Permutation Rotations

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Abstract

In this paper, we discuss certain properties of permutation rotations on each other.

Definition

Let S(N) be the set of all permutations for the integers 1 to N. Define the sum of two permutations (i.e., x + y) as the permutation y performed on x in S; or in simple terms, for every element in x (i.e. x[j] is the j^{th} element of x, where $1 \leq j \leq N$), find its current position in y (i.e., y[j]) and move it to the new position assigned by the value of the j^{th} element of y (i.e., (x + y)[j] = x[y[j]]).

Thus, x + y = (x[y[1]], …, x[y[N]]).

Note:

- The positions are counted from 1 to N, while the permutation positions are counted from 0 to N! − 1.
- The set x can be a group of anything since it is the y that is performing the permutation upon it.

Example

For N = 4, let x = (a, b, c, d) and y = (2, 1, 4, 3). Then x + y = (b, a, d, c).

Table 1 – S(4) defines the ordered permutation number (n) of all permutations of the set S when N = 4.

For the previous example, y = 7.

Table 1 – S(4)

n	1	2	3	4
0	1	2	3	4
1	1	2	4	3
2	1	3	2	4
3	1	3	4	2
4	1	4	2	3
5	1	4	3	2
6	2	1	3	4
7	2	1	4	3
8	2	3	1	4
9	2	3	4	1
10	2	4	1	3
11	2	4	3	1
12	3	1	2	4
13	3	1	4	2
14	3	2	1	4

n	1	2	3	4
15	3	2	4	1
16	3	4	1	2
17	3	4	2	1
18	4	1	2	3
19	4	1	3	2
20	4	2	1	3
21	4	2	3	1
22	4	3	1	2
23	4	3	2	1

Thus, we get the following sums:

- 0 + n = n + 0 = n, since this is the identity permutation. The first sum is changing the identity permutation to itself, while the latter is keeping the current permutation as is.
- n + n = 0, when only pairs of elements are switched with each other (i.e., for zero pairs, n = 0; for one pair, n = 1, 2, 5, 6, 14, 21; for two pairs, n = 7, 16, 23).
- n + m = m + n = 0, when m is the negative of n (i.e., m reverses the rotations in n; (n, m) = (3, 4), (4, 3), (8, 12), (12, 8), (9, 18), (18, 9), (10, 13), (13, 10), (11, 19), (15, 20), (17, 22), (19, 11), (20, 15), (22, 17)). It remains to be shown that if m reverses the permutation in n, then n reverses the permutation in m.

Assuming that x, y are the ordered permutation numbers (n) of the sets from Table 1 – S(4) with 0 ≤ x, y ≤ N! – 1, calculating all the sums for N = 4, we get the following table.

Addition: If 7 = (2, 1, 4, 3) and 8 = (2, 3, 1, 4), then 7 + 8 = (1, 4, 2, 3) = 4, since (7 + 8)[1, 2, 3, 4] = 7[8[1, 2, 3, 4]] = 7[2, 3, 1, 4] = (1, 4, 2, 3) = 4.

Table 2 – All Permutation Mappings (x + y = n)

x\y	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	r
0	**0**	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	**23**
1	**1**	0	**4**	**5**	2	3	**7**	6	10	11	8	9	18	19	20	21	22	23	12	13	14	15	16	17	**17**
2	**2**	**3**	0	1	5	4	**12**	13	**14**	15	16	17	6	7	8	9	10	11	19	18	22	23	20	21	**21**
3	3	2	5	4	0	1	**13**	12	**16**	**17**	14	15	19	18	22	23	20	21	6	7	8	9	10	11	**11**
4	4	5	1	0	3	2	**18**	19	**20**	**21**	22	23	7	6	10	11	8	9	13	12	16	17	14	15	**15**
5	5	4	3	2	1	0	**19**	18	**22**	**23**	20	21	13	12	16	17	14	15	7	6	10	11	8	9	**9**
6	**6**	**7**	**8**	**9**	**10**	**11**	0	1	2	3	4	5	14	15	12	13	17	16	20	21	18	19	23	22	**22**
7	7	6	10	11	8	9	1	0	4	5	2	3	20	21	18	19	23	22	14	15	12	13	17	16	**16**
8	8	9	6	7	11	10	14	**15**	12	13	17	16	0	1	2	3	4	5	21	20	23	22	18	19	**19**
9	9	8	11	10	6	7	15	14	17	16	12	13	21	20	23	22	18	19	0	1	2	3	4	5	**5**
10	10	11	7	6	9	8	20	21	18	19	23	22	1	0	4	5	2	3	15	14	17	16	12	13	**13**
11	11	10	9	8	7	6	21	20	23	22	18	19	15	14	17	16	12	13	1	0	4	5	2	3	**3**
12	12	13	14	15	16	17	2	3	0	1	5	4	8	9	6	7	11	10	22	23	19	18	21	20	**20**
13	13	12	16	17	14	15	3	2	5	4	0	1	22	23	19	18	21	20	8	9	6	7	11	10	**10**
14	14	15	12	13	17	16	8	9	6	7	11	10	2	3	0	1	5	4	23	22	21	20	19	18	**18**
15	15	14	17	16	12	13	9	8	11	10	6	7	23	22	21	20	19	18	2	3	0	1	5	4	**4**
16	16	17	13	12	15	14	22	23	19	18	21	20	3	2	5	4	0	1	9	8	11	10	6	7	**7**
17	17	16	15	14	13	12	23	22	21	20	19	18	9	8	11	10	6	7	3	2	5	4	0	1	**1**
18	18	19	20	21	22	23	4	5	1	0	3	2	10	11	7	6	9	8	16	17	13	12	15	14	**14**
19	19	18	22	23	20	21	5	4	3	2	1	0	16	17	13	12	15	14	10	11	7	6	9	8	**8**
20	20	21	18	19	23	22	10	11	7	6	9	8	4	5	1	0	3	2	17	16	15	14	13	12	**12**
21	21	20	23	22	18	19	11	10	9	8	7	6	17	16	15	14	13	12	4	5	1	0	3	2	**2**
22	22	23	19	18	21	20	16	17	13	12	15	14	5	4	3	2	1	0	11	10	9	8	7	6	**6**
23	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	**0**

Note:

- The yellow highlights are when x + y = s and x, y < s ∀ x, y, s ∈ S.
- The bold is when max(x, y) is at a minimum for every x + y = constant and y ≤ x. For N = 4, x = 0, 1, 2, 6 are their own minimum permutation breakdowns.
- The minimum value is not always unique (i.e., 1 + 6 = 6 + 1 = 7).
- The solid lines group ranges of numbers across rows and columns together. These groupings exist because the permutations in Table 1 change across rows and columns in pairs.
- Every pair of consecutive numbers in a row has a difference of plus or minus one. The column pair differences have a different pattern.
- The last column (r) shows the reverse sequence to the first column (#) which is the last numbered column (i.e., N! − 1 when counting from 0). The rows for each column (like the columns for each row) are unique.
- To calculate the table with minimal effort, one need only keep track of the rows in reverse order to get the pairs of sequences moving forward and backward. Thus, for any pair (x, r), one needs only calculate the row x when x ≤ r or $N!/2$ calculations. If r < x, then that pair would have already been calculated.

For this last statement, one can see from Table 2 that we need only calculate the rows x for the pairs (n, m) = (0, 23), (1, 17), (2, 21), (3, 11), (4, 15), (5, 9), (6, 22), (7, 16), (8, 19), (10, 13), (12, 20), (14, 18) since the missing n are generated by the reverse pair.

Properties

Not (Usually) Commutative

x + y = y + x if x = 0, ±y; otherwise, not usually.

Proof

Plenty of examples of both equality and inequality.

For non-trivial equality: ∀ n < m ∋ n ≠ 0 and n ≠ ±m and n + m = m + n, then (n, m) =:

Table 3 – All Non-Trivial (n, m) where n + m = m + n

n	m	n + m
1	6	7
1	7	6
6	7	1
2	21	23
2	23	21
21	23	2
5	14	16
5	16	14
14	16	5
7	16	23
7	23	16
16	23	7
7	17	22
7	22	17
9	16	18
16	18	9

n	m	n + m
10	23	13
13	23	10

Note: For the missing pairs (9, 18), (10, 13), and (17, 22), they all sum to zero making them also symmetric. However, we are ignoring these inverses since they are trivial; nor do they form a triple with the third number.

Associative

(x + y) + z = x + (y + z)

Proof

By definition, the operation works by performing a permutation of the right-hand operand on the left-hand operand. The left-hand side performs a y permutation on x followed by a z permutation on the result. The right-hand side performs a z permutation on y and uses that compounded permutation on x. Since the operation is just a position transfer, the right-hand side takes the original y positions and transfers them to their new z-positions and uses that end result to transfer the x positions. This is basically a short-hand step to the left-hand side.

Not Distributive

The distributive property is not defined in this case.

Additive Inverse

For every x, there exists a unique y such that x + y = 0 (i.e., 0 = {1, …, n}).

Proof

Let $\{r_1, \ldots, r_m\}$ be a subset of {1, …, n} where $m \le n$ such that each r_k, $1 \le k \le m$, consists of a closed rotation of numbers. Then for each r_k, there exists a unique rotation that reverses the rotational mapping performed for that subset.

Since x + y = 0 = {1, …, n}, the simplest way of calculating the additive inverse is to determine the position of each integer of x in relation to its order in 0 (or its last order position to undo the last sum).

x = k! or k! − 1 is always its own inverse. These are not unique.

Example

S = {1, 2, 3, 4}

x = 19 = {4, 1, 3, 2}

r_1 = {4, 1, 2} and r_2 = {3}

Since 1 was moved to the second place, 2 to the fourth, 3 stayed at third, and 4 to the first place, we need the reverse sequence to move them back. 2 moves the second number (1) back to first, 4 moves the fourth number (2) back to second, 3 stays fixed, and 1 moves the first number (4) back to fourth.

The additive inverse mapping of {4, 1, 2} is {2, 4, 1}, while {3} is its own inverse.

Thus, the additive inverse of x is y = 11 = {2, 4, 3, 1}.

Closed Rotations

Define the operation $(x +)_k + y$ as (x + … (x + y)) where x is reused k times. Similarly, define $x + (+ y)_k$.

Both of these operations will reveal the closed rotations for each x in S. Thus, for some k, $(x +)_k + y = x + y$. This is obvious due to the finiteness of the set. The maximum value of k for a fixed N has not been estimated but is conjectured to be at most N.

Table 4 – S(4) Minimum Permutations

#	x	y
0	0	
1	1	
2	2	
3	2	1
4	1	2
5	1	3
6	6	
7	1/6	6/1
8	6	2
9	6	3
10	7	2
11	7	3
12	2	6
13	3	6
14	2	8
15	8	7
16	3	8
17	3	9
18	4	6
19	5	6
20	4	8
21	4	9
22	5	8
23	5	9

Thus, for N = 4, the minimum permutations needed to generate all sets are x = {1, 2, 6}, while the minimum permutations needed to generate all sets using at most two permutations are x = {1, 2, 3, 4, 5, 6, 7, 8, 9}.

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0 + 0 = 1 + 1 = 2 + 2 = 5 + 5 = 6 + 6 = 7 + 7 = 14 + 14 = 16 + 16 = 21 + 21 = 23 + 23 = 0

( 3 +  3) +  4 =  4 +  4 =  3 + ( 3 +  4) =  3 +  0 =  3
( 8 +  8) + 12 = 12 + 12 =  8 + ( 8 + 12) =  8 +  0 =  8
( 9 +  9) + 16 = 16 + 16 =  9 + ( 9 + 16) =  9 + 18 =  0
(10 + 10) + 23 = 23 + 23 = 10 + (10 + 23) = 10 + 13 =  0
(11 + 11) + 19 = 19 + 19 = 11 + (11 + 19) = 11 +  0 = 11
(13 + 13) + 23 = 23 + 23 = 13 + (13 + 23) = 13 + 10 =  0
(15 + 15) + 20 = 20 + 20 = 15 + (15 + 20) = 15 +  0 = 15
(17 + 17) +  7 =  7 +  7 = 17 + (17 +  7) = 17 + 22 =  0
(18 + 18) + 16 = 18 + 16 = 18 + (18 + 16) = 18 +  9 =  0
(22 + 22) +  7 =  7 +  7 = 22 + (22 +  7) = 22 + 17 =  0
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Completeness

Looking at the bold cells in Table 2, one can see that we can generate the entire set S from just a base set of numbers after a few summations. This generator set consists of {1, 2, 6}.

Table 5 – Number of Permutations Count

Sum	Permutation	Count
1 + 1 =	0	2
1 + 2 =	4	2
1 + 6 =	7	2
2 + 1 =	3	2
2 + 6 =	12	2
1 + 3 = 1 + (2 + 1)	5	3
1 + 12 = 1 + (2 + 6)	18	3
2 + 7 = 2 + (1 + 6)	13	3
6 + 2 =	8	2
6 + 3 = 6 + (2 + 1)	9	3
6 + 4 = 6 + (1 + 2)	10	3
6 + 5 = 6 + (1 + (2 + 1))	11	4
2 + 8 = 2 + (6 + 2)	14	3
8 + 7 = (6 + 2) + (1 + 6)	15	4
3 + 8 = (2 + 1) + (6 + 2)	16	4
3 + 9 = (2 + 1) + (6 + (2 + 1))	17	5
4 + 6 = (1 + 2) + 6	18	3
4 + 8 = (1 + 2) + (6 + 2)	20	4
4 + 9 = (1 + 2) + (6 + (2 + 1))	21	5
5 + 6 = (1 + (2 + 1) + 6	19	4
5 + 8 = (1 + (2 + 1) + (6 + 2)	22	5
5 + 9 = (1 + (2 + 1) + (6 + (2 + 1))	23	6

Conjecture 1: The maximum number of permutations needed to generate all permutations using only the generator set is (N − 1)!.

Conjecture 2: The maximum number of iterations k needed to get back the original sum (i.e., $(x +)_k + y = x + y$) is k = N.

Note: For Conjecture 2, since x is fixed, the iterative sum permutations cannot be a closed loop outside of the original sum since both the last iteration outside of the closed loop starting from the original sum and some iterative sum inside the loop will both yield the same permutation. For a fixed x or a fixed y, x + y is unique.

Conjecture 3: The maximum number of iterations k needed for all iterative sum permutations to get back to their original sum with the exact same k for all is when k = N!/2. This is related to why only N!/2 calculations are needed to determine the full table with the reverse pattern covering the other half.

Theorem

Let S(N) be the set of all ordered permutation numbers of the first N ∈ ℕ and enumerate them starting with 0. Thus, S(N) = (0, 1, …, N! − 1).

If g is the generator set of S[1], then g = {1!, 2!, …, (N − 1)!} satisfies this condition, where g consists only of the N − 1 factorial numbers.

Proof

For each element k in g, $1 \le k \le N - 1$, k! is the ordered position that rotates the $(N - k)$ and $(N - k + 1)$ elements of a set. Since every element gets rotated with the element next to it, all permutations are generated. Furthermore, the generator set is not unique. It just needs to be able to move any element to any other position and back after a finite number of steps.

Final Thoughts

Since the numerical order of these permutations $(0, \ldots, N! - 1)$ can be calculated algorithmically by their indices (and vice-versa), is it possible to generalize these summation tables for any N?

References

[1] Wikipedia, The Free Encyclopedia (2025), https://en.wikipedia.org/wiki/Generating_set_of_a_group, ***Generating set of a group***