SURFACE AREA OF THE MÖBIUS STRIP

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Abstract. The (half) area of the surface of the Möbius strip is the expected product of the length of the circular spine times the width of the sweep line times a positive correction factor. The manuscript writes down this factor as a Taylor series of the ratio of width over circle radius; it approaches one if that ratio approaches zero.

1. Incentive

The Guldin rule (Pappus' theorem) provide a formula for the surface generated by revolving a planar curve with known center of mass around a circle [1, (8.72)]. The naïve expectation is that the Möbius strip has an area equal to the product of length of a circular center line by the width. This manuscript corrects this hypothesis and evaluates a correction factor for this product.

2. Mathematical Model, Coordinates

We look at a Möbius strip of guide line radius R located in the $x-y-$plane with a paddle of width w staying with its middle at the guide line. A point on the guide line has the Cartesian coordinates

$$\begin{pmatrix} R\cos\lambda \\ R\sin\lambda \\ 0 \end{pmatrix} \tag{1}$$

parameterized by an azimuthal angle $0 \le \lambda \le 2\pi$. The tangent line to the circle points into the orthogonal direction

$$\begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}. \tag{2}$$

A point on the strip at a distance t to the guide line has a torsion angle θ relative to the $x-y$-plane, such that its z-coordinate is $t\sin\theta$ in the range $-w/2 \le t \le w/2$. This leaves the factor $t\cos\theta$ for the x and y coordinates. Since the paddle is obtained by rotation around the tangent (2), its direction must be orthogonal to that, so dispersion of the $t\cos\theta$ factor gives a paddle vector of

$$\begin{pmatrix} t\cos\theta\cos\lambda \\ t\cos\theta\sin\lambda \\ t\sin\theta \end{pmatrix}. \tag{3}$$

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Attaching it to the circle (1) gives the Cartesian coordinates of a point on the strip parameterized by λ and t:

$$\vec{r}(\lambda,t) = \begin{pmatrix} R\cos\lambda \\ R\sin\lambda \\ 0 \end{pmatrix} + \begin{pmatrix} t\cos\theta\cos\lambda \\ t\cos\theta\sin\lambda \\ t\sin\theta \end{pmatrix} = \begin{pmatrix} (R+t\cos\theta)\cos\lambda \\ (R+t\cos\theta)\sin\lambda \\ t\sin\theta \end{pmatrix}. \tag{4}$$

The principle of the definition now lets the torsion angle θ increase linearly with λ such that a point of constant t initially at

$$\vec{r}(0,w/2) = \begin{pmatrix} R+w/2 \\ 0 \\ 0 \end{pmatrix} \tag{5}$$

ends up at

$$\vec{r}(2\pi,w/2) = \begin{pmatrix} R-w/2 \\ 0 \\ 0 \end{pmatrix} \tag{6}$$

after one λ-rotation through the circle. This is achieved by setting

$$\theta = \lambda/2. \tag{7}$$

Continuous surfaces with larger numbers of twists as in Figure 1 can be constructed by selecting other positive integers k:

$$\theta = k\lambda/2. \tag{8}$$

Insertion into (4) defines a family of Möbius strips [2, 5]:

$$\vec{r} = \begin{pmatrix} (R+t\cos\frac{k\lambda}{2})\cos\lambda \\ (R+t\cos\frac{k\lambda}{2})\sin\lambda \\ t\sin\frac{k\lambda}{2} \end{pmatrix}. \tag{9}$$

3. Gaussian Parameters

Two tangential directions on the surface are constructed as the partial derivatives:

$$\frac{\partial\vec{r}}{\partial t} \equiv \vec{r}_t = \begin{pmatrix} \cos\frac{k\lambda}{2}\cos\lambda \\ \cos\frac{k\lambda}{2}\sin\lambda \\ \sin\frac{k\lambda}{2} \end{pmatrix}; \quad E = |\vec{r}_t| = 1; \tag{10}$$

$$\frac{\partial\vec{r}}{\partial\lambda} \equiv \vec{r}_\lambda = \begin{pmatrix} -\frac{tk}{2}\sin\frac{k\lambda}{2}\cos\lambda - R\sin\lambda - t\sin\lambda\cos\frac{k\lambda}{2} \\ -\frac{tk}{2}\sin\frac{k\lambda}{2}\sin\lambda + R\cos\lambda + t\cos\lambda\cos\frac{k\lambda}{2} \\ \frac{tk}{2}\cos\frac{k\lambda}{2} \end{pmatrix}. \tag{11}$$

These are orthogonal:

$$F = \vec{r}_\lambda \cdot \vec{r}_t = 0. \tag{12}$$

The cross product (direction of the surface normal, not of unit length) is

$$\vec{r}_t \times \vec{r}_\lambda = \begin{pmatrix} \frac{tk}{2}\sin\lambda - R\sin\frac{k\lambda}{2}\cos\lambda - t\cos\lambda\sin\frac{k\lambda}{2}\cos\frac{k\lambda}{2} \\ -\frac{tk}{2}\cos\lambda - R\sin\frac{k\lambda}{2}\sin\lambda - t\sin\lambda\sin\frac{k\lambda}{2}\cos\frac{k\lambda}{2} \\ (R+t\cos\frac{k\lambda}{2})\cos\frac{k\lambda}{2} \end{pmatrix}. \tag{13}$$

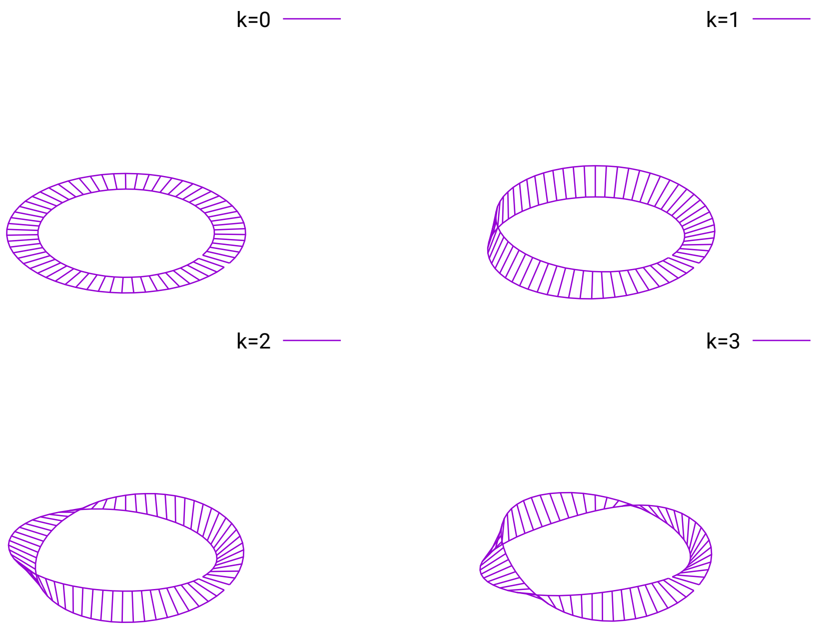

FIGURE 1. Möbius ribbons for twist numbers 0 to 3.

The length of the cross product is

$$|\vec{r}_t \times \vec{r}_\lambda| = |\vec{r}_\lambda| = \sqrt{G} = \sqrt{(R + t\cos\frac{k\lambda}{2})^2 + (\frac{tk}{2})^2}. \tag{14}$$

4. Area

The area is [6, (8.19)][1, (3.498b)]

$$\begin{aligned} A_k = \iint \sqrt{EG - F^2} d\lambda dt = \iint |\vec{r}_t \times \vec{r}_\lambda| d\lambda dt \\ = \int_0^{2\pi} d\lambda \int_{-w/2}^{w/2} dt \sqrt{(R + t\cos\frac{k\lambda}{2})^2 + (\frac{tk}{2})^2} \\ = R\int_0^{2\pi} d\lambda \int_{-w/2}^{w/2} dt \sqrt{(1 + \frac{t}{R}\cos\frac{k\lambda}{2})^2 + (\frac{tk}{2R})^2} \\ = \frac{wR}{2}\int_0^{2\pi} d\lambda \int_{-1}^{1} dx \sqrt{(1 + \frac{xw}{2R}\cos\frac{k\lambda}{2})^2 + (\frac{xwk}{4R})^2}. \end{aligned} \tag{15}$$

Remark 1. *Optionally one could multiply this by 2 to cover the 'back-side' area, i.e., to sweep this in the range* $0 \le \lambda \le 4\pi$.

The λ-integral leads to Elliptic integrals which we shall avoid here.

Remark 2. *The t-integral may be executed* [3, 2.262.1,2.262.2]

$$\int_{-w/2}^{w/2} dt\sqrt{R^2+2Rt\cos\frac{k\lambda}{2}+t^2\cos^2\frac{k\lambda}{2}+\frac{t^2k^2}{4}} \tag{16}$$
$$= \frac{(\cos^2\frac{k\lambda}{2}+k^2/4)t+R\cos\frac{k\lambda}{2}}{2(\cos^2\frac{k\lambda}{2}+k^2/4)}\sqrt{(R+t\cos\frac{k\lambda}{2})^2+\frac{t^2k^2}{4}}$$
$$+\frac{R^2k^2}{8(\cos^2\frac{k\lambda}{2}+2k^2)^{3/2}}\operatorname{arsinh}\frac{(\cos^2\frac{k\lambda}{2}+k^2/4)t+R\cos\frac{k\lambda}{2}}{kR/2}\,|_{t=-w/2}^{w/2}$$

but since this still leaves a pending λ-integration, this analysis is not continued from there.

Remark 3. *The case $k=0$ is the trivial planar hollow circle with $A_0=\pi[(R+w/2)^2-(R-w/2)^2]=2\pi wR$.*

The further strategy is to expand the square root in the kernel into a series of small w.

Definition 1.

$$\hat{w}=w/R \tag{17}$$

is the unitless ratio of the strip width by the radius of the backbone circle.

$$\sqrt{(1+\frac{xw}{2R}\cos\frac{k\lambda}{2})^2+(\frac{xwk}{4R})^2} \tag{18}$$
$$=1+\frac{x}{2}\cos\frac{k\lambda}{2}\hat{w}+\frac{x^2k^2}{32}\hat{w}^2-\frac{x^3k^2}{64}\cos\frac{k\lambda}{2}\hat{w}^3$$
$$+\frac{x^4k^2}{2048}(4\cos\frac{k\lambda}{2}-k)(4\cos\frac{k\lambda}{2}+k)-\frac{x^5k^2}{2096}\cos\frac{k\lambda}{2}(16\cos^2\frac{k\lambda}{2}-3k^2)\hat{w}^5+\cdots$$

and integration over λ and x is easy then. The terms with odd powers of x disappear while integrating because the $x-$limits are symmetric. And because $\hat{w}$ appears with the same power as x in each term, A_k is $2\pi Rw$ multiplied by an even function of $\hat{w}$.

5. Results

Insertion of the previous expansion into (15) and term-by-term integration over $-1\le x\le 1$ and $0\le\lambda\le 2\pi$ yields

$$A_1=2\pi wR\big[1+\frac{1}{96}\hat{w}^2+\frac{7}{10240}\hat{w}^4+\frac{25}{458752}\hat{w}^6+\frac{25}{25165824}\hat{w}^8$$
$$-\frac{2793}{2952790016}\hat{w}^{10}-\frac{53277}{223338299392}\hat{w}^{12}+\cdots\big] \tag{19}$$

There is an apparent discrepancy between this formula and the usual manual construction of a Möbius model which attaches two ends of a rectangular stripe of dimension $2\pi R\times w$ after bending/twisting. In fact the paper model does not keep the center line of the rectangular stripe on a planar circle; its 2-dimensional surface is even more complex than the mathematical model (4) [7, 4, 8].

No new aspect arises in the analysis if twist numbers $k \geq 2$ are computed—besides the fact that for even k the computed area is indeed the area of only one of two sides.

(20)
$$A_2 = 2\pi w R\left[1 + \frac{1}{24}\hat{w}^2 + \frac{1}{640}\hat{w}^4 - \frac{1}{3584}\hat{w}^6 - \frac{5}{98304}\hat{w}^8 + \frac{21}{1441792}\hat{w}^{10} + \frac{205}{27262976}\hat{w}^{12} + \cdots\right];$$

(21)
$$A_3 = 2\pi w R[1 + \frac{3}{32}\hat{w}^2 - \frac{9}{10240}\hat{w}^4 - \frac{783}{458752}\hat{w}^6 + \frac{4115}{8388608}\hat{w}^8 + \frac{267183}{2952790016}\hat{w}^{10} - \frac{28573965}{223338299392}\hat{w}^{12} + \cdots];$$

(22)
$$A_4 = 2\pi w R\left[1 + \frac{1}{6}\hat{w}^2 - \frac{1}{80}\hat{w}^4 - \frac{5}{1792}\hat{w}^6 + \frac{25}{6144}\hat{w}^8 - \frac{1533}{720896}\hat{w}^{10} - \frac{399}{6815744}\hat{w}^{12} + \cdots\right];$$

(23)
$$A_5 = 2\pi w R[1 + \frac{25}{96}\hat{w}^2 - \frac{85}{2048}\hat{w}^4 + \frac{1825}{458752}\hat{w}^6 + \frac{309625}{25165824}\hat{w}^8 - \frac{56366625}{2952790016}\hat{w}^{10} + \frac{3746147475}{223338299392}\hat{w}^{12} + \cdots].$$

6. Summary

The (quasi one-sided) surface area of the Möbius strip of width w with a planar guide line of radius R is given by (19), where (17) denotes the unitless ratio of the two main parameters.

Appendix A. Embedding

The parameters of the second quadratic fundamental normal form are listed here [1, (3.503c)][6, (8.26)]. The normal vector of the plane is

$$\vec{n} = \frac{1}{\sqrt{G}}\vec{r}_t \times \vec{r}_\lambda \tag{24}$$

The products of partial derivatives are

$$L = -\vec{n}_\lambda \cdot \vec{r}_\lambda = \frac{1}{\sqrt{G}}\sin\frac{k\lambda}{2}\left[(R + t\cos\frac{k\lambda}{2})^2 + \frac{t^2k^2}{2}\right]; \tag{25}$$

$$N = -\vec{n}_t \cdot \vec{r}_t = 0; \tag{26}$$

$$M = -(\vec{n}_\lambda \cdot \vec{r}_t + \vec{n}_t \cdot \vec{r}_\lambda)/2 = \frac{kR}{2\sqrt{G}}. \tag{27}$$

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