Wallis-type product formulas and associated Wallis integrals

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Abstract

Variants of the Wallis product formula are established using simplicial polytopic numbers. These are then used to represent the Wallis integrals.

Consider the following product:

$$\prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\cdot\frac{2n+r-1}{2n+r}\right) \quad \text{where } r \text{ is a positive integer.} \tag{1}$$

For $r=1$ we have the Wallis product formula, identified by John Wallis in 1656 [1]:

$$\prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots=\frac{\pi}{2}$$

For odd values of r, we can substitute $r=2k-1$, where k is a positive integer:

$$\prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\cdot\frac{2n+2k-2}{2n+2k-1}\right)=\frac{2}{1}\cdot\frac{2k}{2k+1}\cdot\frac{4}{3}\cdot\frac{2k+2}{2k+3}\cdot\frac{6}{5}\cdot\frac{2k+4}{2k+5}\cdots$$

For example, if $k=3$:

$$\frac{2}{1}\cdot\frac{6}{7}\cdot\frac{4}{3}\cdot\frac{8}{9}\cdot\frac{6}{5}\cdot\frac{10}{11}\cdot\frac{8}{7}\cdots$$

To make this example equal to $\frac{\pi}{2}$ (the $r=1$ case) we would need to multiply it by $\frac{2}{3}\cdot\frac{4}{5}$.

In general, for the product to equal $\frac{\pi}{2}$ we would need to multiply it by $\frac{(2k-2)!!}{(2k-1)!!}$ (or $\frac{(r-1)!!}{r!!}$ for odd r).

Therefore,

$$\prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\cdot\frac{2n+r-1}{2n+r}\right)=\frac{\pi}{2}\cdot\frac{r!!}{(r-1)!!} \quad \text{where } r \text{ is an odd positive integer.}$$

Similarly, for even values of r, we can substitute $r=2k$, where k is a positive integer:

$$\prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\cdot\frac{2n+2k-1}{2n+2k}\right)=\frac{2}{1}\cdot\frac{2k+1}{2k+2}\cdot\frac{4}{3}\cdot\frac{2k+3}{2k+4}\cdot\frac{6}{5}\cdot\frac{2k+5}{2k+6}\cdots$$

For example, if $k = 2$:

$$\frac{2}{1} \cdot \frac{5}{6} \cdot \frac{4}{3} \cdot \frac{7}{8} \cdot \frac{6}{5} \cdot \frac{9}{10} \cdot \frac{8}{7} \cdot \frac{11}{12} \cdot \frac{10}{9} \cdots$$

When $n \geq k+1$, the $\frac{2n}{2n-1}$ terms will equal the reciprocal of the $\frac{2n+2k-1}{2n+2k}$ terms and cancel out.

In general, the product will equal $\frac{(2k)!!}{(2k-1)}$ (or $\frac{r!!}{(r-1)!!}$ for even r).

Therefore,

$$\prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n+r-1}{2n+r} \right) = \frac{r!!}{(r-1)!!} \quad \text{where } r \text{ is an even positive integer.}$$

For all positive integer values of r we can multiply the numerator and denominator of (1) by equal terms and demonstrate that these infinite products can be constructed using simplicial polytopic numbers [2], $P_r(n)$, which are defined as:

$$P_r(n) = \binom{n+r-1}{r} = \frac{n(n+1)(n+2)\dots(n+r-1)}{r!}$$

For $r = 2$:

$$\prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n+1}{2n+2} \right) = \prod_{n=1}^{\infty} \left(\frac{2n(2n+1)}{(2n-1)2n} \cdot \frac{2n(2n+1)}{(2n+1)(2n+2)} \right) = \prod_{n=1}^{\infty} \left(\frac{\frac{2n(2n+1)}{2!}}{\frac{(2n-1)2n}{2!}} \cdot \frac{\frac{2n(2n+1)}{2!}}{\frac{(2n+1)(2n+2)}{2!}} \right)$$

Which are the triangular numbers, $P_2(n)$:

$$\prod_{n=1}^{\infty} \left(\frac{\frac{2n(2n+1)}{2!}}{\frac{(2n-1)2n}{2!}} \cdot \frac{\frac{2n(2n+1)}{2!}}{\frac{(2n+1)(2n+2)}{2!}} \right) = \prod_{n=1}^{\infty} \left(\frac{P_2(2n)}{P_2(2n-1)} \cdot \frac{P_2(2n)}{P_2(2n+1)} \right) = \frac{3}{1} \cdot \frac{3}{6} \cdot \frac{10}{6} \cdot \frac{10}{15} \cdot \frac{21}{15} \cdot \frac{21}{28} \cdots = 2$$

Likewise, for $r = 3$ we can construct a product using the tetrahedral numbers, $P_3(n)$:

$$\prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n+2}{2n+3} \right) = \prod_{n=1}^{\infty} \left(\frac{\frac{2n(2n+1)(2n+2)}{3!}}{\frac{(2n-1)(2n)(2n+1)}{3!}} \cdot \frac{\frac{2n(2n+1)(2n+2)}{3!}}{\frac{(2n+1)(2n+2)(2n+3)}{3!}} \right) = \frac{4}{1} \cdot \frac{4}{10} \cdot \frac{20}{10} \cdot \frac{20}{35} \cdot \frac{56}{35} \cdot \frac{56}{84} \cdots = \frac{3}{4} \cdot \pi$$

Or more generally:

$$\prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n+r-1}{2n+r} \right) = \prod_{n=1}^{\infty} \left(\frac{\frac{2n(2n+1)(2n+2)\dots(2n+r-1)}{r!}}{\frac{(2n-1)(2n)(2n+1)\dots(2n+r-2)}{r!}} \cdot \frac{\frac{2n(2n+1)(2n+2)\dots(2n+r-1)}{r!}}{\frac{(2n+1)(2n+2)(2n+3)\dots(2n+r)}{r!}} \right)$$

$$= \prod_{n=1}^{\infty} \left(\frac{P_r(2n)}{P_r(2n-1)} \cdot \frac{P_r(2n)}{P_r(2n+1)} \right) = \frac{r!!}{(r-1)!!} \begin{cases} \frac{\pi}{2}, & \text{when } r \text{ is odd.} \\ 1, & \text{when } r \text{ is even.} \end{cases} \tag{2}$$

Additionally, we can consider a comparison with the Wallis integrals [3]:

$$I_w = \int_0^{\frac{\pi}{2}} \sin^w x\, dx = \frac{(w-1)!!}{w!!} \begin{cases} 1, & \text{when } w \text{ is odd.} \\ \frac{\pi}{2}, & \text{when } w \text{ is even.} \end{cases}$$

For $w = r$, the ratio of I_r to (2) is:

$$\frac{\frac{(r-1)!!}{r!!}}{\frac{\pi r!!}{2(r-1)!!}} = \frac{2}{\pi} \cdot \left(\frac{(r-1)!!}{r!!} \right)^2 \quad \text{when } r \text{ is odd.}$$

And:

$$\frac{\frac{\pi(r-1)!!}{2r!!}}{\frac{r!!}{(r-1)!!}} = \frac{\pi}{2} \cdot \left(\frac{(r-1)!!}{r!!} \right)^2 \quad \text{when } r \text{ is even.}$$

Therefore:

$$I_r = \int_0^{\frac{\pi}{2}} \sin^r x\, dx = \left(\frac{(r-1)!!}{r!!} \right)^2 \cdot \prod_{n=1}^{\infty} \left(\frac{P_r(2n)}{P_r(2n-1)} \cdot \frac{P_r(2n)}{P_r(2n+1)} \right) \begin{cases} \frac{2}{\pi}, & \text{when } r \text{ is odd.} \\ \frac{\pi}{2}, & \text{when } r \text{ is even.} \end{cases}$$

And by substituting $r!!/(r-1)!!$ from (2):

$$I_r = \left(\frac{1}{\prod_{n=1}^{\infty} \left(\frac{P_r(2n)}{P_r(2n-1)} \cdot \frac{P_r(2n)}{P_r(2n+1)} \right)} \right)^2 \cdot \prod_{n=1}^{\infty} \left(\frac{P_r(2n)}{P_r(2n-1)} \cdot \frac{P_r(2n)}{P_r(2n+1)} \right) \begin{cases} \frac{2}{\pi} \cdot \frac{\pi^2}{4}, & \text{when } r \text{ is odd.} \\ \frac{\pi}{2}, & \text{when } r \text{ is even.} \end{cases}$$

$$I_r = \int_0^{\frac{\pi}{2}} \sin^r x\, dx = \frac{\pi}{2} \cdot \prod_{n=1}^{\infty} \left(\frac{P_r(2n-1)}{P_r(2n)} \cdot \frac{P_r(2n+1)}{P_r(2n)} \right)$$

References

[1] John Wallis. *Arithmetica Infinitorum*. Oxonii: Typis Leon. Lichfield Academiae Typographi, Impensis Tho. Robinson, Oxford, 1656.

[2] OEIS Foundation Inc. Simplicial polytopic numbers. `https://www.oeis.org/wiki/Simplicial_polytopic_numbers`, 2002. OEIS Wiki, Accessed on [2025-03-17].

[3] Eric W. Weisstein. Wallis cosine formula. From MathWorld–A Wolfram Web Resource, 2025. URL `https://mathworld.wolfram.com/WallisCosineFormula.html`.