Extended Diophantine Equations: An Overview and Applications of the Formula $ab = k(a + b) + c$

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Abstract

This paper presents a comprehensive overview of extended Diophantine equations—high-degree equations that admit infinitely many solution sets, with each set comprising infinitely many elements. We focus on the effective application of the formula

$$ab = k(a + b) + c,$$

where a, b, k, and c are integers, to transform complex equations into symmetric forms. This transformation facilitates the derivation of polynomial expansions and enables the systematic construction of solution sets with distinct and primitive elements.

1 Introduction

In classical number theory, Diophantine equations are typically studied in terms of the existence of finitely or infinitely many integer solutions. The notion of *extended Diophantine equations* pushes this framework further by considering high-degree equations with intricate structures, where not only are there infinitely many sets of solutions, but each solution set itself contains infinitely many elements. This expanded perspective reveals the rich and diverse internal structure of such equations and opens new avenues for research in algebra and number theory.

A powerful tool in the analysis of these extended Diophantine equations is the formula

$$ab = k(a + b) + c,$$

with a, b, k, and c being integers, and where k and c are determined based on the particular equation under study. This formula allows for the conversion of complex expressions into symmetric forms, thereby enabling a systematic expansion of related polynomials in terms of the sum $a + b$. In turn, such expansions reveal intricate relationships among the coefficients and support the construction of solution sets characterized by distinct, primitive, and arithmetically significant properties.

By applying the formula $ab = k(a+b)+c$, one can transform high-degree Diophantine equations into either binomial or symmetric polynomial forms. This approach not only simplifies the analysis but also demonstrates the existence of infinitely many solution sets—each containing infinitely many elements that satisfy stringent conditions such as mutual distinctness, non-vanishing, and coprimality. The aim of this paper is to provide a detailed overview of the methodology that utilizes the aforementioned formula to solve extended Diophantine equations, thereby offering new insights and potential applications in both theoretical and applied mathematics.

2. Expansion of $a^n + b^n$ in terms of $X = a + b$

2.1. General Formula

When a and b are connected through the identity:

$$ab = k(a + b) + c$$

We can express $a^n + b^n$ as a polynomial in the variable X, where $X = a + b$. Let:

$$a^n + b^n = \sum_{d=0}^{n} E(n - d, n) \cdot X^{n-d},$$

with $E(n - d, n)$ be the coefficient of X^{n-d} in the expansion of $a^n + b^n$.

From the identity

$$a^n + b^n = [a^{n-1} + b^{n-1}](a + b) - ab[a^{n-2} + b^{n-2}]$$

and the relation $ab = k(a+b)+c$, we obtain the recurrence formula:

$$E(n-d,n) = E(n-d-1,n-1) - k\cdot E(n-d-1,n-2) - c\cdot E(n-d,n-2)$$

with initial values:

$$\begin{aligned} E(n,n) &= 1, \\ E(n-1,n) &= -nk, \\ E(n-2,n) &= \frac{n(n-3)}{2}k^2 - nc, \\ E(n-3,n) &= -\frac{n(n-4)(n-5)}{6}k^3 + n(n-3)kc. \end{aligned}$$

A closed form formula has been proven:

$$\boxed{E(n-d,n) = (-1)^d n \sum_{j=0}^{\lfloor d/2 \rfloor} (-1)^j \frac{(n-d+j-1)!}{(n-2d+2j)!(d-2j)!j!} \cdot k^{d-2j} c^j}$$

We now present the expansions for $n = 4, 5, 6$:

Expansion for $n = 4$

$$a^4 + b^4 = \sum_{d=0}^{4} E(4-d,4).X^{4-d}$$

$$\begin{aligned} E(4,4) &= 1 \\ E(3,4) &= -4k \\ E(2,4) &= 2k^2 - 4c \\ E(1,4) &= 4kc \\ E(0,4) &= 2c^2 \end{aligned}$$

$$a^4 + b^4 = X^4 - 4kX^3 + (2k^2 - 4c)X^2 + 4kcX + 2c^2.$$

Expansion for $n = 5$

$$a^5 + b^5 = \sum_{d=0}^{5} E(5-d,5)X^{5-d}$$

$$
\begin{aligned}
E(5,5) &= 1\\
E(4,5) &= -5k\\
E(3,5) &= 5k^2 - 5c\\
E(2,5) &= 10kc\\
E(1,5) &= 5c^2\\
E(0,5) &= 0
\end{aligned}
$$

$$a^5 + b^5 = X^5 - 5kX^4 + (5k^2 - 5c)X^3 + 10kcX^2 + 5c^2X.$$

3. Relationship: $(X - k)^n = a^n + b^n + P(X)$

Using the binomial theorem:

$$(X - k)^n = \sum_{j=0}^{n} \binom{n}{j} X^{n-j}(-k)^j.$$

We define the polynomial $P(X)$ as:

$$P(X) = (X - k)^n - (a^n + b^n).$$

This gives a way to shift between the binomial expansion and the symmetric power sum using $P(X)$ as the adjustment term.

3. Specific examples: $n = 4$ and $n = 5$

3.1. Case $n = 4$

a) Binomial Expansion:

$$(X - k)^4 = X^4 - 4kX^3 + 6k^2X^2 - 4k^3X + k^4.$$

b) Symmetric Expansion:

$$a^4 + b^4 = X^4 - 4kX^3 + (2k^2 - 4c)X^2 + 4kcX + 2c^2.$$

c) Polynomial $P(X)$:

$$
\begin{aligned}
P(X) &= (X - k)^4 - (a^4 + b^4)\\
&= (4k^2 + 4c)X^2 - 4k(k^2 + c)X + (k^4 - 2c^2).
\end{aligned}
$$

Final identity:

$$\boxed{(X-k)^4 = a^4 + b^4 + \left[(4k^2+4c)X^2 - 4k(k^2+c)X + (k^4-2c^2)\right]}$$

3.2. Case $n = 5$

a) Binomial Expansion:

$$(X-k)^5 = X^5 - 5kX^4 + 10k^2X^3 - 10k^3X^2 + 5k^4X - k^5.$$

b) Symmetric Expansion:

$$a^5 + b^5 = X^5 - 5kX^4 + (5k^2 - 5c)X^3 + 10kcX^2 + 5c^2X.$$

c) Polynomial $P(X)$:

$$\begin{aligned} P(X) &= (X-k)^5 - (a^5+b^5) \\ &= (5k^2+5c)X^3 - 10k(k^2+c)X^2 + 5(k^4-c^2)X - k^5. \end{aligned}$$

Final identity:

$$\boxed{(X-k)^5 = a^5 + b^5 + \left[(5k^2+5c)X^3 - 10k(k^2+c)X^2 + 5(k^4-c^2)X - k^5\right]}$$

6. Conclusion

This report presented the connection between the symmetric expansion of $a^n + b^n$ and the binomial expansion of $(X-k)^n$, where $ab = kX + c$. We have the following:

1. Recursive and closed-form formulas for the coefficients of $a^n + b^n$ in terms of X.

2. The binomial expansion of $(X-k)^n$ using Newton's formula.

3. A definition of the polynomial $P(X)$ such that:

$$(X-k)^n = a^n + b^n + P(X).$$

4. Explicit examples for $n = 4$ and $n = 5$ that illustrate the structure and pattern of $P(X)$.

   This identity provides a powerful tool for analyzing and transforming symmetric expressions involving powers of two variables.

Problem 1:

Prove that there exist infinitely many tuples of integers $(a_1, a_2, \ldots, a_n)$ with $n > 3$, such that:

- Each $a_i \neq 0$
- No two numbers are negatives of each other
- All a_i are pairwise distinct
- $\gcd(a_1, a_2, \ldots, a_n) = 1$

and such that

$$\sum_{i=1}^{n} a_i^2 = b^2 \quad \text{and} \quad \sum_{i=1}^{n} a_i = c^2$$

for $b, c \in \mathbb{Z}$.

Solution

Step 1: Analyze the sum of squares

Let $a_1, a_2, \ldots, a_n$ be integers. We aim to find such numbers such that:

$$a_1^2 + a_2^2 + \cdots + a_n^2 = (x - k)^2,$$

where $x = a_1 + a_2 + \cdots + a_n$ is a perfect square. To simplify the problem, we first consider $n = 3$, then extend it to $n > 3$.

Step 2: Relationship between a_1, a_2, a_3

For any integers a_1, a_2, we always have:

$$a_1 \cdot a_2 = k(a_1 + a_2) + c, \tag{1}$$

where $k, c \in \mathbb{Z}$. Let $a_1 + a_2 = x$:

$$x^2 = a_1^2 + a_2^2 + 2kx + 2c.$$

Then:

$$(x - k)^2 = a_1^2 + a_2^2 + k^2 + 2c. \tag{2}$$

Set $k = a_3$ and choose $2c = a_4^2 + \cdots + a_n^2$. Thus, we have:

$$(x-k)^2 = a_1^2 + a_2^2 + a_3^2 + a_4^2 + \cdots + a_n^2. \tag{3}$$

This ensures that $a_1^2 + a_2^2 + \cdots + a_n^2$ is a perfect square.

Step 3: Choosing a_1, a_2 appropriately

From equation (1), assume $a_1 - k = 1$. Then:

$$a_2 - k = k^2 + c. \tag{4}$$

This gives:

$$a_1 = k + 1, \quad a_2 = k^2 + k + c. \tag{5}$$

With this choice, a_1 and a_2 are integers that are distinct.

Step 4: The sum $a_1 + a_2 + a_3 + \cdots + a_n$

The total sum of $a_1, a_2, a_3, \ldots, a_n$ is:

$$a_1 + a_2 + a_3 + \cdots + a_n = k^2 + 3k + 1 + c + a_4 + \cdots + a_n. \tag{6}$$

Let:

$$q = 1 + c + a_4 + \cdots + a_n,$$

we get:

$$a_1 + a_2 + a_3 + \cdots + a_n = k^2 + 3k + q. \tag{7}$$

Rewriting this sum as a perfect square:

$$k^2 + 3k + q = (k+2)^2 - k + q - 4. \tag{8}$$

To make the sum a perfect square, choose $k = q - 4$. Then:

$$a_1 + a_2 + a_3 + \cdots + a_n = (k+2)^2. \tag{9}$$

Step 5: Conclusion

With the choices:

$$a_1 = k + 1, \quad a_2 = k^2 + k + c, \quad a_3 = k$$

and appropriate $c, a_4, \ldots, a_n$, we ensure:

1. $$\sum_{i=1}^{n} a_i^2 = b^2 \quad \text{and} \quad \sum_{i=1}^{n} a_i = c^2$$

Thus, there exist infinitely many such sets $(a_1, a_2, \ldots, a_n)$ that satisfy the problem's requirements.

Example

For $n = 5$, we wish to prove that

$$\sum_{i=1}^{5} a_i^2 = b^2 \quad \text{and} \quad \sum_{i=1}^{5} a_i = c^2.$$

We begin by choosing a_4 and a_5 such that $a_4^2 + a_5^2$ is divisible by 2. For instance, let

$$a_4 = 1 \quad \text{and} \quad a_5 = 3.$$

Then we compute

$$c = \frac{a_4^2 + a_5^2}{2} = \frac{1^2 + 3^2}{2} = \frac{1 + 9}{2} = 5.$$

Next, define

$$q = 1 + c + a_4 + a_5 = 1 + 5 + 1 + 3 = 10.$$

It follows that

$$k = q - 4 = 10 - 4 = 6.$$

We then determine the remaining values as follows:

$$a_1 = k+1 = 6+1 = 7, \quad a_2 = k^2+k+c = 6^2+6+5 = 36+6+5 = 47, \quad a_3 = k = 6.$$

Thus, the solution set is

$$(a_1, a_2, a_3, a_4, a_5) = (7, 47, 6, 1, 3).$$

We can verify that

$$\sum_{i=1}^{5} a_i^2 = 7^2 + 47^2 + 6^2 + 1^2 + 3^2 = 49 + 2209 + 36 + 1 + 9 = 2304 = 48^2,$$

and

$$\sum_{i=1}^{5} a_i = 7 + 47 + 6 + 1 + 3 = 64 = 8^2.$$

Hence, the solution set $(7, 47, 6, 1, 3)$ satisfies the required conditions.

Problem 2:

Prove that there exist infinitely many tuples of integers $(a_1, a_2, \ldots, a_n)$ with $n > 3$, such that:

- Each $a_i \neq 0$,
- No two elements are negatives of each other,
- All a_i are distinct,
- $\gcd(a_1, a_2, \ldots, a_n) = 1$

  $$\sum_{i=1}^{n} a_i^3 = b^3$$

  for $b \in \mathbb{Z}$.

Solution

Step 1: Start from a known identity

For any integers a_1 and a_2, there always exist integers k and c such that:

$$a_1 a_2 = k(a_1 + a_2) + c. \tag{1}$$

Step 2: Use the identity to simplify x^3

Let $x = a_1 + a_2$. Then:

$$x^3 = a_1^3 + a_2^3 + 3a_1a_2x.$$

Substituting equation (1):

$$x^3 = a_1^3 + a_2^3 + 3kx^2 + 3cx.$$

Rewriting:

$$(x-k)^3 = a_1^3 + a_2^3 + (-k)^3 + 3(k^2+c)x.$$

Step 3: Choose additional cube terms

Choose integers $b_4, b_5, \dots, b_n$ such that:

$$3(k^2+c) = b_4^3 + b_5^3 + \cdots + b_n^3.$$

Define:

$$q = \frac{b_4^3 + b_5^3 + \cdots + b_n^3}{3} = k^2 + c.$$

This is possible by choosing appropriate values for $b_4, \dots, b_n$ so that q is an integer.

Step 4: Construct remaining terms

Let $x = d^3$ for some integer d, and define:

$$a_3 = -k, \quad a_4 = b_4d, \quad \dots, \quad a_n = b_nd.$$

Then:

$$(x-k)^3 = a_1^3 + a_2^3 + a_3^3 + a_4^3 + \cdots + a_n^3.$$

Step 5: Choose a_1 and a_2

From equation (1), set:

$$a_1 - k = 1, \quad a_2 - k = k^2 + c = q.$$

Then:

$$a_1 = k+1, \quad a_2 = k+q.$$

Step 6: Solve for k

We know:

$$x = a_1 + a_2 = 2k + 1 + q = d^3.$$

Solving for k:

$$k = \frac{d^3 - 1 - q}{2}.$$

So k is an integer if d and q are chosen appropriately.

Step 7: Ensure conditions are met

To ensure the conditions (nonzero, not negatives of each other, distinct, and gcd equal to 1), choose d and the b_i's such that:

- All $a_i \neq 0$,
- No pair $a_i = -a_j$,
- All a_i are distinct,
- $\gcd(a_1, \ldots, a_n) = 1$.

Conclusion

We now have:

$$a_1 = k + 1, \quad a_2 = k + q, \quad a_3 = -k, \quad a_4 = b_4 d, \quad \ldots, \quad a_n = b_n d.$$

Since d and b_i can be freely chosen (with infinitely many valid combinations), there exist infinitely many such tuples satisfying

$$\sum_{i=1}^{n} a_i^3 = b^3$$

Example

For $n = 5$, we wish to prove that

$$\sum_{i=1}^{5} a_i^3 = b^3.$$

We proceed as follows:

1. Choose b_4 and b_5 such that $b_4^3 + b_5^3$ is divisible by 3. For instance, let

$$b_4 = 1 \quad \text{and} \quad b_5 = 5.$$

2. Compute

$$q = \frac{b_4^3 + b_5^3}{3} = \frac{1^3 + 5^3}{3} = \frac{1 + 125}{3} = \frac{126}{3} = 42.$$

3. Choose $d = 3$ and determine k by

$$k = \frac{d^3 - 1 - q}{2} = \frac{3^3 - 1 - 42}{2} = \frac{27 - 1 - 42}{2} = \frac{-16}{2} = -8.$$

4. Define the components of the solution:

$$a_4 = b_4 \cdot d = 1 \cdot 3 = 3, \quad a_5 = b_5 \cdot d = 5 \cdot 3 = 15.$$

Also, set

$$a_1 = k+1 = -8+1 = -7, \quad a_2 = k+q = -8+42 = 34, \quad a_3 = -k = 8.$$

Thus, the solution set is

$$(a_1, a_2, a_3, a_4, a_5) = (-7,\ 34,\ 8,\ 3,\ 15).$$

We now verify the result:

$$\begin{aligned}\sum_{i=1}^{5} a_i^3 &= (-7)^3 + 34^3 + 8^3 + 3^3 + 15^3 \\ &= -343 + 39304 + 512 + 27 + 3375 \\ &= 42875,\end{aligned}$$

and indeed

$$35^3 = 42875.$$

Therefore, the solution set $(-7, 34, 8, 3, 15)$ satisfies the condition

$$\sum_{i=1}^{5} a_i^3 = 35^3.$$

Problem 3:

Prove that there exist infinitely many tuples of integers $(a_1, a_2, \ldots, a_n)$ with $n > 3$, such that:

- Each $a_i \neq 0$
- No two numbers are negatives of each other
- All a_i are pairwise distinct
- $\gcd(a_1, a_2, \ldots, a_n) = 1$

and such that

$$\sum_{i=1}^{n} a_i^3 = b^2$$

for $b \in \mathbb{Z}$.

Solution

Step 1: Basic Analysis

For any integers a and b, we can always write:

$$a \cdot b = k \cdot (a + b) + c, \quad \text{where } k, c \in \mathbb{Z}.$$

Let $x = a + b$, then:

$$a \cdot b = kx + c. \tag{1}$$

The sum of cubes of a and b can be expressed as:

$$a^3 + b^3 = (a + b)^3 - 3ab(a + b) = x^3 - 3ab \cdot x.$$

Substituting $ab = kx + c$ from (1), we get:

$$a^3 + b^3 = x^3 - 3(kx + c)x = x^3 - 3k \cdot x^2 - 3c \cdot x. \tag{2}$$

Let us rewrite this as:

$$a^3 + b^3 = (px + q)^2 + x^3 - 3k \cdot x^2 - 3c \cdot x - (px + q)^2. \tag{3}$$

To ensure the expression is a perfect square, we choose:

$$3k + p^2 = 0 \quad \text{and} \quad 3c + 2pq = 0.$$

Solving these equations, we get:

$$k = -\frac{p^2}{3}, \quad c = -\frac{2pq}{3}.$$

Step 2: Constructing the Initial Solution

Substituting $k = -\frac{p^2}{3}, c = -\frac{2pq}{3}$ into (3), we get:

$$a^3 + b^3 + (-x)^3 + q^2 = (px + q)^2. \tag{4}$$

By choosing $q = 1$, we have:

$$a^3 + b^3 + (-x)^3 + 1^3 = (px + 1)^2. \tag{5}$$

Thus, the quadruple $(a, b, -x, 1)$ is a solution.

Step 3: Extending the Solution

Let $p = 3d$ and $q = 1$. Then:

$$k = -3d^2, \quad c = -2d.$$

From (1), we have:

$$(a - k)(b - k) = k^2 + c = 9d^4 - 2d.$$

For a specific d, solving this equation yields integer pairs (a, b).

Step 4: Iterative Process

Consider an additional pair (a_1, b_1) with $x_1 = a_1 + b_1$. Then:

$$(a_1 - k)(b_1 - k) = k^2 + c = 9d^4 - 2dq_1,$$

where $q_1 = px + q = px + 1$.

The sum of cubes for the new pair is:

$$a_1^3 + b_1^3 + (-x_1)^3 + q_1^2 = (px + q_1)^2. \tag{6}$$

Substituting (5) into (6), we obtain:

$$(px + q_1)^2 = a_1^3 + b_1^3 + (-x_1)^3 + a^3 + b^3 + (-x)^3 + 1^3.$$

Thus, we extend the set $(a, b, -x, 1)$ to $(a, b, -x, 1, a_1, b_1, -x_1)$.

Step 5: Infinite Iteration

By choosing $q_2 = px + q_1$, we repeat the process, extending the set to:

$$(a, b, -x, 1, a_1, b_1, -x_1, \dots, a_n, b_n, -x_n).$$

At each step, we ensure that the integers are pairwise coprime and distinct, and their cubes sum to a perfect square.

Conclusion

By iterating this process infinitely, we construct infinitely many sets of integers $(a_1, a_2, \dots, a_n)$ such that:

$$\sum_{i=1}^{n} a_i^3 = b^2$$

Example: Infinite Extension of Solutions

We illustrate the method of extending solutions infinitely for a certain class of Diophantine equations. In this example, we aim to extend a base solution by using specific parameter choices.

Step 1: Base Equation

Let

$$q = 1 \quad \text{and} \quad k = -3d^2.$$

Then, we have the relation

$$(a-k)(b-k) = 9d^4 - 2d.$$

Taking $d = 1$, we obtain

$$(a+3)(b+3) = 7.$$

Solving this equation gives the solutions:

$$(a,b) = (-2,\, 4) \quad \text{or} \quad (a,b) = (-4,\, -10).$$

We choose the pair $(-4, -10)$. Then,

$$x = a + b = -4 + (-10) = -14.$$

It is verified that

$$(-4)^3 + (-10)^3 + 14^3 + 1^3 = 41^2.$$

Step 2: Extending the Solution

Now, set

$$q_1 = 41.$$

Then, consider the modified equation:

$$(a_1 - k)(b_1 - k) = 9d^4 - 2d\, q_1.$$

Choose $d = 2$ so that

$$k = -3d^2 = -12.$$

The equation becomes

$$(a_1 + 12)(b_1 + 12) = -20.$$

Solving this yields possible solutions such as

$$(a_1, b_1) = (-11,\, -32) \quad \text{or} \quad (a_1, b_1) = (-13,\, 8), \ldots$$

We select the pair $(-13,\, 8)$, so that

$$x = a_1 + b_1 = -13 + 8 = -5.$$

It can then be verified that

$$(-13)^3 + 8^3 + 5^3 + 41^2 = (-13)^3 + 8^3 + 5^3 + (-4)^3 + (-10)^3 + 14^3 + 1^3 = 11^2.$$

Conclusion of the Example

We have thus extended the initial solution set from

$$(-4,\ -10,\ 14,\ 1)$$

to a larger solution set

$$(-4,\ -10,\ 14,\ 1,\ -13,\ 8,\ 5).$$

This process can be repeated indefinitely, thereby generating an infinite family of solutions.

Problem 4:

Prove that there exist infinitely many tuples of integers

$$(a_1, a_2, a_3, a_4) \quad \text{and} \quad (b_1, b_2, \ldots, b_t)$$

such that:

- Each $a_i \neq 0$
- No two numbers are negatives of each other
- All a_i are pairwise distinct
- $\gcd(a_1, a_2, \ldots, a_n) = 1$

and

$$a_1^4 + a_2^4 - a_3^4 - a_4^4 = \sum_{i=1}^{t} b_i^3$$

Solution

Let a, b be integers. We have the relation:

$$a \cdot b = k(a + b) + c \quad (1)$$

where k, c are integers. Define $x = a + b$, then:

$$a \cdot b = kx + c$$

Step 1: Express $a^4 + b^4$

We have:

$$a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2$$

Notice that:

$$a^2 + b^2 = x^2 - 2ab$$

Substituting into the equation and expanding:

$$a^4 + b^4 = (x^2 - 2ab)^2 - 2a^2b^2 = x^4 - 4x^2(kx + c) + 2a^2b^2$$

This simplifies to:

$$a^4 + b^4 = x^4 - 4kx^3 - 4cx^2 + 2a^2b^2$$

Continuing to expand:

$$a^4 + b^4 = x^4 - 4kx^3 + (2k^2 - 4c)x^2 + 4kcx + 2c^2$$

We now have:

$$a^4 + b^4 + k^4 + 4(k^2 + c)x^2 - 4k(k^2 + c)x - 2c^2 = (x - k)^4$$

Step 2: Define $k^2 + c = m$

Thus, $c = m - k^2$, and we have:

$$(x - k)^4 = a^4 + b^4 + k^4 + 4mx^2 - 4kmx - 2(m - k^2)^2$$

Expanding further:

$$(x - k)^4 = a^4 + b^4 + k^4 + 4mx(x - k) - 2(m^2 - 2mk^2 + k^4)$$

Step 3: Define P

Let:

$$P = 4mx(x-k) - 2m^2 + 4mk^2$$

From the above, we get:

$$(x-k)^4 + k^4 = a^4 + b^4 + P$$

From equation (1), we have:

$$(a-k)(b-k) = k^2 + c = m$$

Let $a - k = n$, then $b - k = \frac{m}{n}$, so:

$$a = k + n, \quad b = \frac{m}{n} + k$$

Thus:

$$x = a + b = 2k + n + \frac{m}{n}$$

Step 4: Compute P

Substitute into the expression for P:

$$P = 4m\left(2k + n + \frac{m}{n}\right)(k + n + \frac{m}{n}) - 2m^2 + 4mk^2$$

Expanding and simplifying:

$$P = \frac{4}{n^2}m^3 + 6m^2\left(1 + \frac{2k}{n}\right) + 4m\left(3k^2 + 3kn + n^2\right)$$

Step 5: Choose $n = 2$

When choosing $n = 2$, we get:

$$P = m^3 + 6m^2(1+k) + 4m(3k^2 + 6k + 4)$$

This simplifies to:

$P = [m + 2(1+k)]^3 - 3m \cdot 4(1+k)^2 - 8(1+k)^3 + 4m(3k^2 + 6k + 4)$

Finally, we have:

$$P = (m + 2k + 2)^3 - 4m\left[3(1 + 2k + k^2) - 3k^2 - 6k - 4\right] - 8(1+k)^3$$

$$P = (m + 2k + 2)^3 + 4m - 8(1+k)^3$$

Step 6: Define $b_1, b_2, \ldots, b_n$

Let:

$$b_1 = m + 2k + 2, \quad 4m - 8(1+k)^3 = b_2^3 + \cdots + b_n^3 \quad (*)$$

Thus:

$$P = b_1^3 + b_2^3 + \cdots + b_n^3 \quad (5)$$

Substitute into equation (3):

$$P = (x-k)^4 + k^4 - a^4 - b^4 = b_1^3 + b_2^3 + \cdots + b_n^3$$

With $n = 2$, we have $a = k + 2, b = \frac{m}{2} + k$, and for b to be an integer, $m/2$ must be an integer.

Step 7: Compute m

From equation (*), we have:

$$m = 2(1+k)^3 + \frac{b_2^3 + \cdots + b_n^3}{4}$$

Thus:

$$\frac{m}{2} = (1+k)^3 + \frac{b_2^3 + \cdots + b_n^3}{8}$$

Since b_n^3 divided by 8 can have remainders of 0, 1, 3, 5, or 7, we choose $b_2^3 + \cdots + b_n^3$ such that it is divisible by 8, ensuring that $m/2$ is an integer and thus b is an integer.

Conclusion:

We have proven that there exist infinitely many distinct integer quadruples (a_1, a_2, a_3, a_4) and $(b_1, b_2, \ldots, b_n)$ that are pairwise coprime such that:

$$a_1^4 + a_2^4 - a_3^4 - a_4^4 = b_1^3 + b_2^3 + \cdots + b_n^3$$

Example: Extending a Solution via Cubic and Quartic Identities

We illustrate an example where we choose parameters b_2, b_3, and b_4 so that the sum of their cubes is divisible by 8. The steps are as follows.

Step 1: Choosing Parameters

Choose

$$b_2 = 1, \quad b_3 = 2, \quad b_4 = 7,$$

so that

$$b_2^3 + b_3^3 + b_4^3 = 1^3 + 2^3 + 7^3 = 1 + 8 + 343 = 352,$$

and 352 is divisible by 8.

Assume $k = 1$.

Step 2: Determining m

We are given the relation

$$\frac{m}{2} = (k+1)^3 + 44.$$

Substitute $k = 1$:

$$\frac{m}{2} = (1+1)^3 + 44 = 2^3 + 44 = 8 + 44 = 52.$$

Thus,

$$m = 104.$$

Step 3: Defining the Variables a, b and b_1

Set

$$a = k + 2 = 1 + 2 = 3,$$
$$b = \frac{m}{2} + k = 52 + 1 = 53,$$
$$b_1 = m + 2k + 2 = 104 + 2 + 2 = 108.$$

Step 4: Constructing the Solution Set

Now define

$$a_1 = a + b - k = 3 + 53 - 1 = 55,$$
$$a_2 = k = 1, \quad a_3 = a = 3, \quad a_4 = b = 53.$$

Thus, we have the solution sets:

$$(a_1, a_2, a_3, a_4) = (55,\ 1,\ 3,\ 53),$$

$$(b_1, b_2, b_3, b_4) = (108,\ 1,\ 2,\ 7).$$

Step 5: Verification

It is claimed that

$$55^4 + 1^4 - 3^4 - 53^4 = 108^3 + 1^3 + 2^3 + 7^3.$$

A direct computation shows that both sides of the identity are equal, hence the solution sets satisfy the required equation.

Thus, the pair of solution sets

$$(a_1, a_2, a_3, a_4) = (55,\ 1,\ 3,\ 53) \quad \text{and} \quad (b_1, b_2, b_3, b_4) = (108,\ 1,\ 2,\ 7)$$

is valid.

Problem 5:

Prove that there exist infinitely many tuples of integers $(a_1, a_2, \dots, a_t)$ with $t > 3$, such that:

- Each $a_i \neq 0$
- No two numbers are negatives of each other
- All a_i are pairwise distinct
- $\gcd(a_1, a_2, \dots, a_t) = 1$

and such that

$$\sum_{i=1}^{t} a_i^3 = b^3 \quad \text{and} \quad \sum_{i=1}^{t} a_i = c^3$$

for $b, c \in \mathbb{Z}$.

Solution:

Let a and b be integers. We have the equation:

$$ab = k(a+b) + c \tag{1}$$

where k and c are integers. Let $a + b = x$.

Then we have:

$$a^3 + b^3 = x^3 - 3kx^2 - 3cx$$

which simplifies to:

$$(x-k)^3 = a^3 + b^3 + (-k)^3 + 3(k^2 + c)x$$

Solution

Step 1: Define auxiliary variables

Define:

$$k^2 + c = m_1 p \quad \text{and} \quad x = m_2 p^2$$

where p is an integer. Then:

$$3m_1 m_2 = b_4^3 + \cdots + b_t^3 \tag{2}$$

This gives us the equation:

$$(x-k)^3 = a^3 + b^3 + (-k)^3 + (b_4 p)^3 + \cdots + (b_t p)^3$$

Step 2: Express a and b

From the equation (1): $(a-k)(b-k) = k^2 + c$, we have:

$$a - k = n \quad \text{and} \quad b - k = \frac{k^2 + c}{n} = \frac{m_1 p}{n}$$

Thus:

$$a + b = 2k + n + \frac{m_1 p}{n} = m_2 p^2$$

Hence:

$$k = \frac{m_2}{2} p^2 - \frac{m_1}{2n} p - \frac{n}{2} \tag{*}$$

Step 3: Compute the sum S

Now compute the sum:

$$S = (a + b) + (-k) + p(b_4 + \cdots + b_t)$$

This simplifies to:

$$S = \frac{m_2}{2} p^2 + \left(\frac{m_1}{2n} + b_4 + \cdots + b_t\right) p + \frac{n}{2}$$

Step 4: Define new variables

Let:

$$\frac{n}{2} = q^3 \tag{3}$$

and:

$$\frac{m_1}{2n} + b_4 + \cdots + b_t = 3q^2 \tag{4}$$

Thus:

$$S = \frac{m_2}{2} p^2 + 3pq^2 + q^3 = (p+q)^3 - p^3 + \left(\frac{m_2}{2} - 3q\right) p^2$$

Step 5: Condition for S to be a perfect cube

For S to be a perfect cube, we need:

$$-p^3 + \left(\frac{m_2}{2} - 3q\right) p^2 = 0$$

This gives:

$$p = \frac{m_2}{2} - 3q \tag{5}$$

Step 6: Solve for m_1 and m_2

From equation (2), we have:

$$m_1 = \left[3q^2 - (b_4 + \cdots + b_t)\right] 2n = \left[3q^2 - (b_4 + \cdots + b_t)\right] 4q^3$$

and:

$$m_2 = \frac{b_4^3 + \cdots + b_t^3}{3m_1} = \frac{b_4^3 + \cdots + b_t^3}{12q^3 \left[3q^2 - (b_4 + \cdots + b_t)\right]} \tag{2}$$

Then the equation for p becomes:

$$p = \frac{b_4^3 + \cdots + b_t^3}{24q^3 \left[3q^2 - (b_4 + \cdots + b_t)\right]} - 3q$$

Step 7: Expression for p

Let:

$$b_5 = c_5 b_4, \ldots, b_t = c_t b_4$$

Then:

$$p = \frac{b_4^3 \left(1 + c_5^3 + \cdots + c_t^3\right)}{24q^3 \left[3q^2 - b_4 \left(1 + c_5 + \cdots + c_t\right)\right]} - 3q$$

Step 8: Express p in terms of d

Let:

$$b_4 = 6dq^2$$

Then:

$$p = \frac{3d^3 q \left(1 + c_5^3 + \cdots + c_t^3\right)}{1 - 2d \left(1 + c_5 + \cdots + c_t\right)} - 3q$$

Assume:

$$1 - 2d \left(1 + c_5 + \cdots + c_t\right) = 3q$$

Then:

$$p = d^3 \left(1 + c_5^3 + \cdots + c_t^3\right) - 3q$$

Thus:

$$d \left(1 + c_5 + \cdots + c_t\right) = \frac{1 - 3q}{2}$$

with q odd, and then choose $d, c_5, \ldots, c_t$ such that the equation is satisfied.

Step 9: Compute the values of $a_1, a_2, \ldots, a_t$

From equation (*), we have:

$$k = \frac{m_2}{2}p^2 - \frac{m_1}{2n}p - \frac{n}{2} = (p + 3q)p^2 - 9q^3.p - q^3$$

Thus:

$$a_1 = k+n = k+2q^3, \quad a_2 = k+\frac{m_1 p}{n} = k+18q^3 p, \quad a_3 = -k, \quad b_4 = 6dq^2, \quad a_4 = b_4 p, \ldots, a_t = b_t p$$

Example: Verification of a Cubic and Cube Sum Identity for $n = 5$

We wish to show that for a certain choice of parameters, the following two identities hold:

$$\sum_{i=1}^{5} a_i^3 = b^3 \quad \text{and} \quad \sum_{i=1}^{5} a_i = c^3.$$

In this example, the parameters are chosen as follows.

Step 1: Parameter Selection

We start with the relation

$$d(1 + c_5) = \frac{1 - 3q}{2}.$$

Choose

$$q = 1 \quad \text{and} \quad d = 1.$$

Then

$$1 + c_5 = -1 \quad \Longrightarrow \quad c_5 = -2.$$

Step 2: Compute p and k

With $c_5 = -2$, we define

$$p = d^3 \left(1 + c_5^3\right) - 3q.$$

Since $d = 1$ and $c_5^3 = (-2)^3 = -8$, we have:

$$p = 1 \cdot (1 + (-8)) - 3 \cdot 1 = (1 - 8) - 3 = -10.$$

Next, compute

$$k = (p + 3q)\, p^2 - 9q^3\, p - q^3.$$

Substitute $p = -10$ and $q = 1$:

$$\begin{aligned} p + 3q &= -10 + 3 = -7, \\ (p + 3q)p^2 &= -7 \cdot (-10)^2 = -7 \cdot 100 = -700, \\ 9q^3\, p &= 9 \cdot 1 \cdot (-10) = -90, \\ \text{so} \quad k &= -700 - (-90) - 1 = -700 + 90 - 1 = -611. \end{aligned}$$

Step 3: Determining the a_i's

We define the first three components as follows:

$$a_1 = k + 2q^3 = -611 + 2 \cdot 1 = -609,$$

$$a_2 = k + 18q^3 p = -611 + 18 \cdot (-10) = -611 - 180 = -791,$$

$$a_3 = -k = 611.$$

Next, we set

$$b_4 = 6d\, q^2 = 6 \cdot 1 \cdot 1^2 = 6,$$

and define

$$a_4 = b_4\, p = 6 \cdot (-10) = -60,$$

$$a_5 = c_5\, a_4 = -2 \cdot (-60) = 120.$$

Step 4: Verification

The resulting solution set is:

$$(a_1, a_2, a_3, a_4, a_5) = (-609,\ -791,\ 611,\ -60,\ 120).$$

We now verify the identities.

Cube Sum of a_i's: It is given that

$$\sum_{i=1}^{5} a_i^3 = (-789)^3.$$

(One can check by direct calculation that the sum of cubes equals $(-789)^3$.)

Sum of a_i's:

$$\begin{aligned}\sum_{i=1}^{5} a_i &= -609 - 791 + 611 - 60 + 120 \\ &= (-609 - 791) + 611 - 60 + 120 \\ &= -1400 + 611 - 60 + 120 \\ &= -789 - 60 + 120 \\ &= -729,\end{aligned}$$

and since

$$(-9)^3 = -729,$$

we have

$$\sum_{i=1}^{5} a_i = (-9)^3.$$

Thus, the solution set

$$(-609,\ -791,\ 611,\ -60,\ 120)$$

satisfies

$$\sum_{i=1}^{5} a_i^3 = (-789)^3 \quad \text{and} \quad \sum_{i=1}^{5} a_i = (-9)^3.$$

Problem 6

Prove that there exist infinitely many tuples of integers $(a_1, a_2, \ldots, a_t)$ with $t > 6$, such that:

- Each $a_i \neq 0$
- No two numbers are negatives of each other
- All a_i are pairwise distinct
- $\gcd(a_1, a_2, \ldots, a_t) = 1$

and such that

$$\sum_{i=1}^{t} a_i^3 = b^s$$

for $b \in \mathbb{Z}$, for any given positive integer s

Step 1: Establishing the General Equation

Consider two integers a, b satisfying the equation:

$$a \cdot b = k \cdot (a + b) + c$$

Let $a + b = x$, then the sum of their cubes is:

$$a^3 + b^3 = x^3 - 3kx^2 - 3cx$$

Rearranging:

$$(x - k)^3 = a^3 + b^3 + (-k)^3 + 3(k^2 + c)x$$

Thus, we obtain:

$$(x - k)^3 + k^3 + (-a)^3 + (-b)^3 = 3(k^2 + c)x$$

From the original equation, we have:

$$(a - k)(b - k) = k^2 + c$$

Setting $a - k = n$, then $b - k = \frac{k^2+c}{n}$, so:

$$x = a + b = 2k + n + \frac{k^2 + c}{n}$$

Let $k^2 + c = m = np$, then:

$$x = 2k + n + p$$

It follows that:

$$3(k^2 + c)x = 3np(2k + n + p) = 6knp + 3np(n + p)$$

Thus:

$$(x-k)^3+k^3+(-a)^3+(-b)^3=6knp+(n+p)^3-n^3-p^3$$

By adding two new numbers n, p and their negative sum:

$$(x-k)^3+k^3+(-a)^3+(-b)^3+n^3+p^3+(-n-p)^3=6knp$$

Step 2: Choosing k to Satisfy the Condition

Choose k such that:

$$k^3=a_1^3+a_2^3+...+a_t^3$$

where $t \neq 2$, ensuring infinitely many choices for k. For each k, there always exist integers n, p such that:

$$6knp=(6d)^s$$

This allows us to compute $a=k+n$, $b=k+p$. Consequently:

$$(x-k)^3+(-a)^3+(-b)^3+n^3+p^3+(-n-p)^3+a_1^3+...+a_t^3=(6d)^s$$

Conclusion

Thus, we have proven the existence of infinitely many integer sets satisfying

$$\sum_{i=1}^{t} a_i^3=b^s$$

for $b \in \mathbb{Z}$, for any given positive integer s

Example: Verification of a Cubic Sum Equal to a Fourth Power for $t=7$, $s=4$

In this example, we demonstrate that for a suitable choice of parameters, the following identity holds:

$$\sum_{i=1}^{7} a_i^3=b^4.$$

Step 1: Establishing the Relation

We start with the relation

$$6k\,n\,p = (6d)^4.$$

Choosing $d = 1$, this becomes

$$6k\,n\,p = 6^4,$$

which implies

$$k\,n\,p = 6^3 = 216.$$

We then select:

$$k = 1, \quad n = 8, \quad p = 27,$$

since $1 \times 8 \times 27 = 216$.

Step 2: Defining Intermediate Variables

Define:

$$a = k + n = 1 + 8 = 9, \quad b = k + p = 1 + 27 = 28.$$

Step 3: Constructing the Solution Set

We now define the seven numbers $a_1, a_2, \ldots, a_7$ as follows:

$$a_1 = a + b - k = 9 + 28 - 1 = 36,$$

$$a_2 = k = 1,$$

$$a_3 = -a = -9,$$

$$a_4 = -b = -28,$$

$$a_5 = n = 8,$$

$$a_6 = p = 27,$$

$$a_7 = -n - p = -8 - 27 = -35.$$

Step 4: Verification

It can be verified that

$$\sum_{i=1}^{7} a_i^3 = 36^3 + 1^3 + (-9)^3 + (-28)^3 + 8^3 + 27^3 + (-35)^3 = 6^4.$$

Hence, the solution set

$$(36,\ 1,\ -9,\ -28,\ 8,\ 27,\ -35)$$

satisfies the identity

$$\sum_{i=1}^{7} a_i^3 = 6^4.$$

Problem 7

Prove that there exist infinitely many tuples of integers

$$(a_1, a_2, a_3, a_4) \quad \text{and} \quad (b_1, b_2, \ldots, b_t) \quad (t > 5),$$

such that:

- Each $a_i \neq 0$
- No two numbers are negatives of each other
- All a_i are pairwise distinct
- $\gcd(a_1, a_2, a_3, a_4) = 1$
- $\gcd(b_1, b_2, \ldots, b_t) = 1$

and

$$a_1^4 + a_2^4 - a_3^4 - a_4^4 = 6 \cdot \left(\sum_{i=1}^{t} b_i^3 \right)^2$$

Solution

Step 1: Given Identity and Assumption

We are given that there exist integers a, b such that:

$$ab = k(a + b) + c.$$

From a previous result, we have:

$$(X - k)^4 = a^4 + b^4 + \left[(4k^2 + 4c)X^2 - 4k(k^2 + c)X + (k^4 - 2c^2)\right].$$

Let us denote:

$$P(X) = (4k^2 + 4c)X^2 - 4k(k^2 + c)X + (k^4 - 2c^2).$$

Step 2: Simplifying $P(X)$

We set:

$$k^2 + c = m \quad \Rightarrow \quad c = m - k^2.$$

Then:

$$P(X) = 4mX^2 - 4kmX + k^4 - 2(m - k^2)^2.$$

Expanding and simplifying:

$$\begin{aligned} P(X) &= m(4X^2 - 4kX + k^2) - mk^2 + k^4 - 2(m^2 - 2mk^2 + k^4) \\ &= m(2X - k)^2 - k^4 + (3mk^2 - 2m^2). \end{aligned}$$

Step 3: Eliminate the Extra Term

We choose m such that:

$$3mk^2 - 2m^2 = 0 \quad \Rightarrow \quad m = \frac{3k^2}{2}.$$

Now, let $k = 2d$, then:

$$m = \frac{3(4d^2)}{2} = 6d^2.$$

Thus,

$$P(X) = 6d^2(2X - 2d)^2 - (2d)^4.$$

Step 4: Expressing $X = a + b$

From the identity:

$$ab = k(a+b) + c \Rightarrow (a-k)(b-k) = k^2 + c = m.$$

Set:

$$a - k = n, \quad b - k = \frac{m}{n}.$$

Then:

$$a = k + n, \quad b = k + \frac{m}{n}, \quad X = a + b = 2k + n + \frac{m}{n}.$$

Thus:

$$2X - k = 3k + 2n + \frac{2m}{n}.$$

Substitute $k = 2d, m = 6d^2$, we get:

$$2X - k = 6d + 2n + \frac{12d^2}{n}.$$

For example, if $n = 6$, then:

$$2X - k = 6d + 12 + 2d^2.$$

Step 5: Final Expression for $P(X)$

Using $m = 6d^2$ and $2X - k = 2d^2 + 6d + 12$, we have:

$$P(X) = 6d^2(2d^2 + 6d + 12)^2 - (2d)^4.$$

Note that:

$$6d^2(2d^2 + 6d + 12)^2 = 6\left[(d+2)^3 + d^3 + (-2)^3\right]^2.$$

Hence:

$$(X - k)^4 = a^4 + b^4 + 6\left[(d+2)^3 + d^3 + (-2)^3\right]^2 - (2d)^4.$$

Step 6: Final Identity

Let us define:

$$X - k = a_1, \quad k = a_2 = 2.d, \quad a = a_3 = 2d + 6, \quad b = a_4 = 2d + d^2,$$

and

$$d + 2 = b_1, \quad -2 = b_2.$$

Choose d such that:

$$d^3 = b_3^3 + b_4^3 + \cdots + b_t^3 \quad (t > 5),$$

then we obtain:

$$a_1^4 + a_2^4 - a_3^4 - a_4^4 = 6\left(\sum_{i=1}^{t} b_i^3\right)^2.$$

Example: A Quartic and Cubic Identity for $t = 7$

We wish to demonstrate an example with $t = 7$ where the following identity holds:

$$a_1^4 + a_2^4 - a_3^4 - a_4^4 = 6\left(\sum_{i=1}^{7} b_i^3\right)^2.$$

The construction is carried out as follows.

Step 1: Choosing the b_i Values

Assume we require that

$$d^3 = b_3^3 + b_4^3 + b_5^3 + b_6^3 + b_7^3.$$

It is known that the tuple

$$(-7, 34, 8, 3, 15)$$

satisfies the necessary condition. Thus, choose

$$b_3 = -7, \quad b_4 = 34, \quad b_5 = 8, \quad b_6 = 3, \quad b_7 = 15.$$

Then we have

$$d^3 = (-7)^3 + 34^3 + 8^3 + 3^3 + 15^3.$$

For this example, it is given that $d = 35$.

Next, choose the remaining values:

$$b_1 = d + 2 = 35 + 2 = 37, \quad b_2 = -2.$$

Step 2: Defining the a_i Values

We now define the parameters for the a_i's. Set

$$a_2 = k = 2d = 2 \times 35 = 70.$$

Also, choose

$$a_3 = 2d + 6 = 70 + 6 = 76,$$

and

$$a_4 = d^2 + 2d = 35^2 + 2 \times 35 = 1225 + 70 = 1295.$$

Finally, define

$$a_1 = a_3 + a_4 - 2d = 76 + 1295 - 70 = 1301.$$

Step 3: Verification of the Identity

With the above choices, the solution set for the a_i's is

$$(a_1, a_2, a_3, a_4) = (1301,\ 70,\ 76,\ 1295),$$

and for the b_i's we have

$$(b_1, b_2, b_3, b_4, b_5, b_6, b_7) = (37,\ -2,\ -7,\ 34,\ 8,\ 3,\ 15).$$

It is then verified that

$$a_1^4 + a_2^4 - a_3^4 - a_4^4 = 6\left(\sum_{i=1}^{7} b_i^3\right)^2.$$

A direct computation shows that the left-hand side equals the right-hand side when the above values are substituted.

Thus, the solution set

$$(1301,\ 70,\ 76,\ 1295) \quad \text{and} \quad (37,\ -2,\ -7,\ 34,\ 8,\ 3,\ 15)$$

satisfies the desired identity.

Problem 8

Prove that there exist infinitely many tuples of integers

$$(a_1, a_2, a_3, a_4, m) \quad \text{and} \quad (b_1, b_2, \dots, b_t) \quad (t > 4),$$

such that:

- Each $a_i \neq 0$
- No two numbers are negatives of each other
- All a_i are pairwise distinct
- $\gcd(a_1, a_2, a_3, a_4, m) = 1$
- $\gcd(b_1, b_2, \dots, b_t) = 1$

and

$$\sum_{i=1}^{4} a_i^5 = 5.m. \sum_{i=1}^{t} b_i^3,$$

2 Step 1: The Core Identity

Let a, b be integers and define $X = a + b$. Suppose:

$$ab = k(a + b) + c. \tag{1}$$

We consider the expansion of $(X - k)^5$, which yields:

$$(X - k)^5 = a^5 + b^5 + P(X),$$

where $P(X)$ is a polynomial in X depending on k and c.

3 Step 2: Deriving the Polynomial $P(X)$

The polynomial $P(X)$ is given by:

$$P(X) = (5k^2 + 5c)X^3 - 10k(k^2 + c)X^2 + 5(k^4 - c^2)X - k^5.$$

This simplifies to:

$$P(X) = 5(k^2 + c)(X^3 - 2kX^2 + (k^2 - c)X) - k^5.$$

4 Step 3: Symmetrization of the Identity

Using the symmetry of powers:

$$(X-k)^5 + (-a)^5 + (-b)^5 + k^5 = 5(k^2+c)(X^3 - 2kX^2 + (k^2-c)X).$$

Let:

$$G(X) = X^3 - 2kX^2 + (k^2 - c)X.$$

5 Step 4: Setting Parameters

Let $k = 3d$ for some integer d, then:

$$G(X) = X^3 - 6dX^2 + (9d^2 - c)X.$$

Now rewrite:

$$G(X) = (X-2d)^3 + (2d)^3 + (-3d^2 - c)X.$$

To simplify this expression, choose:

$$c = -3d^2 \Rightarrow G(X) = (X-2d)^3 + (2d)^3.$$

6 Step 5: Constructing the b_i's

Let:

$$X - 2d = b_1, \quad (2d)^3 = \sum_{i=2}^{t} b_i^3 \quad \text{for } t > 4.$$

Then:

$$G(X) = \sum_{i=1}^{t} b_i^3.$$

7 Step 6: Constructing the a_i's

Recall that:

$$(X-k)^5 + (-a)^5 + (-b)^5 + k^5 = 5(k^2+c)\sum_{i=1}^{t} b_i^3.$$

Define:

$$a_1 = X - k, \quad a_2 = k, \quad a_3 = -a, \quad a_4 = -b.$$

From the identity (2), we have:

$$(a - k)(b - k) = ab - k(a + b) + k^2 = k^2 + c.$$

Using $k = 3d$, $c = -3d^2$, we get:

$$k^2 + c = 6d^2.$$

Choose:

$$a = 3d + 6, \quad b = d^2 + 3d \Rightarrow a + b = X = 3d + 6 + d^2 + 3d = d^2 + 6d + 6.$$

Thus:

$$X - k = a_1 = d^2 + 6d + 6 - 3d = d^2 + 3d + 6, \quad a_2 = 3d,$$

$$a_3 = -a = -3d - 6, \quad a_4 = -b = -d^2 - 3d.$$

8 Step 7: Final Identity

From all of the above, we get:

$$a_1^5 + a_2^5 + a_3^5 + a_4^5 = 5(k^2 + c)\sum_{i=1}^{t} b_i^3 = 5 \cdot m \cdot \sum_{i=1}^{t} b_i^3.$$

Hence, this identity holds:

$$\sum_{i=1}^{4} a_i^5 = 5 \cdot m \cdot \sum_{i=1}^{t} b_i^3,$$

after choosing d such that $(2d)^3 = \sum_{i=2}^{t} b_i^3$, and setting $b_1 = X - 2d$.

9 Conclusion

We have demonstrated a method to construct infinitely many tuples of integers satisfying:

$$\sum_{i=1}^{4} a_i^5 = 5 \cdot m \cdot \sum_{i=1}^{t} b_i^3,$$

Example: A Fifth-Power and Cubic Identity with $t = 6$

In this example, we show that for a certain choice of parameters, the following identity holds:

$$\sum_{i=1}^{4} a_i^5 = 5m \cdot \sum_{i=1}^{6} b_i^3,$$

with $m = 96$.

Preliminary Observation

It is given that

$$(-7)^3 + 34^3 + 8^3 + 3^3 + 15^3 = 35^3.$$

Hence, we can deduce

$$8^3 = (2 \cdot 4)^3 = 35^3 + 7^3 + (-34)^3 + (-3)^3 + (-15)^3.$$

Step 1: Choice of Parameters for the b_i's

Take

$$d = 4.$$

Then choose:

$$b_2 = 35, \quad b_3 = 7, \quad b_4 = -34, \quad b_5 = -3, \quad b_6 = -15.$$

Define

$$b_1 = d^2 + 4d + 6.$$

Substituting $d = 4$, we have:

$$b_1 = 4^2 + 4 \cdot 4 + 6 = 16 + 16 + 6 = 38.$$

Thus, the vector for the b_i's is:

$$(b_1, b_2, b_3, b_4, b_5, b_6) = (38,\ 35,\ 7,\ -34,\ -3,\ -15).$$

Step 2: Choice of Parameters for the a_i's

We define the a_i's by setting:

$$a_1 = d^2 + 3d + 6, \quad a_2 = 3d, \quad a_3 = -3d - 6, \quad a_4 = -d^2 - 3d.$$

For $d = 4$:

$$\begin{aligned} a_1 &= 4^2 + 3 \cdot 4 + 6 = 16 + 12 + 6 = 34, \\ a_2 &= 3 \cdot 4 = 12, \\ a_3 &= -3 \cdot 4 - 6 = -12 - 6 = -18, \\ a_4 &= -4^2 - 3 \cdot 4 = -16 - 12 = -28. \end{aligned}$$

We are also given $m = 96$.

Thus, the vector for the a_i's is:

$$(a_1, a_2, a_3, a_4) = (34,\ 12,\ -18,\ -28).$$

Step 3: Verification of the Identity

We claim that the chosen vectors satisfy the identity

$$\sum_{i=1}^{4} a_i^5 = 5 \cdot 96 \cdot \sum_{i=1}^{6} b_i^3.$$

A direct (albeit lengthy) computation confirms that the sum of the fifth powers of the a_i's equals $5 \cdot 96$ times the sum of the cubes of the b_i's.

Thus, the solution set

$$(a_1, a_2, a_3, a_4) = (34,\ 12,\ -18,\ -28) \quad \text{and} \quad (b_1, b_2, b_3, b_4, b_5, b_6) = (38,\ 35,\ 7,\ -34,\ -3,\ -15)$$

satisfies the desired identity.

Problem 9

Prove that there exist infinitely many tuples of integers

$$(a_1, a_2, a_3, a_4, m) \quad \text{and} \quad (b_1, b_2, \ldots, b_t)$$

such that:

- Each $a_i \neq 0$
- No two numbers are negatives of each other
- All a_i are pairwise distinct
- $\gcd(a_1, a_2, a_3, a_4, m) = 1$
- $\gcd(b_1, b_2, \ldots, b_t) = 1$

and

$$\sum_{i=1}^{4} a_i^5 = 5.m. \sum_{i=1}^{t} b_i^4,$$

Step-by-step Construction

Step 1: Consider the identity:

$$a \cdot b = k(a+b) + c \tag{1}$$

This implies:

$$(a-k)(b-k) = k^2 + c$$

Step 2: From previous expansion, we have:

$$(X-k)^5 = a^5 + b^5 + P(X)$$

where:

$$P(X) = (5k^2 + 5c)X^3 - 10k(k^2 + c)X^2 + 5(k^4 - c^2)X - k^5$$

which can be rewritten as:

$$P(X) = 5(k^2 + c)\left[X^3 - 2kX^2 + (k^2 - c)X\right] - k^5$$

$$(X-k)^5 + (-a)^5 + (-b)^5 + k^5 = 5(k^2 + c)X(X^2 - 2kX + k^2 - c)$$

Step 3:

$$G(X) = X^2 - 2kX + k^2 - c = (X-k)^2 - c$$

Suppose:

$$X - k = b_1^2, \quad -c = \sum_{i=2}^{t} b_i^4 \Rightarrow G(X) = \sum_{i=1}^{t} b_i^4$$

Step 4: Use the relation $(a-k)(b-k) = k^2 + c$

Let $a = k+1$ and define b accordingly:

$$b = k^2 + k + c$$

Then:

$$X = a + b = (k+1) + (k^2 + k + c) = k^2 + 2k + c + 1$$

So:

$$X - k = k^2 + k + c + 1 = (k+1)^2 - k + c = b_1^2$$

Step 6: Choose $c = k$ then:

$$X - k = (k+1)^2 = b_1^2 \Rightarrow b_1 = k+1$$

Thus, we define:

$$a_1 = X - k, \quad a_2 = k, \quad a_3 = -a, \quad a_4 = -b$$

Compute $m = (k^2 + c).X = (k^2 + k).X$.
Finally, we get:

$$\sum_{i=1}^{4} a_i^5 = 5m \sum_{i=1}^{t} b_i^4$$

Conclusion

We have shown that for any integer k, by choosing:

$$a = k+1, \quad b = k^2 + k + c, \quad c = k$$

and defining $X = a + b$, $X - k = b_1^2$, and selecting $b_2, \dots, b_t$ so that:

$$-c = \sum_{i=2}^{t} b_i^4$$

we construct infinitely many sets (a_1, a_2, a_3, a_4) and $(b_1, \dots, b_t)$ such that:

$$a_1^5 + a_2^5 + a_3^5 + a_4^5 = 5m \sum_{i=1}^{t} b_i^4$$

Example: A Fifth-Power and Fourth-Power Identity for $t = 3$

We wish to demonstrate an example with $t = 3$ for which the following identity holds:

$$\sum_{i=1}^{4} a_i^5 = 5\,m \sum_{i=1}^{3} b_i^4.$$

The construction is carried out as follows.

Step 1: Determine c and k

We start by requiring that

$$-c = b_2^4 + b_3^4.$$

Choose

$$b_2 = 1, \quad b_3 = 2.$$

Then

$$b_2^4 + b_3^4 = 1^4 + 2^4 = 1 + 16 = 17,$$

so that

$$-c = 17 \quad \Longrightarrow \quad c = -17.$$

We also set

$$k = c = -17.$$

Step 2: Define a, b, and b_1

Define

$$a = k + 1 = -17 + 1 = -16,$$

$$b = k^2 + 2k.$$

Since

$$k^2 = (-17)^2 = 289 \quad \text{and} \quad 2k = -34,$$

we have

$$b = 289 - 34 = 255.$$

Also, let

$$b_1 = k + 1 = -16.$$

Step 3: Construct the a_i Values

Define the four a_i's by

$$a_1 = a + b - k, \quad a_2 = k, \quad a_3 = -a, \quad a_4 = -b.$$

Substituting the computed values:

$$a_1 = (-16) + 255 - (-17) = 239 + 17 = 256,$$

$$a_2 = k = -17,$$

$$a_3 = -(-16) = 16,$$

$$a_4 = -255.$$

Thus, the a_i vector is

$$(a_1, a_2, a_3, a_4) = (256,\ -17,\ 16,\ -255).$$

Step 4: Define m and the b_i Vector

We define the parameter m by

$$m = (k^2 + k)(a + b).$$

Here,

$$k^2 + k = 289 - 17 = 272,$$

and

$$a + b = (-16) + 255 = 239.$$

Thus, $m = 272 \times 239$ (its exact numerical value is not necessary for our identity).

The b_i vector is given by

$$(b_1, b_2, b_3) = (-16,\ 1,\ 2).$$

Step 5: Verification

It can be verified (by direct computation) that the chosen values satisfy the identity

$$\sum_{i=1}^{4} a_i^5 = 5\,m \sum_{i=1}^{3} b_i^4.$$

That is, the sum of the fifth powers of the a_i's equals five times m times the sum of the fourth powers of the b_i's.

Thus, the solution set

$$(a_1, a_2, a_3, a_4) = (256,\ -17,\ 16,\ -255) \quad \text{and} \quad (b_1, b_2, b_3) = (-16,\ 1,\ 2)$$

satisfies the desired identity.

Problem 10

Prove that there exist infinitely many tuples of integers

$$(c_1, c_2, \ldots, c_7) \quad \text{and} \quad (d_1, d_2, \ldots, d_t) \quad (t > 11 \text{ or } t = 9)$$

such that:

- Each $c_i, d_i \neq 0$
- No two numbers are negatives of each other
- All c_i are pairwise distinct
- All d_i are pairwise distinct

- $\gcd(c_1, c_2, \ldots, c_7) = 1$
- $\gcd((d_1, d_2, \ldots, d_t) = 1$

and

$$\sum_{i=1}^{7} c_i^5 = \frac{5np}{2} \sum_{i=1}^{t} d_i^3.$$

Let $n, p \in \mathbb{Z}$ be arbitrary integers

10 Step 1: The Core Identity

Let a, b be integers and define

$$X = a + b.$$

Suppose that

$$ab = k(a + b) + c, \tag{2}$$

for some fixed integers k and c. One may show that the fifth power of $X - k$ can be written as

$$(X - k)^5 = a^5 + b^5 + P(X),$$

where $P(X)$ is a polynomial in X (with coefficients depending on k and c).

11 Step 2: Deriving the Polynomial $P(X)$

It can be proved that

$$P(X) = (5k^2 + 5c)X^3 - 10k(k^2 + c)X^2 + 5(k^4 - c^2)X - k^5.$$

Using the transformation

$$(a - k)(b - k) = k^2 + c,$$

we write

$$(a - k)(b - k) = n \cdot p,$$

so that

$$c = np - k^2.$$

With the substitutions

$$a = k + n, \quad b = k + p,$$

we obtain

$$X = a + b = 2k + n + p.$$

After a lengthy calculation, one deduces the alternative form:

$$\begin{aligned} P(X) &= 20np\,k^3 + 30np\,k^2(n+p) + 20k\,np\,(n^2+p^2) + 30k\,(np)^2 \\ &\quad + 5n^4p + 5np^4 + 10(np)^2(n+p) - k^5 \\ &= (n+p)^5 - n^5 - p^5 \\ &\quad + 10np\,k\Big[2k^2 + 3k(n+p) + 2(n^2+p^2) + 3np\Big] - k^5. \end{aligned}$$

12 Step 3: Forming the Symmetric Fifth-Power Identity

By definition,

$$(X-k)^5 = a^5 + b^5 + P(X).$$

Thus,

$$\begin{aligned} (X-k)^5 &= a^5 + b^5 + (n+p)^5 - n^5 - p^5 \\ &\qquad + 10np\,k\Big[2k^2 + 3k(n+p) + 2(n^2+p^2) + 3np\Big] - k^5. \end{aligned}$$

Now, by adding the fifth powers of the negatives of a and b and of $(n+p)$,
we obtain:

$$(X-k)^5+(-a)^5+(-b)^5+(-n-p)^5+n^5+p^5+k^5 = 10np\,k\Big[2k^2+3k(n+p)+2(n^2+p^2)+3np\Big].$$

This expression may be rewritten as

$$\frac{5np}{2}\Big[8k^3+12k^2(n+p)+8k(n^2+p^2)+12knp\Big] = \frac{5np}{2}\Big[(2k+n+p)^3+2k(n^2+p^2)+(-n-p)^3\Big]. \qquad (*)$$

13 Step 4: Reformulation via New Variables

Assume that

$$2k(n^2+p^2) = 6n_1p_1k_1 \quad \text{or equivalently} \quad k(n^2+p^2) = 3n_1p_1k_1.$$

Then one can prove the identity

$$6n_1p_1k_1 = (a_1+b_1-k_1)^3 + (-a_1)^3 + (-b_1)^3 + k_1^3 + n_1^3 + p_1^3 + (-n_1-p_1)^3,$$

where

$$k_1^3 = d_9^3 + d_2^3 + \cdots + d_t^3 \quad (t > 11 \text{ or } t = 9),$$

and the remaining cubes are denoted by $d_3^3, d_4^3, \ldots, d_8^3$. Thus,

$$6n_1p_1k_1 = d_3^3 + d_4^3 + \cdots + d_t^3.$$

Setting

$$d_1 = 2k + n + p \quad \text{and} \quad d_2 = -n - p,$$

equation $(*)$ becomes

$$(X-k)^5 + (-a)^5 + (-b)^5 + (-n-p)^5 + n^5 + p^5 + k^5 = \frac{5np}{2}\sum_{i=1}^{t} d_i^3.$$

Finally, define

$$c_1 = X-k, c_2 = -a, c_3 = -b, c_4 = -(n+p), c_5 = n, c_6 = p, c_7 = k.$$

Then, the above identity can be written in compact form:

$$\sum_{i=1}^{7} c_i^5 = \frac{5np}{2}\sum_{i=1}^{t} d_i^3.$$

14 Conclusion and Infinitude of the Solutions

It remains to show that there exist infinitely many tuples

$$(c_1, c_2, \ldots, c_7) \quad \text{and} \quad (d_1, d_2, \ldots, d_t)$$

satisfying:

- Each $c_i, d_i \neq 0$,
- No two numbers are negatives of each other,
- All c_i are pairwise distinct,
- All d_i are pairwise distinct,
- $\gcd(c_1, c_2, \ldots, c_7) = 1$,
- $\gcd(d_1, d_2, \ldots, d_t) = 1$.

Because the parameters n, p, k can be chosen arbitrarily (subject to the constraint in (2) and the derived identities) and the transformation

$$(a-k)(b-k) = n \cdot p$$

yields infinitely many representations, the identity

$$\sum_{i=1}^{7} c_i^5 = \frac{5np}{2} \sum_{i=1}^{t} d_i^3$$

Let $n, p \in \mathbb{Z}$ be arbitrary integers

Example: A Fifth-Power and Cubic Identity with $n = 1$, $p = 2$, and $t = 7$

We wish to verify the following identity:

$$\sum_{i=1}^{7} c_i^5 = 5 \sum_{i=1}^{7} d_i^3,$$

with parameters chosen as follows.

Step 1. Determination of k:
It is given that

$$k(n^2 + p^2) = 3\, n_1\, p_1\, k_1 = 5k.$$

We choose

$$n_1 = 5, \quad p_1 = 1, \quad k_1 = 3.$$

Then,

$$3 \cdot 5 \cdot 1 \cdot 3 = 45 = 5k \quad \Longrightarrow \quad k = \frac{45}{5} = 9.$$

Step 2. Relating Higher Powers:
We have the relation

$$2k(n^2 + p^2) = 6\, n_1\, p_1\, k_1.$$

Moreover, it is assumed that

$$2k(n^2+p^2) = (n_1+p_1+k_1)^3+k_1^3+n_1^3+p_1^3+(-k_1-n_1)^3+(-k_1-p_1)^3+(-n_1-p_1)^3.$$

Substituting $n_1 = 5$, $p_1 = 1$, $k_1 = 3$ gives:

$$\begin{aligned}
(n_1 + p_1 + k_1)^3 &= (5 + 1 + 3)^3 = 9^3, \\
k_1^3 = 3^3, \quad n_1^3 &= 5^3, \quad p_1^3 = 1^3, \\
(-k_1 - n_1)^3 &= (-3 - 5)^3 = (-8)^3, \\
(-k_1 - p_1)^3 &= (-3 - 1)^3 = (-4)^3, \\
(-n_1 - p_1)^3 &= (-(5 + 1))^3 = (-6)^3.
\end{aligned}$$

Thus, the relation becomes:

$$2k(n^2 + p^2) = 9^3 + 3^3 + 5^3 + 1^3 + (-8)^3 + (-4)^3 + (-6)^3.$$

Step 3. Defining the d_i Values:
We set

$$d_1 = 2k + n + p.$$

With $k = 9$, $n = 1$, and $p = 2$, we have:

$$d_1 = 2 \cdot 9 + 1 + 2 = 18 + 3 = 21.$$

Also, let

$$d_2 = -n - p = -1 - 2 = -3.$$

The remaining d_i are taken as given by the decomposition:

$$(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = (21,\ 9,\ 5,\ 1,\ -8,\ -4,\ -6).$$

Step 4. Defining the c_i Values:
We define

$$c_1 = k + n + p = 9 + 1 + 2 = 12,$$

$$c_2 = -k - n = -9 - 1 = -10,$$

$$c_3 = -k - p = -9 - 2 = -11,$$

$$c_4 = n = 1, \quad c_5 = p = 2, \quad c_6 = k = 9,$$

$$c_7 = -n - p = -1 - 2 = -3.$$

Thus, the c_i vector is:

$$(c_1, c_2, c_3, c_4, c_5, c_6, c_7) = (12,\ -10,\ -11,\ 1,\ 2,\ 9,\ -3).$$

Step 5. Verification:
A direct computation shows that

$$\sum_{i=1}^{7} c_i^5 = 5 \sum_{i=1}^{7} d_i^3.$$

That is, the sum of the fifth powers of the c_i's equals five times the sum of the cubes of the d_i's.

Conclusion:
The solution set

$$(c_1, c_2, c_3, c_4, c_5, c_6, c_7) = (12,\ -10,\ -11,\ 1,\ 2,\ 9,\ -3)$$

and

$$(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = (21,\ 9,\ 5,\ 1,\ -8,\ -4,\ -6)$$

satisfies the identity

$$\sum_{i=1}^{7} c_i^5 = 5 \sum_{i=1}^{7} d_i^3.$$

Conclusion

In this work, we have laid the theoretical foundation for a new branch of Diophantine analysis, which we refer to as the *extended Diophantine equations*, through the application of the symmetric identity

$$ab = k(a + b) + c.$$

This identity allows high-degree symmetric expressions such as $a^n + b^n$ to be rewritten in terms of the variable $X = a + b$, establishing a deep connection between symmetric expansions and binomial expansions via a correction polynomial $P(X)$. As a result, we derived both recurrence and closed-form formulas for the coefficients, enabling the construction of infinitely many sets of integer solutions that satisfy strict conditions—nonzero, mutually distinct, not negatives of each other, and pairwise coprime.

The contributions of this work are not only theoretical in nature but also open several promising avenues for further development:

- **Extension to multivariable systems:** The method can potentially be generalized to Diophantine systems in more than two variables, making it applicable to more complex symmetric structures.
- **Connections to combinatorics and abstract algebra:** The coefficients appearing in the symmetric expansions may relate to combinatorial quantities such as Catalan numbers or binomial coefficients, bridging number theory with other mathematical disciplines.
- **Applications in solution generation algorithms:** The explicit structure of the identities and formulas supports the creation of computational tools that can systematically generate and verify solutions.
- **Inspiration for new Diophantine problems:** This framework enables the formulation and resolution of new problems involving sums or differences of powers under symmetric or structural constraints.

Thus, this work not only broadens the existing understanding of classical Diophantine equations but also establishes a solid foundation for a rich and expandable branch of number theory, with both theoretical significance and practical potential.

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