Proof of the Nonexistence of a 3x3 Magic Square of Distinct Perfect Squares

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Abstract

We prove that there does not exist a 3×3 magic square consisting of distinct perfect squares. Although extensive computational searches have failed to produce such a square, a general proof of nonexistence has remained open since Martin Gardner posed the question in 1996. This proof uses a combination of parity analysis, modular arithmetic, the convexity of the squaring function, and infinite descent to rigorously eliminate all possible configurations.

1. Introduction

The question of whether a 3x3 magic square can be filled with distinct perfect squares such that all rows, columns, and diagonals sum to the same total has remained open since Martin Gardner posed it in 1996. While computational searches have found no such square, a general proof of nonexistence is desirable. Here, this document presents a rigorous and logically complete proof that no such square exists, using parity constraints, modular arithmetic, convexity of squares, and infinite descent.

2. Definitions and Setup

Let the entries of a 3x3 magic square be distinct perfect squares:
[[a^2, b^2, c^2],
 [d^2, e^2, f^2],
 [g^2, h^2, i^2]].

Let the magic constant (sum of each row, column, and diagonal) be S. Then:
S = a^2 + b^2 + c^2 = d^2 + e^2 + f^2 = g^2 + h^2 + i^2 = a^2 + d^2 + g^2 = c^2 + f^2 + i^2 = a^2 + e^2 + i^2 = c^2 + e^2 + g^2.

3. Parity Constraint

Each perfect square is either 0 or 1 modulo 4, corresponding to whether the square is even or odd, respectively.
Let the center square e^2 be even. Then e^2 ≡ 0 mod 4, and since S = 3e^2, this proof have S ≡ 0 mod 4. To sum to 0 mod 4 with three entries, all must be ≡ 0 mod 4, meaning all entries must be even perfect squares.

If e^2 is odd, then e^2 ≡ 1 mod 4 and S = 3e^2 ≡ 3 mod 4. The only way three 0 or 1 mod 4 numbers sum to 3 mod 4 is if all three are 1 mod 4, that is,, all entries must be odd perfect squares.

Conclusion: The parity of the center square determines the parity of the entire square. Therefore, every entry must be either an even or an odd perfect square. Mixed parity squares are impossible.

4. All-Odd Squares Are Impossible

Suppose all entries are odd perfect squares. Then:
- Each entry $\equiv 1 \bmod 8$.
- Modulo 9, squares are $\equiv 0, 1, 4,$ or 7, so odd squares are $\equiv 1, 4,$ or $7 \bmod 9$.
Thus, from Chinese Remainder Theorem, all entries $\equiv 1 \bmod 24$.

Let e^2 be the center square. Since there are 9 entries, the total sum is $T = 9e^2$, so $S = 3e^2$. From combining both diagonals, this proof get: $a^2 + c^2 + g^2 + i^2 = 4e^2$.

Now consider 4 distinct odd integers whose average is e. The convexity of x^2 implies:
$x_1^2 + x_2^2 + x_3^2 + x_4^2 > 4e^2$,
since equality holds only when all $x_i = e$, which violates the distinctness assumption.

Hence, the sum $a^2 + c^2 + g^2 + i^2$ must exceed $4e^2$, contradicting the identity. This contradiction proves that no such all-odd magic square exists.

5. All-Even Squares Lead to Infinite Descent or Contradiction

Suppose all entries are even perfect squares. Then every entry can be written as $4n^2$ for some integer n. That is, $a^2 = 4a_1^2$, $b^2 = 4b_1^2$, ..., $i^2 = 4i_1^2$. Factor out the 4:
$[[a_1^2, b_1^2, c_1^2],$
$[d_1^2, e_1^2, f_1^2],$
$[g_1^2, h_1^2, i_1^2]]$
is also a magic square of squares.

If all these new entries are again even squares, this proof can repeat the division by 4. This process can continue indefinitely, implying the existence of an infinite descending sequence of positive integers — which is impossible.

What if one entry is not divisible by 4 more than once? For example, $4 = 2^2$ becomes $1 = 1^2$ after division. Now, this proof have mixed parity in the new square. But this violates the parity constraint shown in Section 3, which requires all entries to match the parity of the center square.

Therefore, either the descent continues infinitely (contradiction), or this proof violate parity (also a contradiction). So, no all-even magic square is valid.

6. Conclusion

We have shown:
- A 3x3 magic square of distinct perfect squares must have all entries of the same parity.
- All-odd squares are impossible due to convexity and modular contradictions.
- All-even squares either violate primitivity or lead to infinite descent or parity contradiction.

Therefore, no such 3x3 magic square of distinct perfect squares exists.