not thick. We have better luck when we look at power-type sequences, which furnish many examples of thick sequences. Then we focus on the primes and prime-like sequences whose thickness can be proved by assuming a weak form of Dickson's conjecture. In the process, we obtain partial results for the *general problem* of determining whether an arbitrary sequence is thick. Finally, we give heuristic evidence that Dickson's conjecture is plausible and a list of problems for further study.

2. GEOMETRIC-TYPE SEQUENCES

Proposition 1. The geometric sequence $10^1, 10^2, 10^3, 10^4$, ... of positive integer powers of 10 is not thick.

Proof. Let $r > 0$ be any positive real number. If this sequence were thick, then $10^{p-q} = 10^p / 10^q$ (for some positive integers p and q) can be made arbitrarily close to r, that is, $10^{p-q} \sim r$. Taking logs of both sides, by continuity of the logarithmic function, we get $p - q \sim \log r$, that is, $p - q$ can be made arbitrarily close to $\log r$, However, take $r = 10^{1/2}$. Then $p - q$ can be made arbitrarily close to ½ = $\log r$. This is a contradiction because $p - q$ is always an integer.

Of course, there is nothing special about the number 10 in this problem; except for being a convenient logarithmic base. We could have used other bases for the function used to define the geometric sequence.

Theorem 1. The geometric sequence a^1, a^2, a^3, a^4, ... of positive integer powers of a, where a is a positive integer, is not thick.

Now let us consider the Fibonacci sequence, which begins 1, 1, 2, 3, 5, 8, ... with terms F_n ($n > 2$) obtained from the recurrence $F_n = F_{n-1} + F_{n-2}$. Writing the golden ratio as $\tau = (1 + \sqrt{5})/2$ and using the Binet approximation to the Fibonacci term F_n (cf. [GS], P. 70),

$$F_n \sim 1/\sqrt{5}\ \tau^n$$

(that is, F_n can be made arbitrarily close to $1/\sqrt{5}\ \tau^n$ by taking n sufficiently large), the argument of Proposition 1 can be applied to show that the sequence of Fibonacci numbers is not thick. The key idea is that Fibonacci numbers are ultimately (multiples of) powers of τ.

Theorem 2. The sequence of Fibonacci numbers is not thick.

We leave the proofs of the following and a few other corollaries as exercises for the reader. Hints to the solutions are provided throughout. The sequence of Lucas numbers begins with the terms 2, 1 and is defined by the same rule as the Fibonacci sequence.

Exercise 1. The sequence of Lucas numbers is not thick.

Hint to Exercise 1. Look for a Binet-style formula for the n-th Lucas number and proceed as in the case of the Fibonacci sequence.

3. POWER-TYPE SEQUENCES

With the sequence of squares, we encounter our first non-trivial example of a thick sequence.

Proposition 2. The sequence of squares 1, 4, 9, 16, 25, 36, ... is thick.

Proof. Let r be a positive real number contained in an open interval I. We need to show that there is a rational number p/q such that p^2/q^2 is in I.

By continuity of the function $f(x) = x^2$ at $x_0 = r^{1/2}$, there is an open interval J containing x_0 such that $f(J)$ is contained in I. By density of the rational numbers in $\mathbb{R}$, there is a (positive) rational number p/q such that p/q is in J. Hence, $f(p/q) = p^2/q^2$ is in $f(J)$, which is contained in I, as required.

We can use the line of reasoning of the previous proposition to conclude the following generalization and partial answer to the general thickness problem. From this, it follows that the sequence of cubes, fourth powers, and higher powers are also thick.

Theorem 3. Let the (positive) integer sequence $s = \{f(n)\}$ be defined for natural numbers n by the function $f(x)$ from the set of positive real numbers to itself. Suppose f is a continuous, injective (i.e. one-one) function of x for which $f(u\, v) = f(u)\, f(v)$, where u, v are any positive real numbers. Then the sequence s is thick.

(The assumption $f(u\, v) = f(u)\, f(v)$ of Theorem 3 states that f is a homomorphism with respect to multiplication. The theorem says that a positive integer sequence defined by a continuous injective homomorphism (with respect to multiplication) of the set of positive reals to itself must be thick.)

Proof. The proof of Proposition 2 can be modified to show this. Since f is injective, its inverse f^{-1} exists, hence, $f^{-1}(r)$ exists for any positive real number r. The assumption that f is a homomorphism with respect to multiplication implies that $f(p) = \mathrm{f}((p/q)\, q) = \mathrm{f}(p/q)\, f(q)$, so that $f(p/q) = f(p)/f(q)$ for positive integers p and q.

Let r be a positive real number contained in an open interval I. By continuity of f at $x_0 = f^{-1}(r)$, there is an open interval J containing x_0 such that $f(J)$ is contained in I. By density of the rational numbers in $\mathbb{R}$, there is a (positive) rational number p/q such that p/q is in J. Hence, $f(p/q) = f(p)/f(q)$ is a sequence quotient of s contained in $\mathrm{f}(J)$, which is contained in I, as required.

It is easy to check that, for a positive integer n, the function $f(x) = x^n$ satisfies the assumptions of Theorem 3. Hence, we can conclude the following.

Corollary 1. If n is a fixed positive integer, then the sequence $1^n, 2^n, 3^n, 4^n$, ... of n-th powers of the positive integers is thick.

Naturally, we would expect a sequence defined by a finite sum of positive integer powers to be thick.

Exercise 2. (i) Show that if the (positive) integer sequence $s = \{P(n)\}$ is defined by a polynomial P with degree $m \geq 1$, then s is thick.

(ii) Show that the sequence of triangular numbers $T_n = n(n + 1) / 2$ is thick.

Hints to Exercise 2. For part (i), let a be the leading coefficient of P. Let p and q be positive integers. Factor p^m from $P(p)$ to show that $P(p) \sim a\, p^m$, that is, $P(p)$ can be made arbitrarily close to $a\, p^m$ by taking p sufficiently large; similarly, $P(q) \sim a\, q^m$. Then $P(p)/P(q) \sim p^m / q^m$. Apply Corollary 1 to conclude the thickness of s. For part (ii), simply apply the result of part (i). (In general, we can show that a sequence of figurate numbers, of which triangular, square, and pentagonal numbers are examples, is thick.)

4. A "HYBRID" SEQUENCE

What about the sequence $s = \{f(n)\}$, where n is a natural number and $f(x) = x^x$? This sequence, which starts as $1^1, 2^2, 3^3, 4^4$, ..., is neither a geometric nor a power sequence, although it superficially resembles both. At time of writing, we do not have a proof of the thickness or non-thickness of s, although we make the following (tentative, though it is hoped, plausible) conjecture.

Conjecture 1. The sequence $s = \{f(n)\}$, where n is a natural number and $f(x) = x^x$, is not thick.

We present a heuristic argument for Conjecture 1. If s were thick, then $f(p)/f(q) \sim r$ for a given real number $r > 0$ and some positive integers p and q, so that $\ln f(p)/f(q) = p \ln p - q \ln q \sim \ln r$. The celebrated Prime Number Theorem gives a rough estimate of the size of the n-th prime number $P(n)$: $P(n) \sim n \ln n$, implying, $\ln r \sim P(p) - P(q)$. (This is the main heuristic step; here "~" means only approximate equality, with an error that does not necessarily approach 0 as n increases, hence would not suffice for a rigorous proof.) However, the right side of this approximate equality is an integer, whereas the left side can assume any real value. This is implausible and casts a long shadow of doubt on the assumption that s is thick.

5. PRIMES AND DICKSON'S CONJECTURE

Recall that Dirichlet's theorem states that if a and b are two coprime natural numbers, then there are infinitely many primes in the arithmetic progression $a\, n + b$, where n is a natural number. For example, if $a = 3$ and b = 4, then $3n + 4$ is prime for $n = 1, 3, 5, 9, 13$, ... Consider the following conjecture, which we call **DC-2.**

Conjecture 2. (**DC-2)** If a and c are natural numbers, then there are infinitely many natural numbers n such that $a\, n + 1$ and $c\, n + 1$ are prime.

Dirichlet's theorem says that the linear condition "$a\, n + b$ is prime" for coprime a, b is satisfied for infinitely many n. Dickson's conjecture **DC** generalizes this and says that a finite system of linear conditions in n is satisfiable for infinitely many values of n, unless there is a congruence condition preventing this. For our purposes in this paper, we only need to work with **DC-2**, a much weaker form of **DC**,

Theorem 4. DC-2 implies that the sequence of primes is thick.

Proof. Let $r > 0$ be a real number and I be an open interval containing r. Without loss of generality, we can assume that I contains only positive numbers. We need to show that there are primes p and q such that p/q is in I.

Since the set of rational numbers is dense in the set of real numbers, there is a rational number t that is contained in I. Write $t = a/c$ in lowest terms, where a and c are coprime positive integers. By **DC-2**, there are infinitely many n such that $a\,n + 1$ and $c\,n + 1$ are prime. Hence, we can obtain an infinite sequence s of rational numbers with prime numerator and prime denominator that converges to $\lim_n (a\,n + 1) / (c\,n + 1) = a/c = t$.

Since s converges to t, there is a positive integer M such that s_n is in I whenever $n \geq M$; in particular, s_M is in I. Write $s_M = p/q$ where p and q are primes. Then p and q are our required primes.

As a partial result for Problem 1, we note that Theorem 4 and its proof can be generalized as follows. A positive integer sequence s is called *prime-like* if for any natural numbers a, c, there are infinitely many natural numbers n such that $a\,n + 1$ and $c\,n + 1$ are terms in s.

Theorem 5. Any prime-like sequence is thick.

The reader is invited to tackle the following exercises.

Exercise 3. Assuming that the sequence of primes is thick, show that the (positive) integer sequence $s = \{\varphi(n)\}$, where $\varphi(n)$ is Euler's totient function, is thick.

Exercise 4. Redo Exercise 3 using the sequence $s = \{\sigma(n)\}$, where $\sigma(n)$ is the sum-of-divisors function.

Hints to Exercises 3 and 4. Recall that Euler's totient function $\varphi(n)$ is defined as the number of positive integers that are coprime to natural number n and $\leq n$. It is computed by the formula

$$\varphi(n) = n \prod_{p|n} (1 - 1/p)$$

The product in this formula is over the distinct prime factors p of n. The sum-of-divisors function $\sigma(n)$ is defined as the sum of the (positive) divisors of n; it is computed by the formula

$$\sigma(n) = \prod_{p|n} (p^{i+1} - 1) / (p - 1)$$

Again, the product in this formula is over the distinct prime factors p of n, and p^i is the highest power of p that divides n. Also, note that a sequence that contains a thick subsequence must also be thick. Show that the subsequences $s' = \{\varphi(p): p$ is a prime number$\}$ and $s'' = \{\sigma(p): p$ is a prime number$\}$ of the respective sequences s in the exercises are thick, hence, s must be thick.

6. HEURISTICS

Is the conjecture **DC-2** plausible? We employ the type of heuristic probabilistic argument motivated by the Prime Number Theorem: the probability that a natural number $n > 1$ is prime is roughly $1/\ln(n)$. If this reasoning is applied to Dirichlet's theorem on arithmetic progressions, then the expected number of primes of the form $a\,n$

+ b, where a, n, b are positive integers with a and b fixed and coprime, is approximately equal to

$$\int_{1}^{\infty} (\ln(ax + b))^{-1}\, dx$$

This diverges to ∞ (by comparison of the integrand $1/\ln(a\,x + b)$ with $1/(a\,x + b)$, the integrand of a divergent integral). Hence, we expect infinitely many primes of the form $a\,n + b$.

Moving on to **DC-2** and using the same heuristic reasoning, the probability that the linear Dirichlet conditions "$a\,n + 1$ is prime" and "$c\,n + 1$ is prime" are both satisfied is at least $1/(\ln N)^2$ where $N = \max\{a\,n + 1, c\,n + 1\}$, *assuming* the independence of these two events/conditions. If, say, we fix a, c such that $a < c$, then $N = c\,n + 1$, and the expected number of primes among these numbers N is approximately equal to

$$\int_{1}^{\infty} (\ln(cx + 1))^{-2}\, dx$$

Again, this diverges to ∞. (This can be seen by considering the integral of $1/(\ln(x))^2$ from 2 to ∞. By applying the substitution $w = \ln(x)$ then integrating by parts, the indefinite integral of $1/(\ln(x))^2$ can be shown to be $= \mathrm{Li}(x) - x\ln(x)$, where $\mathrm{Li}(x)$ is the so-called *logarithmic integral*, i.e. the integral of $x/\ln(x)$.) Thus, we expect $a\,n + 1$ and $c\,n + 1$ to be both prime for infinitely many n.

A similar (and more involved) heuristic argument supports the plausibility of Dickson's conjecture **DC**. (For example, consider the case with three Dirichlet conditions. By applying the substitution $w = \ln(x)$ then integrating by parts, the indefinite integral of $1/(\ln(x))^3$ can be expressed in terms of the previous integral for $1/(\ln(x))^2$. Again, the corresponding improper integral diverges to ∞.) However, since the argument for **DC** generally requires multiple independence assumptions rather than the single one made in the case of **DC-2**, the former seems to be on less solid ground than the latter.

As numerical supporting evidence, the sequence of terms $n < 200$ such that $2n + 1$ and $3n + 1$ are both prime begins:

2, 6, 14, 20, 26, 36, 50, 54, 74, 90, 116, 140, 146, 174

This is the sequence A130800 in the Online Encyclopedia of Integer Sequences [IS] (oeis.org). For $n < 10{,}000$, there are already 318 such numbers. As a more random-looking example, the sequence of terms $n < 300$ such that $500n + 1$ and $1001n + 1$ are both prime begins:

6, 8, 38, 42, 48, 102, 108, 138, 180, 186, 192, 242, 246, 252

For $n < 100{,}000$, there are already 1,761 such numbers. It is not hard to come by numerical evidence that **DC-2** is plausible.

The quality and amount of numerical evidence for the plausibility of **DC** is less clear. Looking at circumstantial evidence for **DC**, we find for example that the sequence of $n < 2500$ making $100n + 1$, $211n + 15$, and $303n + 17$ prime begins:

4, 28, 292, 628, 694, 778, 904, 1678, 1918, 2332, 2422

However, for larger coefficients and number of conditions, the computations can become prohibitively long and numerical evidence increasingly harder to find. There is no $n < 10^6$, for example, that makes $101n + 1$, $202n + 1$, and $303n + 111$ prime. But omitting the third condition, we find that there are already 73 numbers $n < 10{,}000$ such that $101n + 1$ and $202n + 1$ are prime.

7. OPEN PROBLEMS

Considering the difficulty of the proof of Dirichlet's theorem on primes in arithmetic progressions, it seems likely that if it is true, a formal proof of **DC**, or even just **DC-2,** will be at an equal or greater level of difficulty. We do not attempt a proof here but only pose the problem as a challenge to astute researchers.

In the same vein, we ask if there is some metric, perhaps related to sequence density, that allows us to discriminate between thick and non-thick sequences. We have seen that sequences such as the power and (apparently) the prime sequences are thick, while sparser sequences such as the Fibonacci and geometric sequences fail to be thick. Is there a threshold that, when crossed, forces a sequence to be thick? We note that, while we have partial results from Theorems 3 and 5, the general problem of determining whether a given sequence is thick remains unsolved.

Can we find seemingly dense sequences that fail to be thick, or apparently sparse sequences that are thick? For example, is there a non-trivial non-thick sequence s, such as the Fibonacci sequence, and a thick sequence t, such that $s(n) \leq t(\text{n})$ for all natural numbers n?

Apart from the primes, are there other non-trivial sequences that can plausibly be claimed to be prime-like? Sequences defined by some sieving procedure, such as S. Ulam's "lucky numbers", A000959 of [IS], spring to mind here.

What other interesting thick or non-thick sequences can we find? The factorials, primorials, Catalan numbers, abundant numbers, deficient numbers, perfect numbers, happy numbers, palindromic numbers, and many other sequences we have not considered in this work invite future investigation. The problem book [UP] is an excellent source of sequences of special interest.

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