On the uncomputability of hydrodynamics

Warren D. Smith*
wds@research.NJ.NEC.COM

May 30, 2005

Abstract **— We construct a rigid and bounded 3D container C with bounded surface area and whose boundary, although complicated, is smooth everywhere except at a single point. It is partly filled with a fluid of constant density and viscosity, having bounded kinetic energy, and indeed, uniformly bounded flow velocities. The container's shape and the initial location-set and velocity field for the fluid (both of which are as smooth as possible) all have finite-length mathematical descriptions.**

We demonstrate that, e.g., predicting which of the two alternatives "$\geq 2\text{cm}^3$ of fluid will flow into basin A during the next minute" and "$\leq 1\text{cm}^3$ will flow into basin A, ever" will happen (one of these may be guaranteed) is at least as hard as solving Turing's general halting problem, i.e. *undecidable*. But a physical system corresponding to C, would solve the problem in 1 minute. This demonstrates the falsity of "Church's thesis" under these laws of physics.

This "demonstration" is not a mathematical proof since it depends on certain unproved – but empirically very well confirmed – assumptions. (It also shamelessly exploits certain mathematical, but unphysical, features of the equations of hydrodynamics, namely: assumption of a perfect continuum all the way down to zero length scale, perfect wall rigidity, and exact constancy of viscosity and density despite any temperature and pressure changes.) Nevertheless we produce genuine theorems at the end whose statements (I argue) signify the failure of hydrodynamics.

Keywords — Fluidics, hydrodynamics, undecidability, Church's thesis, Turing's halting problem, algorithmization of physics, non-existence of hydrodynamic limit, failure of hydrodynamics, water hammer, slug flow, fault tolerance, Toom rule.

Contents

1 Introduction

FOR A FLUID WITH CONSTANT density $\rho > 0$ and constant kinematic viscosity $\nu > 0$ (for water, $\rho \approx 1\text{milligram}\,\text{mm}^{-3}$ and $\nu \approx 1\text{mm}^2/\text{second}$), and assuming that all thermal effects can be neglected (either because the fluid has infinite thermal conductivity, or infinite specific heat, or because the viscosity and density do not depend on temperature), the Navier-Stokes equations of 3D fluid flow become

$$\vec{\nabla} \cdot \vec{u} = 0 \tag{1}$$

and

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = \frac{\vec{F}_{\text{ext}} - \vec{\nabla} p}{\rho} + \nu \nabla^2 \vec{u}, \tag{2}$$

customarily called the equations of **"hydrodynamics"** [4][9][36][64][71]. Here $\vec{u}(\vec{x}, t)$ is the fluid flow velocity field, $\vec{x}$ is position, t is time, and $p(\vec{x}, t)$ is the pressure scalar field. $\vec{F}_{\text{ext}}(\vec{x}, t)$ are the externally applied forces (if any) per unit volume. These are 4 equations expressing the 4 unknown fields ($\vec{u}$ and p) at time $t + \mathrm{d}t$ in terms of same at time t.

At solid boundaries, one customarily demands the "no slip" condition $\vec{u} = 0$ (R.P.Feynman: "it is a common observation

*NECI, 4 Independence Way Princeton NJ 08540 USA

that the blade of a fan will collect a thin layer of dust")[1]. If $\nu = 0$ (inviscid flow) these equations are instead called the "Euler equations."

It has sometimes been wondered what is the computational complexity of simulating the hydrodynamics PDEs.

It seems impossible to answer that with full mathematical rigor since at present the more fundamental questions of the *existence* (and smoothness properties) of solutions to these equations remain open[2]. I will recount the mainstream conjectural beliefs about that in §2*. Impatient readers could skip directly from §1 to §6, (or might only wish to examine the skipped sections *after*wards). The present paper argues (nonrigorously, but convincingly) starting in §6 that certain classes of hydrodynamics problems are *undecidable*, i.e. there exists no algorithm for their solution. We *do* get a wholy rigorous theorem that either (1) this undecidability happens, or (2) nonexistence happens, or (3) hydrodynamics predicts behavior on some heavily experimentally-investigated fluid-dynamic scenarios, underlying established technologies, different from what those experiments find. In any of these 3 cases, I conclude that hydrodynamics is a failure. I believe that the mode of failure is (1), and it is due to the lack of existence of what I call the "hydrodynamic limit" – see §13 for interpretation of results and moral.

However, maybe the situation is partially salveagable? Perhaps hydrodynamics still can succeed for some important subset of hydrodynamical problems – in which case effort should be devoted to determining what that subset is and defining what "success" means. In §5, we argue that some attempted "quick fix" salvages won't work.

2 Summary of the usual conjectural beliefs about existence, uniqueness, and smoothness properties for hydrodynamics*

A[3] few theorems [19][9] are known. They say, roughly, that

1. Smooth solutions to the equations of hydrodynamics, given smooth and bounded initial data and externally-applied force fields, will exist, uniquely, for some nonzero (but possibly problem-dependent) amount of time T into the future.
2. If the "bound" is small enough, then $T = \infty$.

The rest of this section summarizes the conventional (albeit conjectural) wisdom about hydrodynamics. But since I have never seen such a summary before, I cannot be sure. We shall use Couette flow, and the Hagen-Poiseuille pipe flow, since they are probably the two most heavily-studied flows ever, as running examples throughout our discussion.

The **"Couette flow"** [60] is a rotatory flow of fluid filling the annular region between two infinitely long concentric cylinders of radii A and B, $0 < A < B$, with the inner cylinder rotated by an external agency at constant angular velocity Ω. At all small Ω, there is a unique solution of the equations of hydrodynamics (after any transients have died), namely

$$u_r = u_z = 0, \quad u_\theta = \frac{\Omega A^2}{r}\frac{B^2 - r^2}{B^2 - A^2} \tag{3}$$

in cylindrical coordinates. (The pressure p is independent of θ and z and obeys $\partial p/\partial r = \rho \vec{u}^2/r$.) This implies that a torque T (representing power consumption $T\Omega$) must be applied to the inner cylinder, per unit length, to keep the rotation going. Here T and E (the total kinetic energy of the flow, per unit length) are

$$\begin{aligned} T &= 4\pi\nu\rho\Omega\frac{B^2A^2}{B^2-A^2}, \\ E &= \frac{\pi}{15}\Omega^2\rho\frac{A^4(B-A)(3A^2+9AB+8B^2)}{(A+B)^2}. \end{aligned} \tag{4}$$

However, above some critical angular velocity Ω_1 [61][71][60] Couette flow becomes unstable in the sense that a certain infinitesimal perturbation of it will grow exponentially with time. Nevertheless, throughout the interval $\Omega_1 < \Omega < \Omega_2$ there is a (different) steady (i.e., time-independent) solution of hydrodynamics. This solution involves, roughly speaking, a superposition of EQ 3 with toroidally shaped "smoke ring" vortices stacked along the z direction at uniform spacing, rotating alternately clockwise and anticlockwise. Note this second solution is infinitely *nonunique* in the sense that it may be translated along the z direction any amount. It is stable against exponential growth of infinitesimal perturbations if $\Omega < \Omega_2$. However, when Ω_2 is exceeded [13][60], it becomes unstable to exponential growth of infinitesimal perturbations which (roughly) cause these toroidal vortices to become "wavy," i.e. their central curves now have z depending on θ in some oscillatory and periodic manner, rather than $z =$ const. But, in some regime $\Omega_2 < \Omega < \Omega_3$ there still appears to be a solution of hydrodynamics (involving stacked toroidal vortices with rotating "waves") which is periodic in time, and which also appears to be stable. Note that this set of solutions is now infinitely nonunique with *two* degrees of nonuniqueness freedom (translation in z and in θ). Both these solutions are observed in numerical and in physical experiments.

Now above some Ω_4 presumably all steady solutions of hydrodynamics become unstable, and above some Ω_5 presumably all time-periodic solutions of hydrodynamics become unstable. Above some Ω_6 presumably *all* solutions become unstable, i.e. exhibit exponential growth of certain infinitesimal

[1] One could instead, unconventionally, have considered "slippery" boundaries on which one merely demands $\vec{u}\cdot\vec{1}_n = 0$ (where $\vec{1}_n$ is a unit normal vector to the surface at that point, if one exists; where one does not exist, demand $\vec{u} = 0$ as usual). I suspect all my results still can be made to work in this case. See [14][73] for recent experimental evidence slip happens. One could also employ surface tension terms at fluid boundaries, although we prefer not to since these will destroy the exact scale invariance symmetry of the Navier-Stokes equations.

[2] The Clay Mathematics Institute is offering a $\$10^6$ prize for a solution [19]. The "intuitionist" and "constructivist" schools of mathematics sneer at non-constructive existence statements of the sort the Clay prize problem seeks. Instead they would not regard a solution of the equations of hydrodynamics as "existing" unless there were an *algorithm* for constructing an arbitrarily close approximation to it. My present paper strongly suggests that for *that* notion of existence, the answer to a similar Clay prize question is "no."

[3] The * in this section's title is intended to indicate that it is inessential to read it. It is here as a service to the community. I had not previously seen a concise discussion of "the usual conjectural beliefs about the mathematical properties of hydrodynamics" and so I am filling that gap.

perturbations, i.e. "chaos." (Observed is: turbulence.) Nevertheless solutions of hydrodynamics presumably exist for all $\Omega \geq 0$.

The conventional wisdom is that these kinds of behavior, in particular the existence of "critical" Reynolds numbers corresponding to our $\Omega_1, \ldots, \Omega_6$ in the Couette problem, also happen generically in *all* forced hydrodynamics problems.

The mathematical cause of this presumed instability is as follows. Consider (EQ 2). It yields $\frac{\partial \vec{u}}{\partial t}$ as a linear operator applied to $\vec{u}$, except for the added nonlinear term $(\vec{u} \cdot \vec{\nabla})\vec{u}$. Upon considering infinitesimal additive *perturbations* $\epsilon\vec{q}$, in the limit $\epsilon \to 0$, to a given velocity field $\vec{u}$, the equations governing $\vec{q}$ are entirely linear (if we ignore terms of order $O(\epsilon^2)$). When $\|\vec{u}\|$ is small this linear operator is dominated by the (well behaved, diffusion equation) term $\nu\nabla^2\vec{q}$, which is self-adjoint and negative-semidefinite, and hence we get stability. But when $\|\vec{u}\|$ is large and the flow $\vec{u}$ is "turbulent" and "random looking," then the $(\vec{u}\cdot\vec{\nabla})\vec{q}+(\vec{q}\cdot\vec{\nabla})\vec{u}$ terms will dominate and make the linear operator behave like a "random matrix with large coefficients." Such a matrix (if it were an $N \times N$ matrix) would be "unstable" (i.e. would have an eigenvalue with positive real part) with very high probability, perhaps roughly $1 - 2^{-N}$. (This would be exact if each eigenvalue independently were equally likely to have a positive or a negative real part.) If we are considering a discretized version of the hydrodynamical PDEs having only a finite number N of degrees of freedom, then that discretization will be a system of N quadratic equations in N variables, and hence should have only a finite number of solutions (at least in the absence of symmetries such as the z-translation in the Couette flow). Meanwhile, the number of these solutions which have reasonably small Sobolev norms and hence act "approximately continuous" and hence are "legitimate" will presumably be a comparatively small finite number – I think much *smaller* than 2^N. This *contrast* is why I think in the limit $N \to \infty$ that essentially *all* solutions will be unstable. The result is "turbulence."

Now hopefully the (presumed) behavior we have just sketched for the Couette flow is "typical." Unfortunately it appears that any direct attempt to write such a statement of universality must fail. That is because it leads to contradictions in the case of another famous flow, the "Hagen-Poiseuille flow" (G.H.L.Hagen & J.L.M.Poiseuille 1839) of fluid through an infinitely long circular pipe, of radius R, driven by a constant pressure gradient G in the axial (z) direction. Let $g = G/\rho$. For small g, the unique solution is

$$u_r = u_\theta = 0, \quad u_z = \frac{g}{4\nu}(R^2 - r^2). \tag{5}$$

which leads to a total flow rate $F = \pi R^2 \overline{u_z}$ given by

$$F = \frac{\pi}{8\nu} R^4 g. \tag{6}$$

Numerous combined analytical and computer-numerical investigations (e.g. [26][56][65]) using several different models of "stability" all have concluded that EQ 6 is *stable* in linearized stability theory (i.e., all infinitesimal perturbations to it shrink exponentially) at *all* g, i.e. *all* Reynolds numbers $\mathrm{Re} \stackrel{\text{def}}{=} 2\overline{u}R/\nu$, no matter how large. (But this is not a theorem at this time.) This stability is a counterexample to the existence of the critical values $\Omega_1, \Omega_2, \ldots, \Omega_6$ that arose in Couette flow.

But... despite the apparent conclusion, from mathematics, of Poiseuille's stability, meanwhile experimentally [54] it seems to be impossible to get non-turbulent pipe flow for any sufficiently large g! (The empirical critical g corresponds to a Reynolds number $\mathrm{Re} \approx 2100$, which for water in a 1cm-radius pipe, corresponds to an average flow rate of only about 3 liters/minute.) Turbulent pipe flow experimentally [38] is very different from EQ 6. The flow velocity $\vec{u}$ is time-*dependent* and is *not* purely in the z direction, and its time average $\overline{u_z}$ has a much flatter dependence on r, roughly proportional (Prandtl) to $(R - r)^{1/7}$. Furthermore, the average flow rate does *not* obey Poiseuille's law (EQ 7), but instead, empirically, obeys the entirely different Darcy-Weisbach (1845) law[4]

$$F = kR^{2.5}g^{0.5} \tag{7}$$

when g is large. Here k depends on the fluid's viscosity and the characteristics of the pipe walls, but, for large Re, seems asymptotically constant.

The subtle resolution of this experimental-theoretical contradiction seems to be:

1. The Poiseuille flow EQ 6, although stable, has smaller and smaller stability exponents, i.e. eigenvalues with negative real parts that get arbitrarily near 0, as $\mathrm{Re} \to \infty$.
2. It is unstable against *finite* perturbations, with the norm of the required perturbation shrinking to 0 as $\mathrm{Re} \to \infty$. Thus eventually, unavoidable microscopic imperfections in the pipe wall, and/or thermal fluctuations, will be large enough to trigger the instability, despite its complete stability against *infinitesimal* perturbations[5]. (This seems a depressing example of the failure of the equations of hydrodynamics *alone* to be a correct and complete description of the physics. It suggests that one also needs a model of the structure of, and the size-distribution of, random small finite perturbations that continually appear.)

Hence we must **modify our above presumptions** that "every" solution of high-Re hydrodynamics is unstable, to "most" are unstable to infinitesimal perturbations, while the rest are unstable to small finite perturbations, with the norm of the required perturbation shrinking to 0 at high Reynolds numbers. The net physical effect of the modified presumptions is the same – fluid flow at high Reynolds numbers will always be "turbulent" and "chaotic."

[4] Please do not confuse this with the unrelated and irrelevant Darcy (1856) law about seeping flow through porous media such as sand.

[5] This view is supported by the fact that the turbulence-transition Reynolds number in pipe flow can be pushed upward from 2100 to values over 40000 depending on the experimenter, the city, and the care with which the pipe was polished smooth and the inlet to the pipe was shaped. Meanwhile the transition Reynolds number in Couette flow seems independent of the experimenter. S.J.Chapman has speculated that the amplitude of the perturbation required to induce turbulence in Poiseuille pipe flow behaves asymptotically proportional to $\mathrm{Re}^{-3/2}$. Also, note that flow in pipes can and does convert from laminar to turbulent and then later *back* to laminar after, e.g. some of the flow is drawn off into a side-channel to reduce the flow rate, or after a widening of the pipe – either one of which can reduce Re back below the transition value.

The reason it is usually thought that solutions of hydrodynamics **always exist** is that the dissipative term $\nu\nabla^2\vec{u}$ tends to "smooth them out," causing hydrodynamics to "act like an elliptic PDE on small enough length scales," implying existence and smoothness. Indeed, there is an approximate statistical model of "isotropic turbulence" by A.N.Kolmogorov [34][21]. It predicts that[6] at length scales smaller than $L_K \approx \nu^{3/4} D^{-1/4}$, (where D is the mean rate of power dissipation per unit mass of the turbulent fluid), fluid flow should be smooth, well-behaved, "viscous" and "laminar," whereas, at length scales L with $L_K \ll L \ll L_M$ (where L_M is the length scale of macroscopic constraints, such as boundaries on the particular flow being considered), the velocity field should exhibit autocorrelations ([21], chapter 5) of the form

$$\overline{[\vec{u}(\vec{x},t) - \vec{u}(\vec{x}+\vec{r},t)]^2} \propto |\vec{r}|^{2/3} \tag{8}$$

and hence have a "$k^{-5/3}$-power spectrum." Kolmogorov's predictions seem supported by experiments, both on real fluids and by computer simulation.

Now, despite the belief that high-Re flows are always turbulent and chaotic, thanks to the dissipative term $\nu\nabla^2\vec{u}$ (which is the only thing preventing unforced hydrodynamics from being a perfect Hamiltonian system and hence from obeying the Liouville law of preservation of "phase space volume") the phase space volume everywhere is **continually exponentially shrinking** – despite the exponential stretching in certain directions responsible for the chaos. This exponential shrinking ultimately falls into a set of zero measure called a **"strange attractor"** [60]. (Such attractors have been observed and studied for various finite-dimensional ODE systems starting in the 1970s.)

So: the usual conjecture about turbulence is that it is a strange attractor in the infinite dimensional phase space of possible flow fields. This strange attractor has measure zero, but essentially all members of it are flows unstable to either infinitesimal or small finite perturbations, so that any actual physical flow will randomly walk around on the strange attractor (with the "randomness" truly arising from random e.g., thermal and quantum fluctuations, which continually get amplified exponentially by the flow dynamics).

About uniqueness, let me say this. The conventional wisdom is that the solution of the hydrodynamics equations (with smooth initial data and force fields obeying appropriate norm bounds) generically exists and is unique. (The present paper may offer grounds for questioning and/or altering this conventional wisdom, but we have not gotten there yet.) The example of infinite non-uniqueness in Couette flow we mentioned is not a counterexample because it was non-generic. I.e. it presumably would not have happened if the inner cylinder had gradually been rotated faster and faster until reaching final speed, and if the initial flow field had been unsymmetric. In that case instabilities such as Taylor's would have started from some state already *finitely* perturbed from the steady flow of EQ 3 and hence would have proceeded unambiguously.

However, despite this presumed uniqueness, in fact, due to our presumptions of instability and chaos, the solutions will be *effectively* very highly non-unique because tiny random perturbations will all the time move one into a different solution. (Also, in our Couette example, the gradual ramp-up of inner cylinder speed, if slow enough, would have had as its first effect, the exponential decay of our initial flow field's difference from EQ 3 to an extremely tiny value, say 10^{-9999}, which we would later be depending on to keep us unique. Ha.)

Finally, let me note that although the conventional wisdom favors existence and uniqueness when $\nu > 0$, in the *Euler* equation $\nu = 0$ case the conventional wisdom seems split, with many betting on existence, and many against it.

3 Some unconventional conclusions arising from the conventional wisdom

If all this be so, in what sense is it useful to "solve the equations of hydrodynamics?" What *is* useful to do? I advocate this answer:

Manifesto: *The output of a hydrodynamics "solver" should be some sort of approximate and discretized (e.g. "pixelized") description of the strange attractor, along with a description*[7] *of the transition probabilities, per unit time, between different pixels in it. Each "pixel" is a flow field.*

At present, *no* Navier-Stokes solver attempts to produce this kind of output – one reason being that it would be enormous. In practice, though, one often is interested in only a few numbers about the strange attractor. (For example, in meteorology "what is the probability it will rain tomorrow?" or in ship design "what is the drag at 5 meter/second ship speed?") In that case the output could be compressed down to only a small number of bits.

The conventional wisdom suggests that solving hydrodynamics (even with my new recommended notion of "solve") should be algorithmic. Specifically, to get accuracy ϵ, it seems plausible that discretizing the PDEs at a length scale of $L_K\epsilon$ or so should suffice, at which point there will be only $\approx (L_K\epsilon)^{-3}L_M^3$ degrees of freedom in the discretized system. Now since volume and total energy are bounded, and we've just concluded that effective dimensionality is bounded, the overall phase space volume is bounded too. Now the (now finite-dimensional polynomial) equations of discretized hydrodynamics may then be solved by brute force by, e.g., methods of "resultants," or by simply exhaustively examining every size-ϵ chunk in the entire phase space. By exhaustive search and exhaustive randomized experiments all the pixel transition probabilities could then be estimated and the pixelized strange attractor determined. This is an extremely slow algorithm – certainly at least exponential time – but it *is* an algorithm, i.e. it will terminate, and its output arguably should be *useful*.

It is conceivable that no significantly faster algorithm exists, because possibly there is no significantly faster way to estimate integrals over attractors, than by exhaustive examination of the attractor, and no significantly faster way to find

[6] For water with $D = 1$watt/gram, this formula yields $L_K = 5.6$microns.

[7] Under some auxiliary model of of the structure of, and the size-distribution of, random small finite perturbations which continually appear, which we need in *addition* to the equations of hydrodynamics.

the attractor, than by exhaustive examination of the whole of phase space[8].

There is a much faster algorithm – running[9] in time polynomial in L_K^{-1} and ϵ^{-1}, which resembles what practitioners actually do – *but* it seems much further from having any rigorous justification, either of its validity (i.e. fact that it succeeds, and hence algorithmicity), or of the usefulness of its output. It is: solve the finite dimensional polynomial equations of discretized hydrodynamics by nonrigorous numerical methods that (hopefully) converge from an initial guess; trying several "random" initial guesses, and then averaging over the resulting few approximate solutions of discretized hydrodynamics (hoping this sampling and averaging procedure is close enough to a true Monte-Carlo integration over the whole strange attractor) to find an approximate answer to whatever the original question (e.g. "what is the drag" or "what is the probability it will rain") was.

4 How hydrodynamics can be nonalgorithmic, despite all that conventional wisdom

Kolmogorov's turbulence model does not apply in *an*isotropic situations such as fluid near a boundary. His autocorrelation and spectral power density laws cannot hold if the interval $L_K \ll L \ll L_M$ is *empty* because the container boundary features "wiggles" at *all* length scales L_M, including arbitrarily tiny ones. Finally, if the initial spectral power distribution had more power at small scales than the Kolmogorov law, then again that law cannot hold. Therefore the "usual" arguments from §2 sketching why hydrodynamics should hopefully "typically" be algorithmic, and why there should be a "smallest length scale that matters" (which is the reason for algorithmicity), both would *not* hold in such scenarios.

The construction we will present in §6 onward indeed has all three of these anti-Kolmogorov and anti-conventional-wisdom properties.

Incidentally, the Clay prize problem statement by Fefferman [19] only considers incompressible *single* fluids filling *all* of $\mathbb{R}^3$ or $(\mathbb{R}/\mathbb{Z})^3$, i.e. forbids solid boundaries, two fluids, and free surfaces. Throughout our upcoming undecidability construction, solid boundaries and free liquid surfaces will be very convenient and helpful tools. It might be much more difficult or impossible to make our same argument go through with those tools denied to us. Thus the present work suggests that, by making this problem choice, in order to "get rid of presumably irrelevant complications," Fefferman actually may have unintentionally sacrificed much of the richness of the problem.

I also point out that in hydrodynamics with free surfaces, a radially collapsing cylindrical cavity – for which there is a trivial exact vorticity-free solution $u_r \propto -r^{-1}$ – actually *does* lead to a near-singularity (exponentially large fluid speeds in exponentially small volumes reached in finite time with finite energy input), see footnote 18. Furthermore, if we also allow unconventional "slippery" boundary conditions (footnote 1) or alternatively, "Euler" fluids of viscosity $\nu = 0$, then one can construct solutions yielding *infinite* flow speeds in finite time with finite energy input: simply push water into a converging pipe with radius proportional to $x^{0.4}$ where x is the length along the pipe (or into a converging wedge with width proportional to $x^{0.8}$).

5 Is there a "quick fix"?

This paper will argue that hydrodynamics is nonalgorithmic. But: perhaps all the *useful* hydrodynamics problems – the ones that arise in "real life" – are algorithmic. If so, that would be very important and desirable. This section will discuss 5 attempts to define such a "good" subclass of hydrodynamics problems. Call any such attempt (if successful) a "quick fix." Unfortunately, all 5 of the quick fix attempts we shall discuss fail, either because (i) they exclude a large class of hydrodynamics problems previously widely regarded as useful, or (ii) because they do not defeat my undecidability construction, or (iii) because the proposed "subclass" of hydrodynamic situations in fact can self-generate situations which lie outside of that class.

Try #1: Lipshitz container walls. One may be led by §4 to the speculation that perhaps hydrodynamics is algorithmic if we demand that the container boundary be smooth (i.e., all derivatives of all orders exist) everywhere *and* have Lipshitz[10] normal vectors everywhere, and similarly demand that the initial velocity field in the fluid also be smooth and Lipshitz everywhere – these conditions tend to prevent imposing microscopic structure at arbitrarily small length scales[11].

Why it fails: But before the physicist/engineer-reader walks away sneering that my whole paper is merely an uninteresting mathematical quibble, a mere byproduct of "rigor mortis," let me remind him that the whole Kutta-Joukowski theory of aerofoils[12] *demands* a "sharp trailing edge" at which boundaries are necessarily non-Lipshitz and non-differentiable. Most physicists are unwilling to sacrifice aerofoil theory[13].

[8] Food for thought: (1) It is easy to set up finite dimensional ODE systems – especially easy if those ODEs are, like hydrodynamics, dissipative – which simulate the operation of universal computers such as "counter machines." (2) Here are three undecidable problems (by similar proofs to the usual demonstration [47] of the insolubility of the Halting Problem). ATTRACTOR MEMBERSHIP: Given a Turing Machine and a state, decide if that state is a member of a perpetual loop the TM will execute. ATTRACTOR CARDINALITY: Decide if that TM will fall into a perpetual loop of length $< k$ (for any specified integer k). ATTRACTOR EXISTENCE: Decide if that TM will fall into a perpetual loop, or if its state sequence will be aperiodic. (3) Finitized versions of the first two of these problems (i.e. with the TM replaced by a computer having only a polynomially large number of memory bits) are PSPACE-complete.

[9] And it probably is possible to reduce the polynomial dependence on ϵ^{-1} to become polynomial in $|\log \epsilon|$ by employing "spectral" numerical methods [28] of "infinite order," rather than fixed-order grid-based discretization schemes. I won't worry about that.

[10] A continuous function $F(x)$ is "Lipshitz" if $|F(x + \Delta) - F(x)| < c|\Delta|$ for some "Lipshitz constant" c.

[11] Refusing to allow perfect incompressibility and perfect independence from thermal effects might be important to allow algorithmicity.

[12] Admittedly, Kutta-Joukowski theory was formulated for inviscid flow.

[13] There is also the well known phenomenon of "shock waves" (*discontinuities* in velocity and density) which arise naturally thoughout the mechanics of compressible fluids.

Because the construction for my undecidability argument will involve boundaries smooth everywhere except at a single *point*, it actually arguably is *nicer* than Kutta-Joukowski aerofoils.

Try #2: forbid small features. Instead of requiring a Lipshitz condition, one could, in some other (unknown) way, try to make a condition forbidding solid boundaries with "features" below some particular "minimum permitted length scale." Aside from the question of how to make that notion precise, there is another problem: there are scenarios, which experimental physicists work with, in which solid boundaries and fluid features arise exhibiting "fractal" behavior down to very small scales. In the case of "diffusion-limited aggregation" [70] (which produces beautiful tree-like fractal structures during the electrodeposition of copper in water, and which is partially responsible for the fractal-like appearance of snowflakes), this fractality seems to continue all the way down to atomic scale. Fractal structure also arises in dielectric breakdown (e.g. lightning, the subject of ≈ 20000 papers) [50]. Surely physicists are unwilling to sacrifice the presumed ability of fluid dynamics to treat flows in the presence of snowflakes and lightning?

Even presuming they *are* willing to sacrifice that, there is another problem. Consider the "viscous fingering" instability [6][16][50][51] arising when a fluid of low viscosity flows (slowly, with the flow always being entirely laminar) under pressure into a fluid of high viscosity. If the interfacial surface tension between the two fluids is small (especially if it is zero) then beautiful fractal tree-like flow structures appear. By making one of the fluids be plaster of Paris slurry [16], one can create fractal *solid* objects which may be studied at leisure. (A fractal dimension 1.6 ± 0.1 was found, extending over at least 3 decades of length scaling [16], with the minimum branch size being of order tens of micrometers – approximately as small as one could hope for, given that treating plaster as a "liquid" is an approximation that presumably breaks down at the size scale of the colloidal plaster particles.) Mathematical models of this suggest viscous fingering will create fractality at every length scale, all the way down to zero, in finite time. (In Bensimon et al. [6], see EQ 1.13 with $T = 0$ and b, U positive constants, and also see their sentence after EQ 3.25.) So one either must admit (a) the need to apply the Navier-Stokes equations in scenarios with boundaries of this kind, or (b) that hydrodynamics is not good enough to handle scenarios it itself can create.

Try #3: forbid high-frequency power: Another attempted quick fix – demanding that the initial spectral power density not be allowed to have more power at small length scales than Kolmogorov's law – also seems inadequate. First of all, it seems possible (see §9) to make our undecidability construction have a bounded and very pleasant initial velocity distribution, with all the strangeness arising from later interaction with the solid boundaries. Second, various interesting and realistic physical scenarios seem constructible in which this scaling demand is false[14]. (Presumably our physicist does not want to sacrifice the ability of hydrodynamics to help analyse such scenarios.)

Try #4: relativity. Might fluid dynamics be salvaged (i.e. again become algorithmic) if we agree to employ *relativistic* hydrodynamics? This suggestion is motivated by the desire to defeat the occurrence, in my upcoming undecidability construction, of unboundedly high fluid speeds.

My replies: First, a technical quibble: there is no such thing as relativistic *hydrodynamics* since the constant-density condition implies infinite speed of sound, faster than the speed of light. But there *is* a relativistic version of the full Navier-Stokes equations [37]. Second, I suspect relativity will *not* salvage algorithmicity (although I have no proof) because merely the size-scaling employed in my uncomputability construction (without the speed scaling) likely is sufficient. Third: However, suppose for the sake of argument that it does salvage algorithmicity. In that case, physicists would, in order to simulate their bathtub, resort to relativistic hydrodynamics. This goes against the usual physicist's belief that "obviously, relativistic effects are not important in bathtubs and may be neglected." (But see footnote 18 and the final paragraph of §13 for examples in which physicist's intuitions of this sort are entirely wrong.) That belief would in fact be incorrect in the sense that they would have the *very* important effect of salvaging algorithmicity. But really, this would be no salvage at all, for practical physics purposes, since obviously, if relativistic effects had a dramatic macroscopic effect on the solutions of the fluids PDEs, then those PDEs would be a very poor approximation of water, so that those solutions would be both physically irrelevant and presumably extremely ill-conditioned. (This all in fact happens: see footnote 18 and the final paragraph of §13.) So, the possible event of relativity "saving" hydrodynamics would be a Pyrrhic victory – really this would be the final nail in its coffin!

Try #5: forbid nasty topology: Luc Tartar (Math dept. CMU) suggested that *topology* could come to the rescue – my upcoming uncomputability construction involves a container whose boundary's topology involves an infinite number of "handles." Perhaps computability could be restored to hydrodynamics by forbidding such? But actually, all of the "pipes" in our construction could be slit by narrow slits, and the resulting pipe-wall fragments attached to an outer spherical wall by means of rods. The resulting container would have an interior topologically equivalent to the interior of a sphere. But if the slits were narrow enough, flows though the pipes would be only negligibly affected by leakage through the slits and so my construction should still work. So apparently Tartar's quick-fix idea does not work – topology is not relevant.

6 Top level plan of undecidability construction

We will follow the standard plan [47] for proving uncomputability. Thus our proof will greatly resemble, in its large scale organization, Conway's proof [1] of the undecidability of the long term behavior of his one-player game LIFE. In that

[14] For example, consider making a fractal solid object, as before, out of an explosive material, then explode it. Similar effects presumably arise in lightning. A less dramatic example is simply the classical thermal acoustic spectrum, which by the "equipartition principle" from statistical mechanics, has a flat power spectrum. Yet another example is "sonoluminescence," see footnote 18.

proof, Conway showed how various LIFE constructs could emulate AND, OR, and NOT logic gates and signal transmitting "wires" and "crossovers." Then, he argued, one could, within the LIFE world, build a computer. That computer could be programmed to solve an arbitrary Turing halting problem and then depending on the answer, set in motion some events that would drastically alter the whole LIFE picture. (For example, erasing everything, or not.) That would prove deciding the long term fate of a LIFE configuration is at least as hard as solving an arbitrary Turing halting problem, i.e. undecidable. Q.E.D. We similarly will show how, using hydrodynamics, to build logic gates and wires, and thus to build computers. We shall use the scaling properties of the Navier-Stokes equations to show how to make that "computer" perform an infinite number of primitive steps in finite time even with a finite energy supply. We shall use error-correction techniques to show it does not matter even if each of our logic-gates suffers from some small-enough-constant probability of making a logic error. We conclude: deciding essentially any question about what a hydrodynamic system will do in some finite time, is at least as hard as solving an arbitrary Turing halting problem, i.e. is undecidable. Q.E.D. The rest of this section outlines this plan in a little more detail, and then the full proof is presented in the rest of the paper.

Henry Cejtin often wished that his computer had just one additional instruction: the "go ten times faster" instruction, which would multiply the clock speed by a factor of 10. Actually, it would suffice to add a "go 1.01 times faster" instruction (or "go s times faster," for any constant $s > 1$) to the instruction set of a Turing machine, to allow it to solve the "Halting problem" (which is insoluble for conventional Turing machines) in finite time[15]. This is because the total time to execute an infinite sequence of instructions, each s times faster than the previous one, is $1 + s^{-1} + s^{-2} + s^{-3} + \cdots = s/(s-1) < \infty$.

For our purposes it is best to imagine having both a "go 1.01 times faster" and a "go 1.01 times slower" instruction.

Today's computers are built out of logical elements which in turn are made of electrical wires, resistors, and transistors. If there were no "atomic size scale" and no discrete "electrons" quantizing electric charge, but instead matter (and electrical charge) were true continua of infinite durability, then there would be no objection to having an infinite set of transistors, resistors, and wires each 1.01 times smaller (and performing each logical operation 1.01 times faster while consuming 1.01^2 times less energy per logical operation) than the previous one, and with the whole infinite set of these components fitting into a finite volume. Indeed it would then be possible to have a computer that would, after doing some instructions, download its state into a 1.01-times smaller and faster and less power hungry computer (but with a 1.001-times larger memory capacity) next door. That computer in turn, after doing some instructions, could download its state into a third computer 1.01-times smaller than it, and so on. E.g., the kth computer would run at clock speed $\propto 1.01^k$, have memory size $\propto 1.001^k$, and perform a number of instructions $\propto 1.001^k$ before downloading itself to computer $k+1$. Each computer, after getting the answer to its problem (if any), would *up*load that answer to the next larger computer. The whole infinite set of computers would occupy finite volume and would perform an infinite number of instructions in finite time (allowing solution of Turing's "Halting Problem," with output by the top-level, macroscopic, computer) and with the consumption of a finite amount of energy in total.

The above scenario is almost precisely the idea I am going to use, except that I am going to base it, not on electrical elements such as wires, resistors, and transistors, but instead on hydraulic elements. Because the PDEs of hydrodynamics, which everything will rest upon, are true-*continuum* equations, there will be no obstacle caused by the discreteness of charge or atoms, and because I am assuming the container is perfectly rigid, there will be no obstacle caused by finite material strengths.

We will depend heavily on the following "scale invariance property" of hydrodynamics (which, incidentally, is well confirmed experimentally – including in turbulent flows).

Lemma 1 (Scale invariance of hydrodynamics). *Let $s > 0$ be real. If all velocities $\vec{u}$ are scaled by a factor of s, all lengths are scaled by a factor s^{-1}, and all times are scaled by a factor of s^{-2} (also pressures p are scaled by s^2, and external forces per unit volume by s^3, and the rigid container is length-scaled), while the fluid's density ρ and kinematic viscosity ν are left unchanged: then the equations EQ 1 and 2 of hydrodynamics are unchanged.*

Note that under this scaling, energies scale as s^{-1}, flow rates as s^{-1}, power dissipation as s, surface areas as s^{-2}, and energy densities as s^2.

In §7 we'll describe the fundamental logic components our hydraulic computers are made from. In §8 we'll sketch how to assemble them into a computer, with more details filled in in §9. Worries re unreliable components and precision requirements for the initial data are dealt with in §11 and §10. In §12 we discuss extensions of our construction to handle "higher levels" of undecidability. §13 concludes. Although many of the arguments in this paper are not mathematically rigorous, the ones that are suffice to yield a *fully rigorous theorem* in §13.

7 Nikola Tesla, "Fluidics," and hydraulic logic components

One of the lesser-known patented inventions of Nikola Tesla (1856-1943) was his "valvular conduit," which later writers have also called the "hydraulic diode" and "fluidic diode."

This is a wholy rigid pipelike device (i.e. it has no moving or flexible solid parts) through which fluid will flow with low resistance in one direction, while flow in the reverse direction incurs high resistance. Flow rates at the same pressure difference can easily differ by over a factor of 10. There are many

[15] Of course [47], the availability of a machine to solve the halting problem in 1 hour, would immediately make it easy to, e.g., settle the Riemann hypothesis, solve Chess, settle the P=NP question, build artificial superintelligences, predict the weather, etc. Even if the machine merely produced a random bit which, with probability $\geq 2/3$, was the correct answer to the halting problem specified to it, that would still be essentially as good. Also, of course the aim of the present paper is not to show how to build a useful device to solve the halting problem – that would be ridiculous – but instead is to show that the usual mathematical formulation of hydrodynamics is unsimulable and leads to Turing-undecidable problems.

possible geometries, but I have illustrated the simplest version of Tesla's conduit in figure 1. Its principle of operation (expressed in modern terms) is as follows. Flow in the forward direction $A \to B$ proceeds roughly in a straight line and is smooth and laminar. If one attempts to force flow in the reverse direction $B \to A$, then some of it will be diverted into the side-tube at D, and its re-emergence at C will induce a transition to turbulence (the side-tube could also be shaped in such a way as to encourage turbulent flow through it) which will greatly increase the hydraulic resistance. This also causes what backwards flow does occur, to be of a randomly whirling, intermittent, and unsteady character (although this can be "smoothed out" by a subsequent conversion back to laminar flow as in footnote 5).

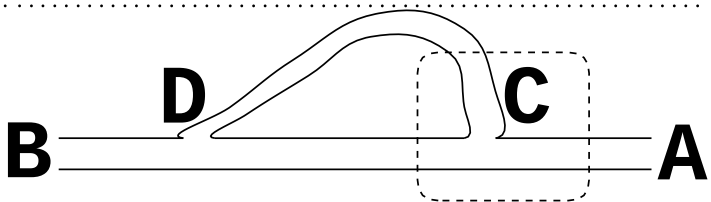


Figure 1. Tesla's valvular conduit (simplified).

If we restrict attention just the region near C (enclosed by dashed boundary), what we have is a more basic *three*-terminal hydraulic component which could be called the "hydraulic transistor." There is no need for the entrance to the side tube (at D) to be drawn off from our main flow. It could instead have been drawn from some other flow in some other pipe (provided the inlet pressure was in the permissible range). In that case, the hydraulic resistance of $B \to A$ flow would be varied over a 10:1 ratio by an *external* control.

This ability to build hydraulic "transistor circuits" analogously to electrical ones (and see table 2 of analogies, plus consider the remark about conversion between laminar and turbulent flows in footnote 5) suggests that it ought to be possible to build a complete family of all-hydraulic digital logic components and, with them, a digital computer.

Electrical	Hydraulic
Voltage	Pressure
Current	Flow rate (volume/time)
Charge	Fluid volume
Resistor	Sinuous capillary, or porous plug (laminar-viscous flow)
Inductor	A pipe (large momentum in flow)
Capacitor charged to some voltage	Tank of water raised to some height
Nonlinear resistor	Length of pipe (turbulent flow)
Diode	Valvular conduit
Bipolar transistor	hydraulic transistor

Figure 2. Electrical-Hydraulic analogies.

This hope is realized. Although I was initially inspired by Tesla's Valvular Conduit patent, further searching in the literature revealed that

1. Hydraulic "amplifiers" or "fluid-controlled-fluid-switches" based on the transition to turbulence had already been invented by Chichester A. Bell in 1892, i.e. before Tesla (and of course, the "diodic" observation that certain bodies have different coefficients of drag when moving forwards instead of backwards, dates to antiquity).
2. The field of "fluidics," rooted in initial discoveries such as these, as well as other principles of operation such as jet diversions, jet interactions, vortices, and the "Coanda effect" [12] is nowadays an established technology.

"Fluidics" has been defined as "techniques which use flowing gases or liquids as an information-carrying medium and as a basis for signal sensing, switching, amplification, and control of fluid flows, and digital and analog logic, employing devices which have no moving parts." An introduction is [3].

Numerous fluidic-logic components including XOR, AND, NAND, NOR, and OR gates, analog and digital amplifiers both inverting and noninverting (including high-gain "op-amps" with differential inputs capable of being used in negative-feedback circuits), binary-adders and counters, Schmitt triggers, oscillators, and bistable "flip-flop" like elements based on the Coanda effect (these seem to have no direct electrical analogue) are commercially available. Typical "fanouts" are ≈ 5. Vortex-based fluidic diodes have diodicity ratios of 40:1. Fluid-controllable variable resistors with resistance ratios of 10:1 are available. Amplifiers can exhibit enormous power and/or amplitude gains if cascaded (thus small fluid flows have been used to steer large rockets) but 10:1 to 200:1 gains are typical of uncascaded amps.

Integrated fluidic logic circuits involving ≈ 100 components have been built using photolithographic techniques with submillimeter line widths. They can run at kilohertz frequencies. (E.g. the frontspiece of [3] pictures a fluidic divide-by-10 circuit.)

The Univac corporation built a complete small (4 bit) fluidic general purpose computer (for experimental purposes) in 1964, using air as the working fluid and plastic parts. It had about 250 NOR gates with fanout and fanin 4 and switching times of around 1-5 milliseconds. It is pictured on page 245 of [31]. Air is the most common working fluid in fluidics technology due to its availability and low mess. At the low pressures ($\leq$ 1 psi) and low subsonic ($\leq$ 0.3 mach) flow speeds often used [33], air may be treated fairly accurately as incompressible; thus the fluidic components mentioned in [3] are claimed to work with most any liquid or gas.

In addition to fluidic digital logic devices, there are also fluidic vacuum pumps (based on Venturi effect), vacuum and pressure sensors, liquid-level sensors, sensors for presence/absence of stream-blocking objects, acoustic sensors and acoustic generators, fluidic timers, flow meters, and pumps and pneumatic-powered mechanical devices, One may purchase a "fluidic logic design kit" containing 24 NOR gates and other devices from `www.air-logic.com`.

The 5 main ideas behind the operations of the most common fluidic logic components are these:

Coanda effect: A jet of fluid is "attracted" toward solid surfaces (due to Bernoulli and "entrainment" effects). One can set up geometries in which a fluid jet emerges vertically upward from a nozzle and then bends rightward until it hits a surface. It then continues to move along that surface (which may also be vertical, or perhaps sloped diagonally to the right). Alternatively, the jet bends leftward until it hits a different surface, then continues along *it.* There are thus two stable states, and a perturbation by a puff of fluid introduced from a separate "control" nozzle can be enough to induce the state to flip. One can also make several control nozzles such that a simultaneous signal from all of them is needed to cause the state-flip (AND) or such that only one signal suffices (OR).

This effect was discovered, and devices based on it patented, by Henri Coanda [12] in 1936. However, already Thomas Young in 1800 had published some observations of this effect, as demonstrated by the sideways "attraction" of a candle flame and its smoke stream to solid objects. (Directions for how to use a candle to build your own fluidic "flip-flop" based on this effect are in [58].)

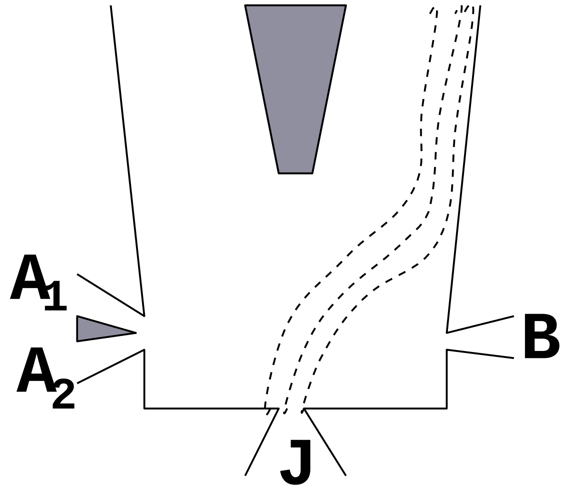
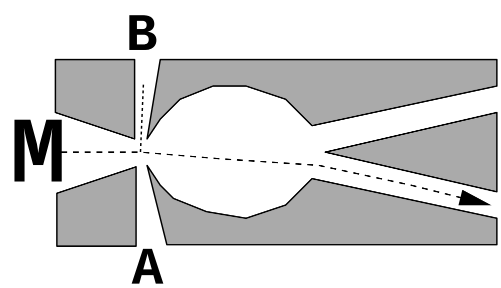


Figure 3. Coanda-based flip-flop. Main jet J is "attracted" to right wall and stays there. After perturbation by puff from control jet B, will move to left wall and stay there. Puffs from A_1 OR A_2 will flip it back. By making the device asymmetric so that the left wall is closer to J than the right wall, one can make a "biased" flip-flop which automatically initializes itself to the left-state, in the absence of a control signal.

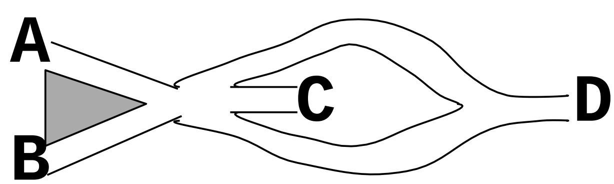
Jet diversions:

Figure 4. Differential op-amp based on diversion of main jet M by control jets B (positive input) and A (negative input) to inverting and non-inverting outputs. (Also implements logical-NOT function.)



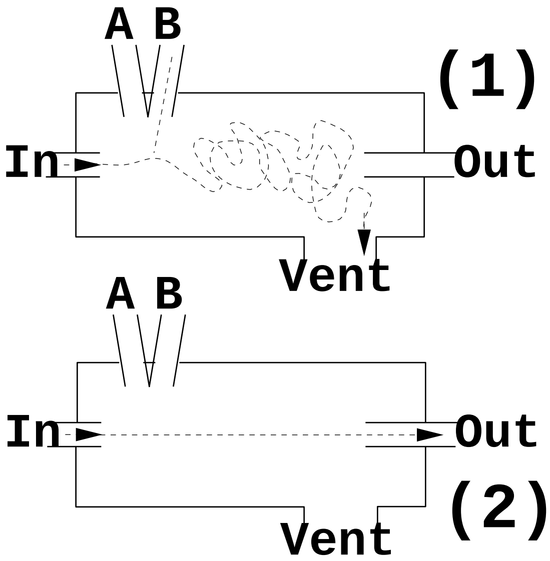
Jet interactions:

Figure 5. Passive AND and XOR gates based on jet interaction (or lack thereof). Jets input at A and B interact to yield output $C = A$ AND B, and output $D = A$ XOR B.



Jet transition to turbulence:

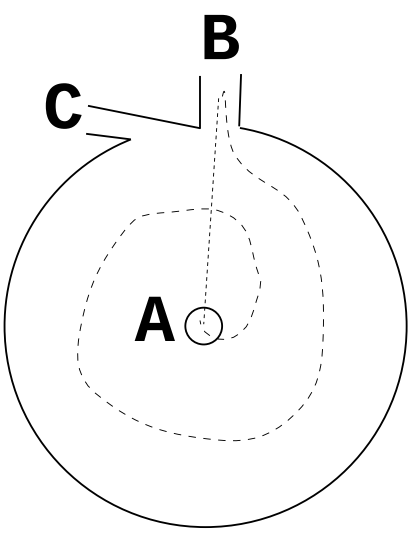
Figure 6. "Turbulence amplifier" implements logical NOR functionality: If fluid emits from neither A NOR B, then main jet flows laminarly from input to output (2). If A OR B control jets are active, then a laminar→turbulent transition is triggered in main jet (1) causing the in→out path to have large hydraulic resistance, and causing most of input flow to exit through the vent.



Vortices:

Figure 7. "Vortex diode." Shut vent B. Fluid flows with low resistance from A to C as fairly straight jet; but if $C \to A$ flow is attempted, then the flow will form a vortex, causing large hydraulic resistance. Indeed in the inviscid limit *zero* flow (in theory) will exit A if centrifugal "pressure threshhold" is not exceeded at input C.

If B is not shut, the same device instead works as an inverting amplifier or "variable resistor." The main jet flows from B to A with low resistance; but if a control jet from C is introduced, it will deflect the main jet to form a vortex (dashed spiral) causing high $B \to A$ hydraulic resistance.



Fluidic technology is the subject of numerous books: [2] [5] [10] [17] [20] [30] [31] [33] [48], as well as the *Journal of Fluid control* (formerly *Fluidics quarterly*). (There also are articles on fluidics in the Brittanica and Americana encyclopedias.)

Thanks to the vast increases in speed and reductions in size, power consumption, and cost of electronic logic circuitry during 1940-2000, fluidics cannot compete with electronics in general purpose logic and computing applications. However, fluidic devices can be built to be very robust and to work in extreme environmental conditions such as:

- heavy radiation,
- huge electrical noise,
- (this one may surprise you) intense vibration,
- high temperature,
- high power levels,
- harsh chemicals,
- in the presence of easily ignited fuels.

These conditions might be incompatible with electronics. Also, in some applications, fluidic power sources are readily available but electrical power sources are not, while great speed is not needed. People are also interested in ways to control and create fluid flows which do not suffer from the unreliability inherent in devices with moving parts. Also, interest in fluidics has recently been reinvigorated by possible applications in "micromachines" and/or biomedical devices, and chemical micro-quantity analysers and processing devices. There may also be musical and acoustical applications. All this causes fluidics to remain viable.

8 Assembling the computer from logic gates

We make a sequence C_0, C_1, C_2, C_3,... of self-contained general purpose Von Neumann architecture fluidic computers, each presumably rather similar to Univac's. Those unfamiliar with how to build computers (as well as simpler constructs, e.g. adders) from elementary logic gates may consult [29]. (Call the complete multicomputer C.) C_{k+1} is similar to C_k, but all of its pipes and fluidic logic components are scaled down in[16]length by a factor of 1.01, while its number of bits of memory is scaled up by a factor of 1.001. We assume our fundamental fluidic logic components are robust enough to handle driving logic gates a few percent larger or smaller than usual without error, so that C_k and C_{k+1} can communicate. The scaleup of the memory size could present some difficulties caused by the necessity of making the pipes between the memory and processor get longer and longer relative to their diameter. Such difficulties need to be circumvented by placing amplifying "repeaters" along such long pipes. Indeed we may imagine everything as built on a grid with no pipe segment longer than a constant number of grid side-lengths [39] and no logic gate output needing to drive more than a constant number of other inputs.

Due to the scaling relations of Lemma 1, C_k's fundamental logic gates will operate 1.01^{2k} times more quickly than C_0's.

[16]This fluid-computer construction is actually extremely similar to the human circulatory system, which is fractal in nature [35] with the aorta (diam. 2.5cm) branching into arteries, branching ultimately down to capillaries (diam. 10μm) which then recombine, ultimately forming large veins. The radii at the branchings approximately obey "Murray's law" $R^3 = \sum_k r_k^3$ which tends to minimize pumping power for a given volume and flow rate [35][49][69]. (Supposedly in turbulent flow the exponent would be 2.33 not 3.) The fact that the heart can pump blood through this system with a complete cycle taking only finite time is due to essentially the same sort of math (scaling relationships and finite sums of geometric series) that allows the present paper's construction to work in finite time and energy.

However, since each memory access could be up to 1.001^k times slower (we are being generous here) than if C_k had a constant size memory, the overall instruction-rate of C_k will be $\geq 1.01^{2k} 1.001^{-k}$.

C_k performs 1.001^k instructions before turning on, copying its state to, and handing control over to, C_{k+1}. (If, however, these instructions included a "halt" instruction, it instead uploads the information "I have halted" to C_{k-1}.) All this takes time proportional to $1.01^{-2k} 1.001^{2k}$. It then sits and waits for an answer (if any) to be uploaded from C_{k+1}; if so it uploads that answer in turn to C_{k-1}. The total waiting time for C_k to wait for all C_j with $j > k$ is also proportional to $1.01^{-2k} 1.001^{2k}$ or less. After this waiting period has expired, C_k may turn itself off. C_k fits inside a cube with sidelength proportional to $1.01^{-k} 1.001^k$. The whole set of cubes may be placed along a line in a self-similar manner (they all are inscribed in a cone, see figure 9) with a limit point C_∞ at the apex of the cone.

The internal plumbing of C_k has everywhere smooth boundary, so that the entire multicomputer is a container with everywhere smooth boundary except at the unique limit point. The surface area of the container due to C_k is proportional to $1.01^{-2k} 1.001^k k^{O(1)}$. The total power consumption of C_k is proportional to $1.01^k 1.001^k$ but it only needs to operate for a time proportional to $1.01^{-2k} 1.001^{2k}$ or less, so its total energy consumption is proportional to $1.01^{-k} 1.001^{3k}$ or less. Hence (by summing geometric series) the entire multicomputer has bounded surface area and bounded energy consumption.

Initially consider C_k to be powered by its own self-contained pressure reservoir containing an amount of fluid proportional to $1.01^{-3k} 1.001^{2k}$ at an input pressure (supplied by an external force pressing downward on the upper surface of the fluid in the reservoir) proportional to $1.01^{2k} 1.001^k$. Fluid from this reservoir flows through the computer during its operation and into a same-volume self-contained drainage basin (at zero pressure) as in figure 8.

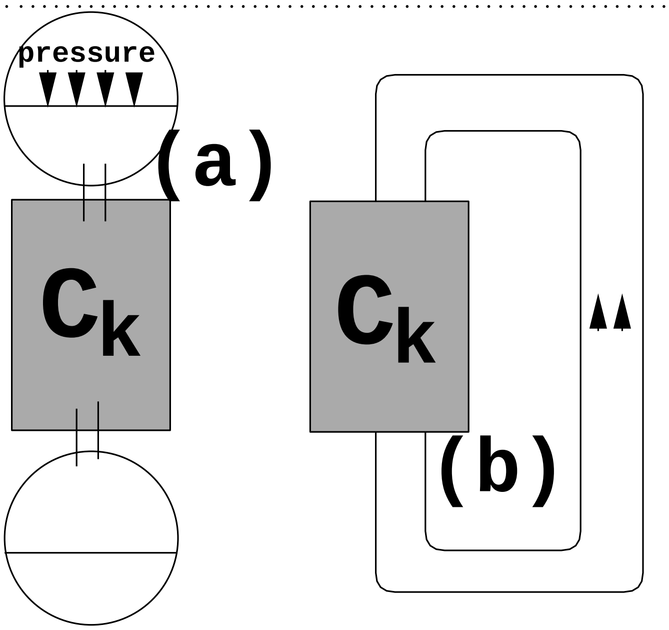


Figure 8. (a) Schematic of simple power supply scheme for computer C_k. (b) Alternatively the flow could be cyclic with the initial energy stored kinetically rather than as pressure (and/or the power could be supplied by external forcing). In §9 we shall abandon these schemes.

To solve the general Turing halting problem: The *input* is assumed to be pre-initialized into the data in the memory bits of the first computer C_k whose memory is sufficiently large to contain it. (I.e., this input data is encoded in the initial conditions of our hydrodynamics problem.). All other C_j start with a blank memory. Thus the whole multicomputer including its "input" and/or "program" is finitely describable (since, e.g., all the C_k are geometrically similar). All the computers C_j do nothing until the input-containing computer is reached, then the real computation starts.

Alternatively, all computers can be initialized identically, but there are order-N blobs of fluid (whose size and speed are appropriately scaled) in a row all flying into a sensor located within C_k, where there are N bits of input and C_k is the first computer capable of reading that much input into its memory. (The queued-up input in this setup would have to be allowed initially to extend outside of C.)

The *output* from the top-level computer C_0 can be used (perhaps after some amplification) to flip the state of a Coanda flip-flop, thus diverting the flow emerging from one source basin into one of two possible destination basins, so that, if the halting problem's answer is "it halts," the diversion will activate and hence most (say $> 90\%$) of that flow will end up in basin A, whereas if the answer is "does not halt" then no diversion will be activated and $> 90\%$ of the water will end up in basin B (see figure 9).

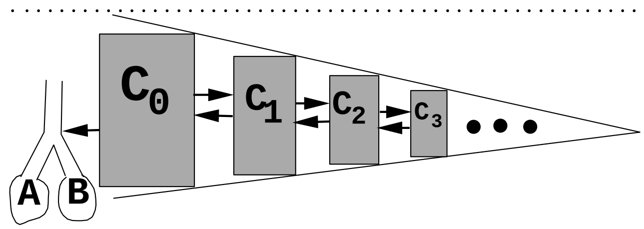


Figure 9. Multicomputer C formed by cubical computers $C_0, C_1, C_2, \ldots$ inscribed in cone with apex C_∞. Arrows denote information-flow channels.

Thus the question of deciding which basin, A or B, will end up containing more water, is Turing-undecidable. We are now done presenting the core of our uncomputability argument, but additional details and extensions will be fleshed out in §9-12, and a theorem statement will be presented in §13.

9 How to power it

Our plan as we've so far described it has flaws. For simplicity I said the kth computer was powered by its own pressure reservoir (the hydraulic equivalent of a "battery"), with an exponentially increasing sequence of required initial pressures, but an exponentially decreasing sequence of required reservoir volumes (and hence exponentially decreasing total stored energies). These pressure reservoirs needed to be "turned on" at appropriate moments. But all this is not realistic if externally applied forces and moving solid parts are disallowed.

Furthermore, some initial reservoir pressures were unboundedly large – despite the fact that their summed energy remained bounded. Some readers would object to that[17].

Fortunately, it seems possible to still make everything work even with all initial energy densities (and pressures and velocities) subjected to *one*, uniform, global upper bound.

The idea is that, because of our assumptions of perfect wall rigidity and perfectly incompressible fluid, it is possible to generate unboundedly large pressures. For example, suppose a mass M of fluid moving with velocity V in a radius-R pipe suddenly collides with a barrier in the pipe (and/or with another, pre-existing stationary equal mass of fluid) and that barrier happens to have been shaped almost exactly the same as the forward boundary of the fluid mass (so that the entire collision happens in time ϵ). Then the time-averaged force exerted on the fluid mass during a time-interval of width ϵ, must be at least MV/ϵ, which can be made arbitrarily large by decreasing $\epsilon \to 0^+$. Similarly (and necessarily) the pressures found in the fluid near the barrier will also be proportional to $MVR^{-2}\epsilon^{-1}$ during a time interval of width $\approx \epsilon$. Such large temporary pressures could be used to power our computers for the small amounts of time they must remain operational. If the fluid mass initially was moving as a solid body separated from the pipe walls by ϵ, then there would be no frictional losses until the collision happened.

The idea we have just described is what underlies the established technology of "two stage light-gas guns" [15][27] in which a piston, driven by a gunpowder explosion, is used to drive a mass of fluid along a pipe and then into a small hole, where it in turn is used to launch projectiles at up to 11km/second. The fluid used to reach these high speeds is hot hydrogen gas, because it is the most incompressible fluid (in the sense of having the highest speed of sound) available[18].

In this scheme, the power source for the C_k would be a mass of fluid moving initially at a velocity *independent* of k, and having a shape largely (but not entirely) independent of k. The linear dimensions of the kth fluid mass, and the pipe containing it, and the separation between that fluid mass and its containing pipe walls, and the collision-time-duration when that mass hit the barrier at the end of its pipe, all would decrease exponentially with k at different rates (permitting everything to fit in finite volume), while the pressures generated at the small ejection nozzles in the barrier (figure 10) would increase exponentially with k. These ejection nozzles would serve as[19] the pressure source for powering computer k.

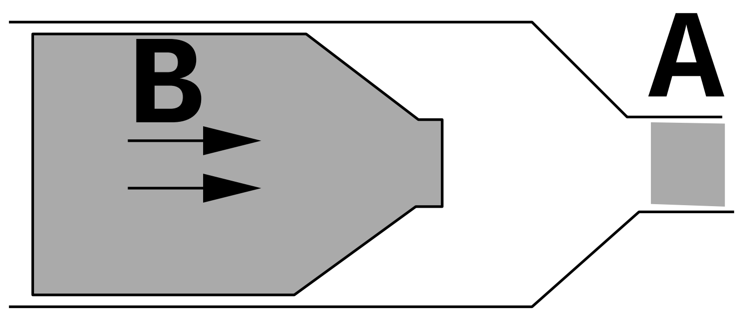


Figure 10. When blob B of incompressible fluid, moving as solid body at fixed speed, hits end of pipe, initially stationary blob A of fluid will be ejected at pressures and speeds which can be made arbitrarily large by decreasing the radius of the hole and the discrepancy between the shapes of B and the pipe-end.

[17] Actually, it is fairly common in the fluid mechanics literature to use the "vortex" flow, which is the same as Couette flow EQ 3 in the limit $A \to 0^+$ with ΩA^2 held constant. This has infinite flow velocity on the vortex's axial line. Note that in this limit the energy E and torque T in EQ 4 approach finite values. This suggests that many mathematical fluid mechanicists do not care about infinite flow speeds if energy remains bounded. Because *linear combinations* of the Couette and Poiseuille (EQ 6) flows are also exact solutions, one can also have "helical vortices." In the limit $\nu \to 0$, $g \to 0$, $g/\nu \to$ constant these also are solutions of the Euler equations for *frictionless* flow. (Also note that high-speed helical vortex-like flows arise naturally as "tornadoes." Science News 155,20 [15 May 1999] claimed that Prof. Joshua Wurman used truck-mounted Doppler Radar to measure a wind speed, ≈ 50 meters up, of about Mach 0.43 in a tornado at 7pm 3 May 1999 near Moore, Oklahoma. In an *incompressible* fluid, Mach 0.43 would be infinite. It is not known whether infinite speeds can arise "naturally" under the equations of hydrodynamics, but this suggests the possibility exists.) It should be possible to tap an Euler helical vortex at one end for power (notice its geometry nicely fits our scaling needs). So *if* fluidic logic still works well enough to build computers even under the *Euler* equations, then this vortex-based power supply scheme ought to be usable to show a theorem analogous to our final result theorem 5 but for Euler equations. However, whereas in that theorem with $\nu > 0$ I feel confident that alternative 2 is the correct one, in the Euler $\nu = 0$ case any of the 3 alternatives seem plausible to me.

[18] The phenomenon of large short-duration pressure surges also arises in water systems after the closing of a valve as a well known hazard called "water hammer" [32][7][57][72]. A related hazard closer to our idea is "slug flow" in which a blob of water appears (due to condensation) in a pipe filled with flowing steam. It is carried along by the gas and collides with a pipe bend at high speed. These phenomena have led to, e.g., nuclear power plant failures. When one uses a hammer to pound a nail, a small amount of force applied over a comparatively long time is converted on impact into a large amount of force for a short time. We want to use the same idea to build hydraulic pressure up-convertors. Another related effect is "sonoluminescence" [53] in which spherical bubbles in liquid collapse. (Similar effects should happen during the radial collapse of a *cylindrical* cavity. However spherical cavities presumably are the only ones with shape-stability if surface tension terms are imposed at free liquid boundaries.) The walls of the bubble, in an ideal incompressible fluid, would reach infinite speed during such a collapse; in actual common liquids they are known to reach 4 times the speed of sound (i.e. about 6km/sec), creating shock waves, pressures of order 10^6Atm, temperatures $> 10^4$Kelvin, and consequent visible 100-picosecond flashes of ultraviolet light. This happens spontaneously in certain common fluid flow scenarios [68] including when shrimp snap their claws in the ocean [41].

[19] Actually it may not be necessary to make the collision "take time ϵ" by carefully shaping the barrier slightly imprecisely. Alternatively the barrier could be shaped precisely (collision time *zero*) but with a small hole in the barrier leading to computer k, with the computer and hole already pre-filled with (stationary) fluid. The size and hydraulic resistance of computer k would then automatically serve to self-adjust the pressure to the correct post-collision value. The situation is analogous to an LR inductor-resistor series electrical circuit with a pre-existing current I in the (large) inductor. The value R of the resistor then will serve to self-adjust the voltage across it to be IR. In other words, view this as a current source rather than a pressure source.

10 Precision requirements

Unfortunately, while the power supply idea of §9 can repair one aesthetic objection (unbounded initial pressures), it does so at an aesthetic cost: we now require very precise timings and hence initial positionings. In the previous picture from §8 with pressure reservoirs supplied via deus ex machina, all the initial velocities, and lengths and volumes of the component pipes, logic gates, and chambers could all have been off by a few percent and everything could still work. This is no longer entirely true since the timing of the pressure surges needs to be extremely precise.

However, it is possible to partially overcome even this objection, by making the moving slugs of fluid, which ultimately will power everything, initially be aligned. Then, C_k will turn on a predictable amount of time (depending on the length of a pipe P_k from C_k to C_{k+1}) before C_{k+1} does. This turn-on may be sensed by C_k, which can delay itself appropriately to compensate for any small relative error in the length of P_k. In this case only the initial alignment of the initial power-supplying fluid mass, and the initial requirement that this fluid be moving at exactly constant velocity ("as a solid body") needs to be enforced ultra-precisely. Unfortunately these remaining alignment requirements are still subject to criticism.

There are several approaches that plausibly allow overcoming that criticism. One of them: C_k could act in such a way as to speed up or slow down the mass of onrushing fluid that is to be used to power the C_j with $j > k$ in such a way as to get the *effect* of artificially enforcing this ultra-precise initial alignment, without actually needing to have it. **Alternative power-supply methods** aimed at this same goal:

1. C_k perhaps could adjust the stores of pre-existing pre-impact stationary fluid inside the C_j with $j > k$, or
2. C_k could "output" the flow used to power C_{k+1} by a diversion and pressure up-conversion (by intentionally creating a "water hammer" pressure surge, as before) from a constant fractional part of its own power store. This idea seems very desirable since it would allow the use of solely *constant ratio* standardized pressure up-convertors and would allow the entire multicomputer to be (essentially) a geometrically self-similar object depending solely on the scale invariance properties of lemma 1.
3. Both of the preceding ideas could be combined with "sensing" by C_k of when C_{k+1} has turned on.

It is not completely clear whether these ideas really can be accomplished, and indeed it is not clear to me how to *precisely state* the desirable notion that our fluid-computer is "self correcting" and "immune to any sufficiently small perturbation in its initial conditions." (The question is: what kind of perturbations should be allowed? If one tries to write down suitable conditions, one soon finds oneself dealing with unusual – and perhaps unnatural and undesirable – new kinds of "norms.")

The question of whether idea #2 can be done, depends on how reproducibly C_k can control the shape of the forward envelope of a body of fluid it intentionally ejects (e.g. into a pipe whose end is shaped, to very high accuracy, identically, analogously to figure 10). I would suspect the answer is "exponentially (in b_k) accurately if b_k stages of logical 'buffering' are placed between C_k and its fluidic ejector device." In that case, idea #2 really should be feasible... but the fluid initially filling those b_k buffer stages would have to have initial speed *exactly* zero and the buffer devices and corresponding pipe-end would have to be shaped *exactly* so that this "solution of the precision problem" would in some ways defeat itself.

Whether or not error-tolerant, self-correcting power supply schemes and fluid computers are possible (and I think they are[20]) we still conclude that at least *some* initial conditions exist for which hydrodynamics is unsimulable.

11 Component reliability

Since some fluidic logic component designs are based on turbulence, it might be thought (if we believe, cf. §3, that turbulence, or hydrodynamics itself, is a randomized phenomenon) that such components may not operate 100% reliably but instead would occasionally err, say with probability $\leq 10^{-10}$ of failure on any particular logic operation. (Empirically they always seem to work[21], but this is not a proof.) This (since our computer has an *infinite* number of components, and performs an infinite number of logic operations) would seem to be the kiss of death. Fortunately there are known "fault tolerance" techniques for designing reliable circuits from unreliable components.

For our purposes, the best[22] available such method is, essentially, due to Andrei L. Toom [63], with the latest and best proof being given by Peter Gács [22] (building on ideas of Gács, Reif, Berman, and Simon [8][25][24]).

Definition 2. *Let a 2D Cellular Automaton (CA), whose cells are the integer lattice points $\mathbb{Z}^2$ and each cell containing 1 bit, obey the "Toom [transition] rule" if the state transitions are: each cell's updated version is the majority vote of itself (x, y), its North neighbor $(x, y+1)$, and its East neighbor $(x+1, y)$.*

Toom's point is that if all cells are initially 0s (or initially all 1s) then repeated application of the Toom rule will keep them that way. Even if some small fraction of the cells are randomized at each time step, the Toom rule will tend to eliminate the errors. A theorem formalizing this can be shown, and was shown, by Toom, and the latest and greatest version by Gács appears as our lemma 3 below[23].

[20] This contrasts with my earlier work [59] settling the computability status of Newtonian mechanics. There, error correcting diminution of all small perturbations seemed impossible. Here, because hydrodynamics is *dissipative*, it *does* seem possible.

[21] Angrist [3] tested one fluidic logic component for 3.5×10^9 cycles with no failure, even in the presence of heavy vibrations.

[22] The upper bound we will give, which involves $O(\log(Nt)^2)$-fold increase in the number of components in an N-gate circuit with t levels of delay, does not quite meet [23]'s lower bound $\Omega(\log(Nt))$. The simplest way I know of making circuits reliable which *does* meet this lower bound, is to replace each wire with a bundle of $m = O(\log(Nt))$ wires and on each employ a m-input, m-output corrector which works by using a "bipartite expander graph" with m inputs and m outputs and constant valency 5 (the "Ramanujan graphs" of [43] will do) where each output takes a majority vote among its 5 corresponding input signals. But this method is not suitable for our purposes since expander graphs cannot be embedded efficiently into 3-space [39].

[23] Note the asymmetry of Toom's N-E-self pattern. This appears to be essential: Attempts to replace it with symmetrical patterns, such as majority vote of oneself and one's 4 (N,S,E,W) neighbors, *fail.*

Gács used the Toom rule as a tool to help build reliable logic circuits (despite components with some small error rate) which could emulate arbitrary circuits (which might not have been reliable if their components had some small error rate). For example:

Lemma 3 (Gács's error bound [22] for Toom rule). *Start with any N-cell long 1D CA. Cross product it with a $\mathbb{Z}_m^2$ "discrete torus" ($m \times m$ grid with horizontal and vertical "wraparound[24]") to get a 3D CA. Use as the transition rule on this 3D CA, the Toom rule in 2D planes, followed by the 1D CA rule in the 3rd dimension. Use the 3D CA to simulate t steps of the 1D CA using the 3D CA with 1D data simply initially copied m^2 times into all entries within the corresponding $m \times m$ plane. Suppose each 3D CA cell's state is toggled, each transition, with a probability of $\leq \epsilon$, where $\epsilon < 12^{-8}32^{-1}$. Then the total error probability at the end of the t-step simulation (i.e., the probability that the final state of the 1D CA, with each 1D CA cell state judged by majority vote within its $m \times m$ plane, will be wrong) will be*

$$\leq 24tm^2N(288\epsilon^{1/12})^m + 24\epsilon. \tag{9}$$

Thus by making m large we get exponential reliability, and m of order $\log(tN)$ suffices to drive the final error probability below 12^{-8}. Note that the probabilities of logic failures do *not* need to be assumed to be independent. All one needs to assume, is that, at any particular timestep at any particular gate, the *conditional* probability (conditioned on whatever the other gates did or are doing) of a logic failure right here and right now (with correctness judged based on the present, local, input signals), is $\leq \epsilon$.

Instead of starting with a 1D CA, we can start with any circuit built out of "wires" and bounded-fanin and fanout logic "gates" in 3D. (Regard any "repeaters" in the middle of long wires simply as 1-input, 1-output "gates.") Replace each wire with a "bundle" of $m \times m$ wires. Replace each logic gate with a Gács-Toom correction step (taking x, y coordinates modulo m) before and after, plus m^2 parallel copies of the appropriate gate operation on disjoint tuples of input wires, one from each input bundle. Note (see footnote 24) that *no wires longer than a constant are required* in the new circuit, if none were required in the original circuit. Note also that this converts any circuit into a circuit that emulates the original one but is *reliable* even if each gate in the new circuit will, when performing any particular logical operation, produce the wrong answer with some probability $\leq \epsilon$. Finally, the new circuit, if built with the same-size components as the original circuit, occupies $O(m^3)$ times its volume, and $O(m)$ times its linear dimensions.

The key message here is that, if the failure probabilities of one's computer components may be brought below some constant threshhold ($12^{-8}32^{-1}$ in Gács's proof, but probably in reality, since that proof undoubtably is weak, 0.001 suffices) then the whole computer may be redesigned to decrease its failure probability to exponentially small levels.

Now, to make our entire multicomputer reliable it suffices to Gácsify C_k using $m = O(k+1)$. This causes the entire *infinite* computation to proceed to completion with $\leq 30\epsilon$ probability (i.e., well below 1/3) that any error at all occurred, ever, on any logical operation. Because, e.g., $\sum_{k\geq 1} k^3 1.01^{-k}$ converges, the total volume, power consumption, surface area, etc. all remain finite and are only increased by a constant factor. (Then the "cone" in figure 9 is no longer a cone, but a slightly different pointed shape, and that picture no longer looks exactly self-similar.)

Note also that this way of decreasing the output-error probability below 1/3 also may be regarded as an *existence proof* for a *zero*-error hydrodynamic computer, if we now again regard hydrodynamics as completely deterministic given by EQ 1 and 2, and if we regard all the "random" errors as really arising from small perturbations in the initial data, and if we assume they still act (if that perturbation had initially been chosen randomly) in the required probabilistic manner. I.e., under all those assumptions we have proven that some set of initial data exist in a tiny neighborhood of the "right" such data (in fact, of relative measure $\geq 2/3$ within that neighborhood) that makes C deterministically work. We could use this to solve the halting problem with high correctness probability by repeatedly rerunning the hydrodynamic system C, starting from random small-norm perturbations of the "right" initial data, each run.

12 Higher levels of undecidability

The ability to solve the general halting problem represents immense computational power (cf. footnote 15). But we can get more. There are "higher levels" of undecidability. The ability to solve the halting problem for general Turing machines is level 1. Level $k+1$ is the ability to solve the halting problem for a general Turing machine which has access to an oracle for solving arbitrary level-k problems in 1 step. By (essentially) simply giving each of our computers C_k their own private copy of the "cone" of computers C_j for all $j > k$ for use as an oracle, hydrodynamics can solve level-2 undecidable problems in finite time. Now if each these private cones is itself made of computers with their own private oracles, we get level-3. In fact, one can similarly achieve level k for any finite k.

The above plan (as illustrated in figure 11) does *not* work to reach $k = \infty$ because the energy requirements would be infinite. However, if instead of making C_{k+1} about 1.01 times smaller than C_k, it is permissible to make it, say, 5 times smaller, and if instead of making C_k have only one descendant C_{k+1} we make it have *two* (forming an infinite binary tree of child-computers), then hydrodynamics gains access to level-∞ undecidability, still with only finite energy, volume, and time resources (since now the appropriate geometric series converge – the factor of 2 caused by the branching of the binary tree is no longer enough to make them diverge).

The only penalty one pays for all this extra compute-power is that the container becomes more peculiar. At level 1, it sufficed to have a container smooth everywhere except at *one* bad point of $\mathbb{R}^3$. At level 2, there are a countable infinity of such bad points, which have one point of accumulation (call

[24] Note: One can embed an $m \times m$ toroidal grid graph in the plane *without* using any wire lengths longer than a constant, no matter how large m is, by the well-known trick of mapping the integers from 0 to $\lfloor (m-1)/2 \rfloor$ into even positions and the integers from $\lceil m/2 \rceil$ to $m-1$ into odd positions in reverse order.

this a "2-bad" point). At level 3, there are a countable infinity of bad points *and* a countable infinity of 2-bad points, and the 2-bad points have a point of accumulation (call that a "3-bad" point). At level k there are a countable infinity of bad, 2-bad,... and $(k-1)$-bad points plus one k-bad point. At level ∞, the container has an *uncountable* infinity of bad points on its boundary, part of which is a fractal-like surface in some ways like the "Koch snowflake" [44].

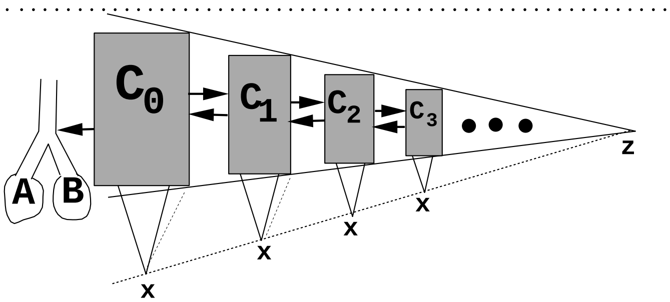


Figure 11. Multicomputer C extended by the addition of oracles (in downward pointing solid cones) for each C_k so that it can solve level-2 undecidable problems. There are now an infinite number of "bad" points x at which the boundary is non-smooth, as well as a "2-bad" point z which is a limit point of bad points. It is also possible to extend C so that it can solve level-3, or indeed level-∞, undecidable problems. This is signified by the widening of the downward pointing oracle-cones (shown dashed for first two cones).

But the story still is not over. It is possible to reach *transinfinite* levels of undecidability in a similar way. (The entire infinite tree of computers may be used as an oracle...) It was unclear to me how far (and how to describe how far!) one can go, but recursion theorist Frank Stephan helped me. According to him, the computational power these techniques allow access to is described by $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$, which is the set of so-called "hyperarithmetic" sets and we follow the notation of Odifreddi ([52] vol. 1, page$\approx$ 362). Here

$$A \in \Sigma^1_1 \iff [x \in A \iff \exists F \text{ such that } P(F, x_1, x_2, \ldots, x_n)] \tag{10}$$

for some computable predicate P, and

$$A \in \Pi^1_1 \iff \Big[x \in A \iff \forall F,\ \ \tilde{P}(F, x_1, x_2, \ldots, x_n)\Big] \tag{11}$$

for some computable predicate $\tilde{P}$, and $x_1, x_2, \ldots, x_n \in \mathbb{N}$, $A \subseteq \mathbb{N}$, and F is a function from $\mathbb{N}$ to $\mathbb{N}$. Here A is the "computational problem" being handed to some computer and sets such as Σ^1_1 or Π^1_1 of problems are the "language being recognized by some computer." To learn about these sets, see [52][55]. My crude understanding of Δ^1_1 (as a non-logician) is that it seems to be everything one can compute using infinite and transinfinite recursive oracle calls to halting-problem solvers for machines of $\leq \Delta^1_1$ power, where the structure of the recursive calls and what you do with them is required to be specifiable by a finite program. The fact that $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$ is a nontrivial theorem.

Going still further, a multicomputer like ours could easily be programmed to access, and return at top level in finite time, any desired bit from an infinite pre-filled *read-only* memory. Thus *any* Boolean-valued function of the input bits could be computed by hydrodynamics, which seems the ultimate in computational power. However, accomplishing this feat requires (what computational complexity theorists call) "nonuniformity." I.e.: for our previous "uniform" hydrodynamic computers, just *one* container-shape C defined a "universal" computer, capable of being programmed to execute any algorithm. The program was specified via a *finite* number of bits, each encoded as, e.g., the presence or absence of (or one of two possible velocity vectors for) a standardized fluid blob (and that encoding appeared to be capable of being made immune to all small relative errors in, e.g., the positioning of the blobs). But in our "non-uniform" construction each computational problem requires a *different* container shape, or the initial data (e.g. the program) needs to be specified via an *infinite* number of bits. Many computer scientists regard such non-uniform results as "cheating."

13 Conclusion and moral

The whole mission of hydrodynamics was the hope that macroscopic fluid flows, involving huge numbers ($N = 10^{23}$-10^{64}) of atoms, could be successfully modeled as a continuum, without having to think about 10^{23} different particles. What might be called the "hydrodynamic limit" was precisely that continuum limit of $N \to \infty$, or numerical pixel size$\to 0$ (there are numerous possible precise definitions and I intentionally do not make one). The present paper has shown (nonrigorously) that *the hydrodynamic limit does not exist.*

In other words, the **moral** is that the usual physicist's assumption that real fluids can be modeled as a continuum without sacrificing accuracy for macroscopic problems, is false. Scenarios within the equations of hydrodynamics can exist in which phenomena happening at arbitrarily tiny length scales, and having arbitrarily tiny energies, cause predictable macroscopic effects (or their absence) of a fixed finite size in a fixed finite amount of time. (Note: this is *not* the same thing as "chaos," in which infinitesimal perturbations in initial conditions merely grow exponentially [and hence by a finite factor] with time – it is something far more severe than that. Our construction does not employ chaos [or if it does, it is only at sub-component length scales which never "leak" into neighboring components].)

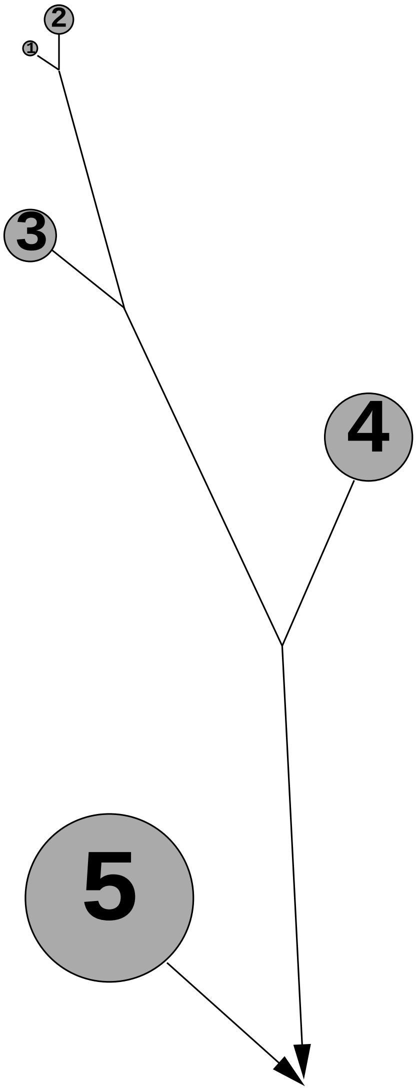


Figure 12. Nonexistence of the hydrodynamic limit with all velocities bounded. An infinite sequence of moving balls of fluid collide in a geometrically self-similar 3D spiral pattern. If tiny ball 1 is present, then 2 will be deflected to hit 3, which will be deflected to hit 4, etc. Thus an infinitesimally tiny fluid ball will cause macroscopic effects in finite time, even if all balls move with unit speed. In this case the hydrodynamics of each collision will *not* be scaled-identical, but all the small balls will effectively be in the "Stokes limit" of hydrodynamics in which viscous effects are (exponentially) enormously dominant, thus each smaller and smaller collision will act more and more like an inelastic collision of solid bodies. Newton's laws alone then should suffice to convince you that it will work.

. . . .

To convince yourself of the *nonexistence of the hydrodynamic limit*, it is not necessary to go through our whole argument about fluidic computers, undecidability, etc.

Theorem 4 (Nonexistence of hydrodynamic limit). *Assume that a fluidic amplifier device exists*[25]. *Then unboundedly small perturbations in the initial state of a fluidic system suffice to cause full size changes in the flow after a bounded amount of time.*

Proof: An infinitely long exponential cascade of geometrically similar fluidic amplifiers will suffice to make our point. We rely on lemma 1. Say each amplifier has "gain 8," meaning (for the present purposes) that a jet with volume $\propto 8$ is diverted by a control jet of volume $\propto 1$. Each amplifier stage, lengths and energies are scaled up by, and velocities are scaled down by, a factor of 2. The delay time for the kth amplifier stage is $\propto 4^k$. Thus the total signal propagation delay, even for an infinitely long amplifier sequence, is finite if the last (biggest volume) stage has volume 1. An infinitesimal jet having infinitesimal energy causes a size-1 effect in finite time. Q.E.D.

Remark. Our uncomputability construction – the main result of this paper – is *more* than merely some disguised version of theorem 4. That is because it invoves the self-generation of an unboundedly small flow perturbation as the result of some unboundedly long computation, and only *then* is that perturbation amplified up to macroscopic size.

Remark. In the scenario of theorem 4, as usual, some readers may dislike the fact that the very small jets have very high speeds (despite the finiteness of total energy). This apparently cannot be avoided if one wants to precisely use the scale invariance properties of hydrodynamics from lemma 1. However, the present scenario is so simple that it seems we can get away with making each amplifier stage be hydrodynamically *in*equivalent and keeping all velocities *bounded*: that is done in figure 12.

Our (nonrigorous) undecidability result represents the utter failure of hydrodynamics from the point of view of a computer scientist. Our moral represents the utter failure of hydrodynamics from the point of view of a physicist.

Because our undecidability result has not been a Theorem, there seems to be an escape hatch for physicists to wriggle out of. But in fact, by wording things appropriately to encapsulate all the unproved assumptions in one nugget, we *can* summarize our arguments with a genuine theorem – which, in my opinion, slams shut that escape hatch, since each of its three alternatives seem to represent (to a physicist) a "failure" of hydrodynamics.

Theorem 5 (Failure of hydrodynamics). *One of the following 3 alternatives must be true:*

1. *Solutions to the equations EQ 1 and 2 of unforced hydrodynamics in rigid bounded containers of fluid do not necessarily exist for all future times. (Here we may demand that the fluid have everywhere-smooth and bounded initial velocity and pressure fields, and that all fluid surfaces and container-boundaries have finite surface area, and that the latter [and all fluid free surfaces] are smooth everywhere except at a single point of* $\mathbb{R}^3$*.)*
2. *There does not exist an algorithm (even a randomized one) for approximating either the velocity field or the location-set of the fluid, at time* $t > 0$ *in the future, to accuracy* $\epsilon > 0$ *(in, e.g., the* L_1*,* L_2 *or* L_∞ *norms;* ϵ *and* t *are part of the input to the algorithm), with correctness probability* $\geq 2/3$.

[25] One could easily make any number of precise mathematical definitions of "fluidic amplifier device," but it seems pointless to bother.

3. *The widely accepted engineering/experimental conclusion that boolean-complete "Fluidic logic" component families exist, which* work[26] *with low mean error rate is incompatible with the predictions of the equations of hydrodynamics. Here, by "low mean error rate," we mean that each logical operation errs on its inputs with conditional probability* $\leq 7\times 10^{-11}$*, conditioned on everything else.*

Morally unsatisfied physicists should also re-read our §5.

Another **moral** is that the Navier-Stokes PDEs of hydrodynamics are unsimulable. You can't build an algorithm to simulate those PDEs. If physicists ask a computer programmer "program a simulation of the hydrodynamics PDEs" then those physicists were asking the impossible. They were asking the wrong question. They should have been asking for something possible. If physicists want to simulate fluids, they have to put in something else into the mathematical description of the set of physical laws other than merely the Navier-Stokes PDEs EQ 1 and 2. I suggested, in §3, also putting in a model of thermal or other "noise," but it appears that this too will be insufficient, and perhaps nothing short of simulating all the atoms will really work.

Let me end with a **bang.** I think this all represents a deathblow to computational hydrodynamics. To make an analogy: tremendous effort has been devoted, and will be devoted, to the task of **debugging computer programs.** But there is no easy way, and no mechanized way, to do that – *and there never will be*, because, as is well known [47], it is a nonalgorithmic task. That is noncontroversial. Tremendous computational effort has been and will be devoted to simulating the Navier-Stokes fluid equations. But these attempts have never been easy and have (almost) never come with any guarantees about solution quality. There has never been a general purpose Navier-Stokes solver with arbitrarily specified guaranteed solution quality *and there never will be.*

And now let me **hit you again.** Here is a typical statement, which I think it is safe to say, could have been made (before the appearance of the present paper) by 90% of all physicists: "The constant temperature and density equations of hydrodynamics are perfectly adequate, to very good approximation, for modeling you playing in your bathtub with your pet shrimp. That is because all flow velocities are going to be well below the speed of sound (1.5km/sec) in water, all pressures are going to be a few atmospheres at most (far below those required to compress water significantly), and all temperatures are going to be constant to within a few centigrade." But in fact, this statement is entirely wrong. The truth is: "You, by playing in your bathtub with your pet shrimp, are entirely capable of easily creating situations with pressures $> 10^6$atm, flow speeds 6km/sec, and temperatures $> 10^4$Kelvin. This has been experimentally observed [53][68]. Indeed, such flows are very common and are biologically important to the shrimp [41][27]. In these situations, the usual equations of hydrodynamics are clearly inapplicable – indeed they predict their own failure. The fact that hydrodynamics often gets certain gross characteristics of the flow (e.g. drag, integrated flow volume) roughly correct, is therefore largely due to luck. Indeed, mathematical scenarios can be created (the point of this paper) in which the gross characteristics of the flow (according to the equations of hydrodynamics) are not predictable by any algorithm."

14 Open problems

1. What happens to algorithmicity if we refuse to permit solid walls, or free surfaces, or both?

Free surfaces seem avoidable if we power everything via the recirculating scheme of figure 8 (b) and use, as the output bit, an integrated flowmeter measurement rather than a volume measurement. In these cases the entirety of C can be completely filled with liquid, *but* timed large external forces are needed to power it. Really the only thing we needed the free surfaces for was the bounded-initial-speed power supply scheme of §9. (Under the *Euler* equations of *inviscid* flow, as we mentioned in footnote 17, one may be able to get power from a vortex without needing free surfaces.)

It is conceivable that the solid walls in our construction could be avoided, with their role somehow instead being played by some very clever initial flow velocity distribution. (Tornado-like vortices could perhaps be used instead of "pipes?" Cf. footnote 17.) Also, if two fluids were permitted, it might be possible to get the effect of having solid walls, by using a fluid of very large density ρ and viscosity ν. (It is often claimed that window glass is a "liquid" with very large viscosity [45].) Perhaps something like our construction could still be made to work even for large *but finite* ρ, ν.

Many physicists justifiably object that our construction, involving a container wall with self-similar "wiggles" at unboundedly small length scales, is "unnatural." If a "natural" fluid flow could be constructed that somehow would automatically *create* such a structure, this physicist's objection would vanish. It is indeed possible to satisfy – at least partially – the desire for such a construction, if we now allow hydrodynamics *plus* moving solid rigid parts, plus some kind of ability to "freeze" certain parts of the liquid into rigid solid[28]. The idea is, roughly, to build a computerized "robot" out of a fluidic computer and hydraulic-powered machinery. This robot

[26] The vague-sounding term "work" here could be given a precise mathematical meaning. Essentially, it means to work in the engineering sense in which electronic logic devices presently commercially available from various manufacturers, are supposed to work – if their inputs are voltages (for us: pressures) within certain allowed ranges, and their power supply pins are supplied with voltages in certain allowed ranges, etc., then they will produce output with certain specified properties and react in certain specified ways to certain stimuli, and this all is demanded to remain true no matter how these devices are interconnected, and no matter how the initial conditions are perturbed, provided those perturbations have sufficiently small norms and provided those interconnections obey certain widely accepted and well known design rules. Those design rules are requirements such as "keep fanout below 5" and "use pipe lengths, curvatures, and input pressures within the following bounds..." Furthermore, it suffices if these logic devices "work" with some sufficiently small constant probability of failure.

[27] The shrimp use the "pop" sounds created by cavity collapses to signal each other. A fluidic signal processor, complete with fluidic microphone, could recognize these sounds, just as the shrimp do, and respond by diverting 100 liters of flow. I believe such a device should be entirely feasible to build. In that case a 100-liter flow diversion would be triggered by a cavity-collapse event for which the PDEs of hydrodynamics were inapplicable.

[28] Again we remark that a fluid with large density ρ and large viscosity ν essentially *is* a rigid solid body. Fluids whose viscosity and/or density change (sometimes permanently) in a history dependent manner are well known ("non-Newtonian fluids;" commonplace examples include human blood and bread dough). Recently it was discovered [18] that nitrogen gas, when subjected to pressures p = 1.4-2.4GPa at room temperature,

is programmed to manufacture a smaller, scaled copy of itself. It would then weld the smaller robot onto itself as a new appendage, download its program into that smaller robot, and turn it on. Due to the scaling properties of lemma 1 and the fact that geometric series converge to finite answers, the entire infinite chain of robot manufacturing will happen (due to our assumption of a true-continuum) down to zero length scale in finite time – but note we only needed to start with *one* robot having all length scales *macroscopic*. The flaw in this idea is the need to *power* all the smaller robots, but conceivably the power-diversion idea of item 2 in the list at the end of §9 can be used to accomplish that[29].

2. Does the set of initial conditions leading to non-algorithmicity, in some sense have "measure zero?" Or not? (I think the answer is "not," at least for some reasonable formulation of the question. See the remarks on "precision" in §10.)

3. What about the non-constructive existence problem for hydrodynamics [9][19]?

4. Try to delineate precisely: In what senses, and when, and how often, can physical validity and algorithmicity be ascribed to hydrodynamics?

Clearly, there are some scenarios (such as Couette and Poiseuille flow at low speeds) when fluid dynamics is algorithmic and works well to describe and predict most or all experimentally measureable properties people care about. But there are other scenarios, such as the one constructed here, where that does not happen. So there is some "dividing line" separating these two possible kinds of scenarios, and the problem is to understand it. §5 suggests that this line will not be easy to draw and may have disquieting properties.

15 Abbreviations used in this paper

CA Cellular automaton;

ODE ordinary differential equation;

PDE partial differential equation;

Re Reynolds number UL/ν (U=typ. speed, L=typ. length, ν=kinematic viscosity);

TM Turing machine (often I have universal ones in mind).

$\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ Reals, integers, and natural numbers.

References

[1] E.R.Berlekamp, J.H.Conway, R.K.Guy: What Is Life?, Ch. 25 in "Winning Ways for Your Mathematical Plays," Vol. 2 Academic Press, London 1982.

[2] B.W.Anderson: The analysis and design of pneumatic systems, Wiley 1967, reprint 1976.

[3] Stanley W. Angrist: Fluid control devices, Scientific American 211,12 (Dec. 1964) 79-88 and 154.

[4] G.K.Batchelor: Introduction to fluid mechanics, Cambridge Univ. Press 1967.

[5] Charles A. Belstering: Fluidic systems design, Wiley-Interscience 1971.

[6] D.Bensimon, L.P.Kadanoff, S.Liang, B.I.Shraiman, C.Tang: Viscous flows in two dimensions, Rev. Modern Physics 58,4 (1986) 977-999.

[7] Louis J.B. Bergeron: Water Hammer in Hydraulics & Wave Surges in Electricity. Wiley 1961; reprinted by UMI Books on Demand.

[8] Piotr Berman & Janos Simon: Investigations of fault-tolerant networks of computers, Sympos. on Theory of Comput. (STOC) 20 (1988) 66-77.

[9] A.Bertozzi & A.Majda: Vorticity and Incompressible Flow, Cambridge University Press 2001.

[10] Thomas Ephraim Brock: Fluidic applications, analysis of the literature and bibliography, British hydrodynamaics research association 1968.

[11] S.Chandrasekhar: Hydromagnetic and hydrodynamic stability, Oxford Univ. Press 1961.

[12] Henri Coanda: Device for deflecting a stream of elastic fluid projected into an elastic fluid, US Patent 2,052,869 (issued 1 Sept 1936, filed 19 April 1935).

[13] D.Coles: Transition in circular Couette flow, J.Fluid. Mech. 21 (1965) 385-425.

[14] V.S.J. Craig, C. Neto, D.R.M. Williams: Shear-dependent boundary slip in an aqueous Newtonian liquid, Physical Review Letters 87 (30 July 2001) 054504 (4 pages).

[15] W. D. Crozier & William Hume: High-Velocity, Light-Gas Gun, Journal of Applied Physics 28 (Aug. 1957) 892-894.

[16] G.Daccord & R.Lenormand: Fractal patterns from chemical dissolution, Nature 325 (1987) 41-43.

[17] G.W.A.Dummer & J.M.Robertson (eds.): Fluidic components and equipment (1st ed.), Pergamon 1968.

[18] Mikhail Eremets, Eugene Gregoryanz, Russell Hemley, Ho-kwang Mao: Semiconducting non-molecular nitrogen up to 240 GPa and its low-pressure stability, Nature 411 (10 May 2001) 170-174.

[19] Charles L. Fefferman: Existence & Smoothness of the Navier-Stokes equation, problem statement for Clay Mathematics Institute $1 million prize `http://www.claymath.org/prizeproblems/navierstokes.htm`.

[20] Keith Foster & G.A.Parker: Fluidic components and circuits, Wiley-Interscience 1970.

[21] Uriel Frisch: Turbulence [subtitled "the legacy of A.N.Kolmogorov"], Cambridge Univ. Press 1996.

[22] Peter Gács: A new version of Toom's proof, Boston University CS tech. report 1995-009-toom-proof.pdf, 8 pages (`http://cs-www.bu.edu/techreports/pdf/1995-009-toom-proof.pdf`).

[23] P.Gács & A.Gal: Lower bounds for the complexity of reliable boolean circuits with noisy gates, IEEE Trans. Info. Theory 40,2 (1994) 579-583.

[24] P.Gács: Reliable cellular automata with self-organization, Sympos. on Foundations of Computer Sci. (FOCS) 38 (1997) 90-99.

becomes a semiconducting solid. At temperatures below 100Kelvin, it remains in solid form even after removal of the pressure. This may be important inside some of the gas-giant planets.

[29] The point I am trying to make with these arguments is not that all this is physically realizable and a good way to solve the halting problem. Instead it is that the mathematical structure of the PDEs of hydrodynamics allows some severe pathologies including non-algorithmicity. Although the "fluidic computers," and "robots" I've used to construct those pathologies are rather baroque, their *existence* indicates the possibility that, quite probably, much more natural flow situations exist in which pathological behavior also happens. Such behavior in the *absence* of my comforting and easily-understood artificial-computer framework, will presumably be even *harder* to understand and simulate, not easier.

[25] P.Gács & John Reif: A simple three-dimensional real-time reliable cellular array, Journal of Computer and System Sciences 36,2 (April 1988) 125-147.

[26] V.K.Garg & W.T.Rouleau: Linear spatial stability of pipe Poiseuille flow, J.Fluid Mech. 54 (1972) 113-127.

[27] Information on the 20-meter-long two-stage light-gas gun at Lawrence Livermore National Labs: `http://www-phys.llnl.gov/H␣Div/GG/`. This was used to create metallic hydrogen [67].

[28] D.Gottlieb & S.A.Orszag: Numerical analysis of spectral methods, SIAM 1977.

[29] John Hennessy and David Patterson: Computer Architecture: A Quantitative Approach, 3rd Edition, Morgan Kaufmann Publishers, 2002.

[30] E.L.Holbrook & P.I.Chen: Design of pneumatic and fluidic control systems, PECH Publishing, Milwaukie, Oregon 1984.

[31] Eugene F. Humphrey & Dave H. Taramuto (eds.): Fluidics, a comprehensive examination (2nd ed.), Fluid amplifier associates, Boston 1968.

[32] J.H. Kim: Water-hammer prevention, mitigation, and accomodation: a perspective, Trans. Amer. Nuclear Soc. 55 (1987) 733-734.

[33] Joseph M. Kirschner & Silas Katz: Design theory of fluidic components, Academic Press 1975.

[34] A. N. Kolmogorov (Translation by V. Levin): The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers, Proc. Royal Soc. London A 434 (1991) 9-13. Original Russian Dokl. Akad. Nauk SSSR 30,4 (1941) 3201. Also see: On degeneration of isotropic turbulence in an incompressible viscous liquid, Compt. Rendus Acad. Sci. URSS 31 (1941) 538.

[35] M.LaBarbera: Principles of design of fluid transport systems in zoology, Science 249 (1990) 992-1000.

[36] Horace Lamb: Hydrodynamics (5th ed.), MacMillan 1932, Dover reprint 1945.

[37] L.D. Landau & E.M. Lifshitz: Fluid mechanics, Pergamon 1987 (course of theor. physics. vol. 6).

[38] John Laufer: The structure of turbulence in fully developed pipe flow, NACA (Natl. Advisory Committee Aeron.) Report 1174 (1954). 10 pages. `http://naca.larc.nasa.gov/reports/1954/naca-report-1174/naca-report-1174.pdf`

[39] F.T. Leighton: Complexity Issues in VLSI, MIT Press 1983.

[40] C.C.Lin: Theory of hydrodynamic stbility, Cambridge Univ. Press 1955.

[41] D.Lohse, B. Schmitz, M. Versluis: Snapping shrimp make flashing bubbles. Nature 413 (4 Oct. 2001) 477-478.

[42] T.C.Lubensky & P.A.Pincus: Superpolymers, ultraweak solids, and aggregates, Physics Today 37 (Oct 1984) 44-50.

[43] A. Lubotzky, R. Phillips and P. Sarnak: Ramanujan graphs, Combinatorica 8 (1988) 261-277.

[44] B.B.Mandelbrot: The fractal geometry of nature, W.H.Freeman 1977.

[45] Steve W. Martin: Is glass really a liquid, pp.215-216 in *Scientific American's Ask the Experts*, Harper Collins 2003.

[46] B.S.Massey: Mechanics of fluids (6th ed.), Van Nostrand Reinhold 1983.

[47] Marvin Minsky: Computation: finite and infinite machines. Prentice-Hall 1967. (Enlarged edition 1988.)

[48] M.J.Moylan: Fluid logic in simple terms, (2nd ed.) Machinery publishing co. 1971.

[49] C.D.Murray: Proc. Nat'l Acad. Sci. USA 12 (1926) 207-214.

[50] L.Niemeyer, L.Pietronero, H.J.Wiesmann: Fractal dimension of dielectric breakdown, Phys. Rev. Lett. 52,12 (1984) 1033-1036.

[51] J.Nittmann, G.Daccord, H.E.Stanley: Fractal growth of viscous fingers, Nature 314 (1985) 141-144.

[52] Piergiorgio Odifreddi: Classical recursion theory, North-Holland (vol 1) 1989, (vol 2) 1999, vol 3 not yet published.

[53] S. Putterman & K. Weniger: Sonoluminescence: How Bubbles Turn Sound into Light. Annual Reviews Fluid Mechanics 32 (2000) 445-

[54] Osborne Reynolds: An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels, Philos. Trans. Royal Soc. London 174 (1883) 935-982.

[55] Gerald E. Sacks: Higher Recursion Theory, Springer-Verlag 1990.

[56] H.Salwen, F.W.Cotton, C.E.Grosch: Linear stability of Poiseuille flow in a circular pipe, J. Fluid Mech. 98,2 (1980) 273-284.

[57] B.B. Sharp, D.B. Sharp: Water Hammer: Practical Solutions, J.Wiley & Sons 1996.

[58] James Sharpsteen: contribution to "The Amateur Scientist" column, 128-135 in Scientific American 214,5 (May 1966).

[59] Warren D. Smith: Church's thesis meets the N-body problem, Tech. report, NEC Research Institute, Princeton NJ 1999. `http://www.neci.nj.nec.com/homepages/wds/church.ps`.

[60] H.C.Swinney & J.P.Gollub (eds.): Hydrodynamic instabilities and the transition to turbulence (2nd ed.) Springer (Topics in Applied Physics #45) 1985. See esp. Chapter 1 on strange attractors and 6 on Couette flow.

[61] G.I.Taylor: Stability of a viscous liquid contained between two rotating cylinders, Philos. Trans. Royal Soc. London A 223 (1923) 289-343.

[62] Nikola Tesla: Valvular Conduit, US Patent #1,329,559 (issued 3 Feb 1920, filed 1916). `http://www.tfcbooks.com/patents/valvular.htm`.

[63] Andrei L. Toom: Stable and attractive trajectories in multicomponent systems, 549-575 in A.L.Dobrushin (ed.): Multicomponent systems, vol. 6 of Advances in Probability, M.Dekker, New York 1980.

[64] D.J.Tritton: Physical fluid dynamics, Van Nostrand Reinhold 1977. (2nd ed. Oxford Univ. Press 1988.)

[65] Anatoli Tumin: Receptivity of Pipe Poiseuille Flow, J.Fluid Mech. 315 (1996) 119-137.

[66] H.Van Damme, F.Obrecht, & 3 others: Fractal viscous fingering in clay slurries, Nature 320 (1986) 731-733.

[67] S.T. Weir, A.C. Mitchell, W.J. Nellis: Metallization of fluid molecular hydrogen at 140 GPa, Phys. Rev. Lett. 76 (1996) 1860-1863.

[68] K.R.Weniger, C.G. Camara, S.J.Putterman: Energy focusing in converging fluid flow: implications for sonoluminescence, Physical review Letters 83,10 (1999) 2081-2084.

[69] G.B. West, J.H. Brown, B.J. Enquist: A general model for the origin of allometric scaling laws in biology, Science 276 (1997) 122-126.

[70] T.A. Witten Jr. & L.M.Sander: Diffusion limited aggregation, a kinetic critical phenomenon, Phys. Rev. Lett. 47 (1981) 1400-1403.

[71] Chia-Shun Yih: Fluid mechanics, a concise introduction to the theory, West River Press, Ann Arbor Michigan, 1979.

[72] Josef Zaruba: Water Hammer in Pipe-Line Systems, Elsevier Science 1993.

[73] Y.Zhu & S.Granick: Rate-dependent slip of Newtonian liquid at smooth surfaces, Physical Review Letters 87 (27 Aug 2001) 096105 (4 pages).