EQUIVALENCE BETWEEN DIRECT MASS AND NFW-TOTAL MASS FORMULA IN MW AND M31 GALAXIES

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Abstract

In the framework of Dark Matter by Quantum Gravitation (DMbQG) theory, the Direct mass gives the total mass into the halo region, growing with the square root of radius. This formula has only one parameter specific for each galaxy.

The NFW profile is a universal method that gives the DM mass function depending on radius at the disk and galactic halos. This function is defined by two parameters.

The DMbQG claims that DM is generated by the own gravitational field and consequently the DM halo is unbounded. In the paper [1] Abarca,M.2024 was proved that it is the Dark energy the mechanism able to counterbalance the DM but at cluster scale.

As the direct mass is linked to the total mass (baryonic plus DM), in the paper has been developed a method to integrate the baryonic matter into the NFW DM function in order to be able to compare both functions connected to the total mass.

Thanks this method has been possible to define the $R_{200\text{-TOTAL}}$ and the $M_{200\text{-TOTAL}}$ both referred to a sphere whose total mass has a mean density equal to 200 times the critic density of the Universe.

The main achievement of this paper is to demonstrate that the Direct mass function and the NFW-total mass function are equivalents into the halo region of MW and the M31.

The equivalence of the two formulas for masses is based on four tests, all of them tested successfully.

Test I. Comparison the $R_{200\text{-TOTAL}}$ and the $M_{200\text{-TOTAL}}$ calculated by the two different formulas, for MW and M31. As the Direct mass as the NFW mass formula gives similar values to $R_{200\text{-TOTAL}}$ and the $M_{200\text{-TOTAL}}$, compatibles with the equality if it is considered the error measures.

Test II. Using the $M_{200\text{-TOTAL}}$, value that is given by the NFW method, is calculated the parameter a^2 which is compared with the one got in the framework of DMbQG theory. This process is made in MW and M31 and the comparison is compatible with the equality if it is considered the error measures.

Test III. Is similar to test III but using the $R_{200\text{-TOTAL}}$ and the $M_{200\text{-TOTAL}}$ and the result is successful as well.

Test IV. The direct mass function and the NFW-total mass function are compared into the halo region up to $R_{200\text{-TOTAL}}$. It is proved that its relative differences are below 5% into a wide region of the galactic halo up to $R_{200\text{-TOTAL}}$ and beyond.

Although the thesis of this paper is for any galaxies, the calculus has been made with MW and M31, because they are the best well studied galaxies and his data have the maximum of accuracy.

The prove reach in this paper is valuable because the NFW is a trustable profile for DM, tested in thousand of galaxies, and although the DMbQG theory claims that DM has an unbounded halo region, it gives similar results in the halo region common for both theories i.e. up to the virial radius.

1. Introduction

Since 2014 up to 2024, I have published several papers studying DM in galactic halos, especially in M31 and Milky Way although also I have published some papers studying other galaxies and clusters.

This paper is focused on the equivalence of NFW mass formula and the Direct mass formula, which is the formula for total mass in the halo region developed in the framework of DMbQG, so the reader has to be at least a general knowledge about this original theory. The paper [1] Abarca,M.2024, is the best work about it, so the reader may consult such paper in order to understand the DMbQG theory.

The chapter 3 is dedicated to introduced the Direct mass and some derived formulas, and the chapter 4 is dedicated to introduced the NFW method and his extension to the total mass (baryonic plus DM). As the reader knows, the NFW is a density profile for DM only. It has been necessary to integrate the baryonic matter into the NFW profile because the Direct mass function gives the total mass into the halo region.

As reader knows, M31 is the twin galaxy of Milky Way in the Local Group of galaxies. According [2] Sofue, Y. 2015 its baryonic masses are $M_{M31} = 1.6\cdot10^{11}\ M_{\Theta}$ and $M_{MILKY\ WAY} = 1.4\cdot10^{11}\ M_{\Theta}$

The DM by Quantum Gravitation, DMbQG hereafter, theory was introduced in [3] Abarca, M.(2014). *Dark matter model by quantum vacuum.* It considers that DM is generated by the own gravitational field according an unknown quantum gravitational phenomenon.

In order to study purely the phenomenon it is needed to consider a radius dominion where it is supposed that baryonic matter is negligible. i.e. radius bigger than 30 kpc for MW and 40 kpc for M31, according some calculus made about it.

This hypothesis has two main consequences: the first one is that the law of dark matter generation, in the halo region, has to be the same for all the galaxies. In the paper [1] Abarca,M.2024 is developed the theory using the rotation curve of M31 published by [2] Sofue,Y.2015 and the rotation curve of MW by [4] Sofue,Y.2020.

The second consequence is that the haloes are unlimited so the total dark matter goes up without limit. In the paper [1] Abarca,M.2024 is solved the divergence of the total mass, thanks to the Dark energy.

As I have mentioned before, this theory has been developed assuming the hypothesis that DM is a quantum gravitational effect. However, it is possible to remain into the Newtonian framework to develop the theory. In my opinion there are two factors to manage the DM conundrum with a quite simple theory.

The first one, that it is developed into the halo region, where baryonic matter is negligible. The second one, that the mechanics movements of celestial bodies are very slow regarding velocity of light, which is supposed to be the speed of gravitational bosons.

It is known that community of physics is researching a quantum gravitation theory since many years ago, but it does not exist yet, however I think that my works in this area support strongly that DM is a quantum gravitation phenomenon.

Use a more simple theory instead the general theory is a typical procedure in physics. For example the Kirchhoff ´s laws are the consequence of Maxwell theory for direct current and remain valid for alternating current, introducing complex impedances, on condition that signals must have low frequencies.

So these reasons support the possibility to study a complex phenomenon as it is the DM with a theory mathematically simple in the framework of Newtonian mechanics.

In the paper [1] Abarca, M.2024 in the framework of DMbQG theory it is calculated by the Direct mass with unbounded dominion for radius the dynamical mass of the Local Group, that according [5] Azadeh Fattahi, Julio F. Navarro.2020 is estimated to be $5 \cdot 10^{12}\ M_{\Theta}$. The result given by the direct mass considering the four main galaxies of the L.G. match perfectly with such estimation, whereas using the virial masses associated to MW,M31, M33 and LMC calculated by NFW the total amount of masses is scarcely $3 \cdot 10^{12}\ M_{\Theta}$. This calculus have been made without considering the dark energy because according some calculus made, into the L.G. the D.E. is important for radius bigger than 1 Mpc, so for the system MW and M31 the DE may be neglected.

The DMbQG theory has been developed successfully in cluster of galaxies in the paper [6] Abarca, M.2024, and there have been found a set of remarkable theoretical results tested in the L.G. and the Virgo cluster, that is the nearest big cluster and consequently the cluster where measures reach the maximum of accuracy. Namely some theoretical finding match perfectly with the results published by [7] Kashibadze, Karachentsev,(2020) and by [8]Karachentsev, I.D., Tully, R.B (2014).

Despite the fact that the DMbQG theory have been tested successfully in galaxies and clusters, the prove reach in this paper is valuable because the NFW is a trustable profile for DM, tested in thousand of galaxies, and although the DMbQG theory claims that DM has an unbounded halo region, it gives similar results in the halo region common for both theories i.e. up to the virial radius.

2.Virial mass and virial radius in galaxies and clusters

In galaxies and clusters, it is a good estimation about virial radius and virial mass to consider $R_{vir} = R_{200}$ and $M_{vir} = M_{200}$. Where R_{200} is the radius of a sphere whose mean density is 200 times bigger than the critic density of Universe

$$\rho_C = \frac{3H^2}{8\pi G} = 9.2055 10^{-27}\, kgm^{-3} \quad (2.1)$$

and M_{200} is the total mass enclosed by the radius R_{200}.

Considering the spherical volume formula, it is right to get the following relation between both concepts.

$$\boldsymbol{R_{200}^3} = \frac{G \cdot M_{200}}{100 \cdot H^2} \quad (2.2) \text{ or } \boldsymbol{M_{200}} = \frac{100 H^2 R_{200}^3}{G} \quad (2.3) \text{ or } \frac{M_{200}}{R_{200}^3} = \frac{100 H^2}{G} \ (2.4)$$

These parameters are common in cluster of galaxies as well. In the chapter 2 of paper [6] Abarca , M. (2024) is checked the above relation between R_{200} and M_{200} on a set of clusters.

In the chapter 4 will be introduced the NFW density profile and the NFW mass function, both linked to DM, so in the framework of NFW the R_{200} and the M_{200} are connected with DM exclusively.

As the Direct mass is linked to the total mass, in the chapter 4 will be developed a procedure to integrate the baryonic mass in the NFW function mass, and this is the reason why R_{200} and the M_{200} are written with the subscript $R_{200\text{-TOTAL}}$ and the $M_{200\text{-TOTAL}}$ when they are linked to the total mass.

3.Virial theorem as a method to get the direct mass formula in galaxies or galaxy clusters

In chapter 9, of paper [1] Abarca,M.2024 was demonstrated that the direct formula $M_{TOTAL}(<r) = \frac{a^2 \cdot \sqrt{r}}{G}$ (3.1) is the most suitable formula to calculate the total mass (baryonic and DM) enclosed by a sphere with a specific radius that ranges into the the galactic halo.

The halo is the region where the density of baryonic matter is negligible versus the d.m. density. e.g. the halo for Milky Way may be a radius bigger than 30 kpc, or the halo for M31 may be a radius bigger than 40 kpc.

3.1 Parameter a² formula depending on virial radius and virial mass.

Due to the fact that the Direct mass formula has one parameter only, is enough to know the mass associated to a specific radius to be able to calculate parameter a^2.

According DMbQG theory is possible to do an equation between $M_{200} (< R_{200}) = M_{DIRECT}(< R_{200})$ i.e.

$$M_{200} \equiv M_{TOTAL}(< R_{200}) = \frac{a^2 \cdot \sqrt{R_{200}}}{G}$$

and clearing up $a^2 = \frac{G \cdot M_{200}}{\sqrt{R_{200}}}$ (3.2)

this formula is called parameter a^2(M_{200}, R_{200}) because depend on both measures.

3.2 Parameter a² formula depending on virial mass only

In chapter 2 was got this formula $R_{200}^3 = \frac{G \cdot M_{200}}{100 \cdot H^2}$ (2.2) as a good approximation between the virial mass and the virial radius. So using that formula and by substitution of the virial radius in $a^2 = \frac{G \cdot M_{VIRIAL}}{\sqrt{R_{VIRIAL}}}$ it is right to get the parameter a^2 depending on M_{200} only $a^2 = (G \cdot M_{200})^{5/6} \cdot (10 \cdot H)^{1/3}$ (3.3)

This formula will be called parameter a^2 (M_{200}) as depend on M_{200} only.
Conversely it is possible to clear up the virial mass from the previous formulas.

$$M_{200-TOTAL} = \frac{a^{12/5}}{G \cdot (10 \cdot H)^{2/5}} \quad (3.4)$$

or using the formula (2.3) and clearing up the virial radius then

$$R_{200-TOTAL} = \left[\frac{a^2}{100 \cdot H^2}\right]^{2/5} \quad (3.5)$$

It is important to insist that the parameter a^2 is linked to the total mass and this is the reason why the radius and mass are written with the subscript 200-TOTAL

In [1] Abarca.2024 using the rotation curve of M31 published by [2] Sofue (2015) was got the parameter a= $4.727513\cdot10^{10}$ $m^{5/4}/s$ or $a^2 = 2.235\cdot10^{21}$ $m^{5/2}/s^2$ and using the rotation curve of [4] Sofue (2020) was got the same parameter for MW: $a^2 = 1.521\cdot10^{21}$ $m^{5/2}/s^2$, so using the previous formulas are got the following values

Table 1 Parameter a^2 I.S.	$M_{200\text{-TOTAL}}$ $M_\odot$	$R_{200\text{-TOTAL}}$ kpc
M31 $2.235\cdot10^{21}$	$1.42\cdot10^{12}$	232.15
MW $1.521\cdot10^{21}$	$8.98\cdot10^{11}$	199

4. The NFW profile for d.m. mass density

The NFW profile for DM density in galaxies is $\rho(r) = \frac{\rho_0}{x\cdot(1+x)^2}$ (4.1)

being ρ_0 a characteristic density, x= r/R_0 a dimensionless magnitude related with radius by R_0, which is called scale radius.

By integration it is right to get the Dark matter enclosed by a sphere with radius r.

$M_{DM}(< r) = K_{NFW} \cdot f(x)$ (4.2)

being $K_{NFW} = 4\pi\rho_0 R_0^3$ (4.3) , with mass dimension and

$f(x) = Ln(1 + x) - x/(1 + x)$ (4.4)

where x= r / R_0 , being Ln the natural logarithm.

Two important parameters for NFW profiles are M_{200} and R_{200} both referred to DM only i.e. the DM enclosed into a sphere with R_{200} as radius whose mean density is 200 times the critic density $\rho_C = \frac{3H^2}{8\pi G}$ =$9.2055\cdot10^{-27}$ kg·m^{-3}

So $M_{200\text{-DM}}$ =M_{DM} (<R_{200}) = $K_{NFW} \cdot f(c)$ (4.5)

where c = R_{200} / R_0 (4.6)

is called the concentration parameter and $R_{200} = R_0\cdot c$

4.1 Calculus of concentration parameter

As $\frac{M_{200}}{R_{200}^3} = \frac{100H^2}{G}$ then $\frac{M_{200}}{c^3R_0^3\cdot} = \frac{100H^2}{G}$ and $\frac{4\cdot\pi\rho_0\cdot R_0^3\cdot f(c)}{c^3R_0^3\cdot} = \frac{100H^2}{G}$ so

$\frac{c^3}{f(c)} = \frac{4\pi G\rho_0}{100\cdot H^2}$ (4.7)

This equation is quite easy to solve numerically, and it is clear that c depend on the characteristic density only.

With this parameter c, it is rightly calculated $M_{200\text{-DM}}$ and R_{200}.

See the example below.

Table 2 [4] Sofue (2020) parameters for the NFW Milky Way profile	
Characteristic density	Scale radius kpc
$\rho_0 = 0,787 \pm 0.037 \mathrm{GeVcm}^{-3} = 1.403 \cdot 10^{-21}\ \mathrm{kg} \cdot \mathrm{m}^{-3}$	$R_0 = 10.94 \pm 1.05$

Using the characteristic density it is right to get the equation

$c^3 / f(c) = 2286.125$ that gives the value $c = 16.348$, and $f(c) = 1.91$

So $R_{200} = R_0 \cdot c = 178.85$ Kpc

Using (4.3) $K_{NFW} = 3.4 \cdot 10^{11} M_\Theta$ then using (4.5) $M_{200\text{-}DM} = 6.498 \cdot 10^{11} M_\Theta$

In the table 3 is checked the density of the sphere with the radius R_{200}

Table 3 Mean density $M_{200\text{-}DM}$ versus R_{200}	$200\rho_C$
$1.837 \cdot 10^{-24} kgm^{-3}$	$1.841 \cdot 10^{-24} kgm^{-3}$

Both values match quite well, the ratio mean density versus $200\rho_C$ is 0.997827

4.2 Determining the NFW profile by R_{200} and the concentration parameter c

Conversely, some authors give the NFW profile using three parameters $M_{200\text{-}DM}$, R_{200} and c.

Using (4.7) and knowing the parameter c is possible to clear up the characteristic density

$$\rho_0 = \frac{100 \cdot H^2 c^3}{f(c) \cdot 4 \cdot \pi \cdot G} \quad (4.8)$$

in addition $R_0 = R_{200} / c$. This way, knowing ρ_0 **and** R_0, it is defined the NFW profile.

Although $M_{200\text{-}DM}$ is derived from the previous ones, as it is very important all the authors publish its value. Namely its value may be calculated by (2.3) or by (4.5)

For example, in the table 4 the author gives the NFW density profile this way:

Table 4 [9] E.karukes,(2020) - Milky Way data			
$M_{200\text{-}DM}$ M_Θ	$M_{200\text{-}TOTAL}$ M_Θ	R_{200} kpc	Concentration factor c
$8.3^{+1.2}_{-0.8} \cdot 10^{11}$	$8.9^{+1}_{-0.8} \cdot 10^{11}$	193^{+9}_{-6}	c=19

The value $M_{200\text{-}TOTAL}$ represents the total mass enclosed by the sphere R_{200} so by subtraction of $M_{200\text{-}TOTAL}$ minus $M_{200\text{-}DM}$ may be calculated the baryonic mass of MW i.e. $M_{BA\text{-}MW} = 6 \cdot 10^{10}\ M_{\Theta}$ according this author.

Obviously using the $M_{200\text{-}TOTAL}$ into the sphere R_{200} does not verify that the mean density is $200\rho_C$ as it si shown in table 5.

Table 5 $Density_{MEAN}$ $M_{200\text{-}DM}$ into R_{200}	$Den._{MEAN}$ $M_{200\text{-}TOTAL}$ into R_{200}	$200\rho_C$
$1.867 \cdot 10^{-24}\ kgm^{-3}$	$2 \cdot 10^{-24}\ kgm^{-3}$	$1.841 \cdot 10^{-24}\ kgm^{-3}$
Match well with $200\rho_C$	Does not match with $200\rho_C$	

As $R_0 = R_{200}/c$ then Ro = 10.1578 kpc

As c= 19 then f(c) =2.04573

As $M_{200\text{-}DM} = M_{DM}(<R_{200}) = \mathbf{K_{NFW} \cdot f(c)}$ then $K_{NFW} = 4.057 \cdot 10^{11}\ M_{\Theta}$

As $\mathbf{K_{NFW} = 4\pi\rho_0 R_0^3}$ then $\boldsymbol{\rho_0 = 2.086 \cdot 10^{-21} kg \cdot m^{-3}}$

According [9] E.Karukes (2020), the above data about masses means that the baryonic mass enclosed by R_{200} is 6 $\cdot 10^{10} M_{\Theta}$. However for Sofue 2015 the baryonic mass for the MW is 1.3 $\cdot 10^{11} M_{\Theta}$ and others authors give different values. It is known that the measures for the baryonic mass of MW has a high imprecision. Similarly the relative differences about the virial masses and radius are not negligible, although both authors give results compatibles if it is considered the range of errors.

Table I DM virial data Sofue Vs Karukes	$M_{200\text{-}DM}$ M_{Θ}	$R_{200\text{-}DM}$ kpc
Using Sofue 2020 data	$6.498 \cdot 10^{11}$	178.85
[9] Karukes 2020 data	$\mathbf{8.3^{+1.2}_{-0.8} \cdot 10^{11}}$	$\mathbf{193^{+9}_{-6}}$
Relative difference %	**21%**	7.8 %

4.3 Calculus of concentration parameter t for the total mass

As in this paper it will be compared the R_{200} for the total masses, in this epigraph it will be developed a method to calculate the $R_{200\text{-TOTAL}}$ in the framework of the NFW profile density for DM. i.e. the radius of the sphere where the total mass has a mean density of $200\rho_C$

The total mass is the addition of baryonic plus the DM, as the baryonic mass is mainly concentrated into the bulge and disk of a galaxy, this amount of mass is a constant quantity into the halo dominion i.e.

$M_{TOTAL}(<r) = M_{BA} + M_{DM}(<r) = M_{BA} + \mathbf{K_{NFW}} \cdot \mathbf{f(x)}$

If it is defined $f_{BA} = M_{BA} / K_{NFW}$ (4.9) then

$M_{TOTAL}(<r) = [\mathbf{f_{BA}} + \mathbf{f(x)}]\mathbf{K_{NFW}}$ (4.10)

As the sphere whose mean density is $200\rho_C$ verify $\frac{\mathbf{M_{200-TOTAL}}}{\mathbf{R^3_{200-TOTAL}}} = \frac{\mathbf{100H^2}}{\mathbf{G}}$ (2.4)

and defining parameter $t = R_{200\text{-TOTAL}} / R_0$ (4.11)

It is got $\frac{[\mathbf{f_{BA}}+\mathbf{f(t)}]\cdot 4\pi\cdot\rho_0\cdot R_0^3}{R_0^3\cdot t^3} = \frac{\mathbf{100H^2}}{\mathbf{G}}$ that leads to the expression

$\frac{t^3}{\mathbf{f_{BA}}+\mathbf{f(t)}} = \frac{4\pi G\cdot\rho_0}{100\cdot H^2}$ that by (4.7) leads to $\frac{t^3}{\mathbf{f_{BA}}+\mathbf{f(t)}} = \frac{c^3}{f(c)}$ (4.12)

This equation allows calculating the concentration parameter for the total mass.

The parameter t depends on the parameter c and the fraction f_{BA}.

4.3.1 $R_{200\text{-TOTAL}}$ and $M_{200\text{-TOTAL}}$ calculus by parameter t using NFW Sofue data

In this epigraph will be used the [4] Sofue,(2020) data to calculate the parameter t

Table 6 [4] Sofue (2020) Ro = $\mathbf{10.94 \pm 1.05}$ kpc K_{NFW}= $3.4\cdot10^{11}$ $M_\odot$	
See epigraph 4.1 for calculus of: c, R_{200}, $M_{200\text{-DM}}$, K_{NFW}	
c= 16.348 R_{200} = 178.85 kpc	$M_{200\text{-DM}} = 6.498\cdot10^{11}$ $M_\odot$
[2] Sofue (2015) M_{BA}= $1.3\cdot10^{11}$ $M_\odot$	$f_{BA} = M_{BA} / K_{NFW}$ = 0.382

So $\frac{t^3}{f_{BA}+f(t)} = \frac{c^3}{f(c)}$ leads to $\frac{t^3}{0.382+f(t)} = \frac{4369.12}{1.911} = 2286.156$ whose solution is t = 17.527

So $R_{200\text{-TOTAL}} = t\cdot R_0$ = 191.745 kpc

and as f(t) =1.9732 the $M_{200\text{-TOTAL}} = [\mathbf{0.382} + \mathbf{f(t)}]\mathbf{K_{NFW}} = 8.0077\cdot10^{11}$ $M_\odot$

4.3.2 $R_{200\text{-TOTAL}}$ and $M_{200\text{-TOTAL}}$ by the parameter t using NFW Karukes data

In this epigraph will be used the [9] Karukes,(2020) data to calculate the parameter t

Table 7 [9] Karukes (2020)		
$\mathbf{M_{200-DM}}$	$\mathbf{R_{200}}$	Concentration factor c
$\mathbf{8.3^{+1.2}_{-0.8}\cdot 10^{11}}\ M_{\odot}$	$\mathbf{193^{+9}_{-6}}$ kpc and R_0 = 193/19 =10.158	c=19 f(c) =2.0457
K_{NFW} =4.057·10^{11} $M_{\odot}$	M_{BA}= 6·10^{10} $M_{\odot}$	$f_{BA} = M_{BA}/K_{NFW}$ =0.1479

So $\frac{t^3}{f_{BA}+f(t)} = \frac{c^3}{f(c)}$ leads to $\frac{t^3}{0.1479+f(t)}$=3352.886 whose solution is t = 19.5191

So $R_{200\text{-TOTAL}}$ =t· R_0 = 198.273 kpc

and as f(t) =2.07 the $M_{200\text{-TOTAL}} = [\mathbf{0.1479+f(t)}]\mathbf{K_{NFW}}$ = 8.998·10^{11} $M_{\odot}$

In the table II are summarized the total mass and the total radius data, although the relative differences are not negligible both match if it is considered the range of error measures.

Table II Total Mass Virial data Sofue Vs Karukes	$M_{200\text{-TOTAL}}$ $M_{\odot}$	$R_{200\text{-TOTAL}}$ kpc
Using Sofue data	8·10^{11}	191.7
Using Karukes data	9·10^{11}	198.3
Realtive differences %	11 %	3.3 %

5. Testing the equivalence between direct mass and NFW mass in MW halo

As the direct mass is referred to the total mass into the halo region, it is needed to extend the NFW for DM formula to the total mass, in order to be able to compare both formulas.

In this chapter will be introduced a set of tests to check the equivalence between the two formulas into the halo region up to the $R_{200\text{-TOTAL}}$ radius.

Although the set of tests developed in this chapter is general, are used the MW data because our galaxy is the best well known with the most accuracy data.

5.1 Comparison between $R_{200\text{-TOTAL}}$ and $M_{200\text{-TOTAL}}$ values got by direct mass and NFW total mass. Test I

In [1] Abarca, M.2024 was got the direct mass (3.1) formula, that in the framework of DMbQG is the formula for the total mass.

As a derived formula was got $M_{200-TOTAL} = \frac{a^{12/5}}{G\cdot(10\cdot H)^{2/5}}$ (3.4) and $R_{200-TOTAL} = \left[\frac{a^2}{100\cdot H^2}\right]^{2/5}$ (3.5) so using the parameter a^2 for MW, see table 1, are got rightly both values.

As in the epigraph 4.3 has been got the $R_{200\text{-TOTAL}}$ and the $M_{200\text{-TOTAL}}$ in the framework of NFW, now it is possible to compare both parameters got with the two different methods.

In the table 8 is shown the three different values for $R_{200\text{-TOTAL}}$ and $M_{200\text{-TOTAL}}$, they match perfectly if it is considered the range of errors, although the values got by the direct mass (parameter a^2) give a mean density which is the nearest to $200\rho_C$

Table 8 MW	$R_{200\text{-TOTAL}}$	$M_{200\text{-TOTAL}}$	Mean density Versus $200\rho_C$
By parameter a^2	199 kpc	$8.98\cdot10^{11}\ M_\Theta$	1.0008
Sofue NFW-total	191.745 kpc	$8.0077\cdot10^{11}\ M_\Theta$	0.9976
Karukes NFW-total	198.273 kpc	$8.998\cdot10^{11}\ M_\Theta$	1.0139

5.2 Calculus of parameter a^2 using $M_{200\text{-TOTAL}}$ got by NFW . Test II

As in the epigraph 4 was calculated $M_{200\text{-TOTAL}}$ using the NFW method, then it is possible to use such result to calculate the parameter a^2 by the formula:

$$a^2 = (G\cdot M_{200})^{5/6}\cdot(10\cdot H)^{1/3} \text{ (3.3)}$$

Then this result may be compared with the parameter a^2 got in [1] Abarca, M.2024 in the framework of DMbQG theory that for MW is $1.527\cdot10^{21}$ $m^{5/2}s^{-2}$ that is the reference value.

Table 9	$M_{200\text{-TOTAL}}\ M_\Theta$	Parameter a^2 $m^{5/2}s^{-2}$	Relative diff. %
Sofue NFW	$8.0077\cdot10^{11}$	$1.383\cdot10^{21}$	9.4
Karukes NFW	$8.998\cdot10^{11}$	$1.524\cdot 10^{21}$	0.2
a^2 [1] Abarca		$1.527\cdot10^{21}$	

The comparison between the reference a^2 and the one got by Sofue NFW data is good (9.4 %), but the comparison with the one got by Karukes data is excellent (0.2 %).

5.3 Calculus of parameter a^2 using $M_{200\text{-TOTAL}}$ and $R_{200\text{-TOTAL}}$ got by NFW. Test III

In this test the parameter a^2 is got by the formula $a^2 = \frac{G \cdot M_{200}}{\sqrt{R_{200}}}$ (3.2),

using $M_{200\text{-TOTAL}}$ and $R_{200\text{-TOTAL}}$ got by NFW method.

Table10 MW	$R_{200\text{-TOTAL}}$ kpc	$M_{200\text{-TOTAL}}$ M_Θ	Formula (3.2) a^2 $\mathbf{m^{5/2}s^{-2}}$	Relative diff. %
Sofue NFW	191.745	$8.0077 \cdot 10^{11}$	$1.3824 \cdot 10^{21}$	9.4
Karukes NFW	198.273	$8.998 \cdot 10^{11}$	$1.5276 \cdot 10^{21}$	0.04
a^2 as reference			$1.527 \cdot 10^{21}$	

The results of comparison are good but the one with Karukes NFW is excellent.

5.4 Comparison of Direct mass formula with the NFW-total mass formula into the halo region up to $R_{200\text{-TOTAL}}$. Test IV

The NFW mass formula extended to the total mass was developed in chapter 4,

M_{TOTAL} (<r) = $[\mathbf{f_{BA}} + \mathbf{f}(\mathbf{x})]\mathbf{K_{NFW}}$ (4.10), being K_{NFW} the constant defined by (4.3), and f_{BA} as the fraction of baryonic matter, $f_{BA} = M_{BA} / K_{NFW}$ (4.9).

In addition f(x) = Ln(1+x)-x/(1+x) (4.4) where x = r/R_0 is the dimensionless variable associated to the variable radius.

The direct mass formula for the total mass in the framework of DMbQG theory is

$$M_{TOTAL}(<r) = \frac{a^2 \cdot \sqrt{r}}{G} \quad (3.1)$$

In order to compare both formulas it is needed to use the same dimensionless variable x for the direct mass so:

$M_{TOTAL}(<r) = \frac{a^2 \cdot \sqrt{r}}{G} = \frac{a^2 \cdot \sqrt{R_0}}{G} \cdot \sqrt{x}$ where x = r/R_0 being R_0 the scale radius.

Defining $K_T = \frac{a^2 \cdot \sqrt{R_0}}{G}$ (5.1)

then the direct mass is: $M_{TOTAL}(<r) = K_T \cdot \sqrt{x}$ (5.2)

For example, for the MW galaxy using R_0 =10.94 kpc, see table 2, and parameter a^2 = $1.527 \cdot 10^{21}$ $m^{5/2}s^{-2}$ then $K_T = 2.11 \cdot 10^{11}$ M_Θ

In order to compare the direct mass (5.2) with the NFW total mass (4.10) it is defined

$f_T = K_T / K_{NFW}$ and so $M_{TOTAL}^{DIRECT}(<r) = f_T \cdot K_{NFW} \cdot \sqrt{x}$ (5.3)

this way this function may be compared with the $M_{TOTAL}^{NFW}(<r) = [f_{BA} + f(x)]K_{NFW}$ (4.10) If it is cancelled the common factor K_{NFW} both functions become as dimensionless functions:

$FM_{TOTAL}^{DIRECT}(<r) = f_T \cdot \sqrt{x}$ (5.4) being $x = r/R_0$ and

$FM_{TOTAL}^{NFW}(<r) = [f_{BA} + f(x)]$ (5.5)

5.4.1 Comparison of Direct mass formula with the NFW-total mass formula using Sofue data

For example using the MW Sofue data, see table 6, may be defined (5.4) and (5.5)

Table 11	f_{BA}	f_T	$R_{200\text{-}TOTAL}$	R_0 kpc
MW	0.382	0.62	192 kpc	10.94

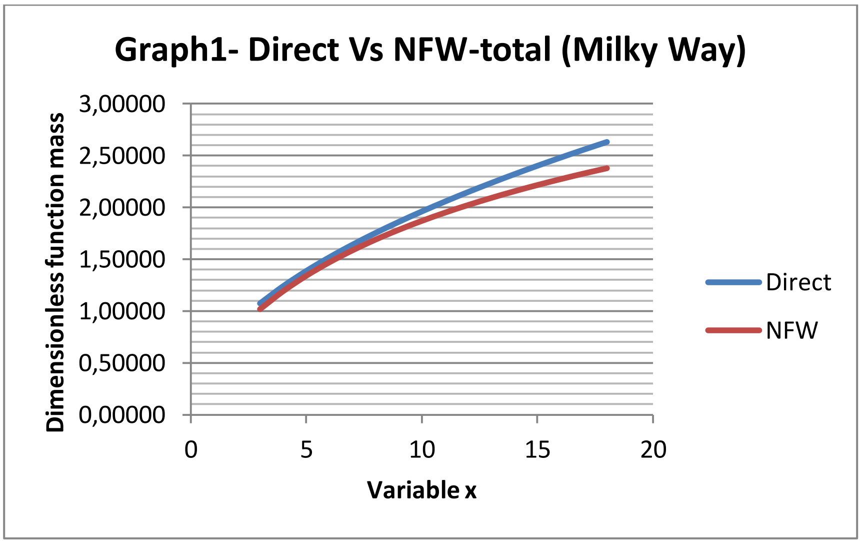
As the dominion for radius is the halo region from 30 kpc up to 200 kpc, the dominion for the variable x is $x \in [3,18]$

In the table 12 are tabulated both dimensionless function and it is shown its relative difference.

Table 12 Radius kpc	Variable X	Dimension less Direct mass	Dimension less NFW-total	Relative Diff. %
32,82	3	1,07387	1,01829	5,18
43,76	4	1,24000	1,19144	3,92
54,7	5	1,38636	1,34043	3,31
65,64	6	1,51868	1,47077	3,16
76,58	7	1,64037	1,58644	3,29
87,52	8	1,75362	1,69034	3,61
98,46	9	1,86000	1,78459	4,05
109,4	10	1,96061	1,87080	4,58
120,34	11	2,05631	1,95024	5,16
131,28	12	2,14774	2,02387	5,77
142,22	13	2,23544	2,09249	6,39
153,16	14	2,31983	2,15672	7,03
164,1	15	2,40125	2,21709	7,67
175,04	16	2,48000	2,27404	8,30
185,98	17	2,55633	2,32793	8,93
196,92	18	2,63044	2,37907	9,56

Although the relative difference increases continuously, remains below 10% in the whole dominion.

This relative difference is acceptable because the f_{BA} is a value with a very high imprecision. In the following epigraph will be used f_{BA} 0.1479 given by [9] E. Karukes.



5.4.2 Comparison of Direct mass formula with the NFW-total mass formula using Karukes data

In order to compare both formulas are needed the new parameters provided by this author, see table 7, R_0=10.158 kpc , f_{BA} =0.1479 and K_{NFW}=$4.057 \cdot 10^{11}$ M_Θ

In addition it is needed the factor $f_T = K_T / K_{NFW}$ where $K_T = \frac{a^2 \cdot \sqrt{R_0}}{G}$

As the parameter a^2 is got by the rotation curve into the halo region and this author did not publish such curve, it is need to use the parameter calculated with [4]Sofue data, see table 1 parameter a^2=$1.521 \cdot 10^{21}$ so K_T = $2.028 \cdot 10^{11}$ M_Θ and f_T = 0.4998

Table 13	f_{BA}	f_T	R_0	$R_{200\text{-}TOTAL}$
MW	0.1479	0.4998	10.158 kpc	198 kpc

As the dominion for radius is the halo region from 30 kpc up to 198 kpc, the dominion for the variable x is $x \in [3,20]$

Comparing the tables 11 and 13 it is clear that parameters f_T are lightly different but parameters f_{BA} are very different because according Karukes the baryonic mass of MW is lower than a half the value considered by Sofue.

$FM_{TOTAL}^{DIRECT}(<r) = f_T \cdot \sqrt{x}$ (5.4) being x = r/R_0 and

$FM_{TOTAL}^{NFW}(<r) = [f_{BA} + f(x)]$ (5.5)

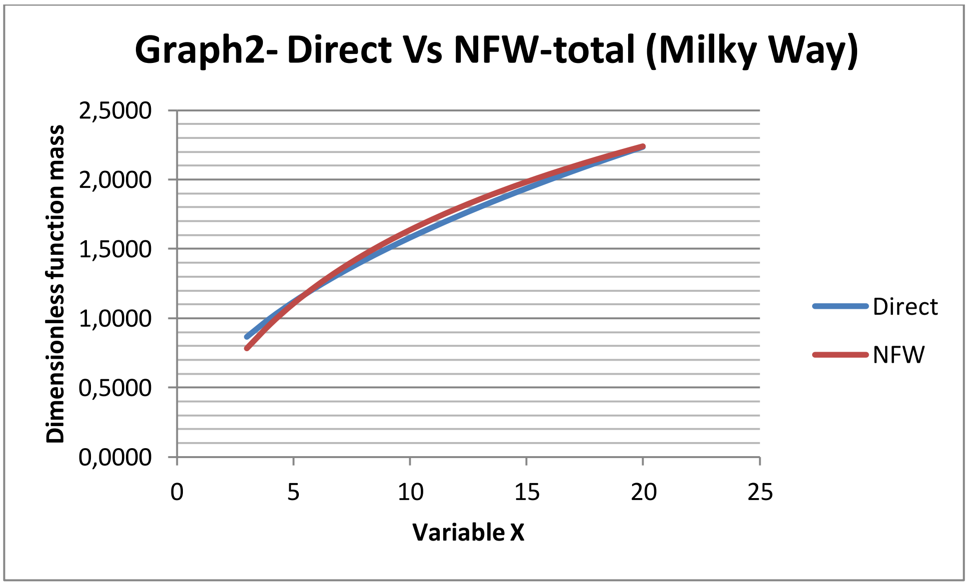
In the table 14 are tabulated both dimensionless mass functions and its relative difference. Excepting the value for radius 30.5 kpc the other ones values have a relative difference below 5%, however with the Sofue data a half of data have its relative difference under 5% and the other half data range between 5% and 10%.

The cause about this discrepancy is that the baryonic mater in MW has a high level of imprecision. E.g. for Karukes $M_{BA} = 6 \cdot 10^{10}\ M_{\Theta}$ and for Sofue $M_{BA} = 1.3 \cdot 10^{11}\ M_{\Theta}$

However, in my opinion both examples demonstrate the main thesis of this paper: The direct mass formula is equivalent to NFW formula extended to the total mass, into the halo region of MW from 30 kpc up to 200 kpc.

Table 14 radius	Variable X	Factor Direct mass	Factor NFW-total	Relative Diff. %
30,474	3	0,8657	0,7842	9,413
40,632	4	0,9996	0,9573	4,228
50,79	5	1,1176	1,1063	1,008
60,948	6	1,2243	1,2367	-1,014
71,106	7	1,3223	1,3523	-2,268
81,264	8	1,4136	1,4562	-3,013
91,422	9	1,4994	1,5505	-3,407
101,58	10	1,5805	1,6367	-3,556
111,738	11	1,6576	1,7161	-3,529
121,896	12	1,7314	1,7898	-3,374
132,054	13	1,8021	1,8584	-3,126
142,212	14	1,8701	1,9226	-2,809
152,37	15	1,9357	1,9830	-2,442
162,528	16	1,9992	2,0399	-2,038
172,686	17	2,0607	2,0938	-1,606
182,844	18	2,1205	2,1450	-1,155
193,002	19	2,1786	2,1936	-0,691
203,16	20	2,2352	2,2400	-0,218

In the graph 2 are shown how close both functions are.



6. Testing the equivalence between direct mass and NFW mass in M31 halo

In this chapter it will made the same four tests made to MW in the previous chapters but to M31 galaxy. It will be used the NFW DM density profile published by [3] Sofue, Y.2015 and the direct mass formula published in [1] Abarca, M.2024

Table 15 From [3] Sofue,Y. 2015	NFW profile	R_0 kpc	ρ_0 kg·m^{-3}
		34.6 ± 2.1	$1.51 \pm 0.16 \cdot 10^{-22}$
	M31 Baryonic mass	$1.6 \cdot 10^{11}$ M_Θ	

So from (4.3) formula $K_{NFW} = 1.16 \cdot 10^{12}$ M_Θ and $f_{BA} = 0.14$

From formula (4,7) it is right to get $\frac{c^3}{f(c)} = 246.048$ the equation to calculate numerically the concentration parameter c whose solution is c= 6.579 and f(c) = 1.15734 so $R_{200\text{-}DM} = R_0 \cdot c = 227.66$ Kpc and from the formula (4.5) it is right to get $M_{200\text{-}DM} = 1.34 \cdot 10^{12}$ M_Θ

The equation (4.12) allows to calculate the concentration parameter for the total mass, so $\frac{t^3}{f_{BA} + f(t)} = \frac{c^3}{f(c)}$ becomes $\frac{t^3}{0.14 + f(t)} = 246.048$ whose numerical solution is t= 6.89639

and so $R_{200\text{-}TOTAL} = t \cdot R_0 = 238.6$ kpc , in addition $0.14 + f(t) = 1.333$

Finally by (4.10) $M_{200\text{-}TOTAL} = [0.14 + f(t)]K_{NFW} = 1.546 \cdot 10^{12}$ M_Θ

These two parameters $R_{200\text{-}TOTAL}$ and $M_{200\text{-}TOTAL}$ are the adequate parameters because are linked to the total mass and they verify that the mean density in the $R_{200\text{-}TOTAL}$ radius sphere is $200\rho_C$. In the table 17 is checked this property almost with mathematical accuracy.

6.1 Comparison between $R_{200\text{-}TOTAL}$ and $M_{200\text{-}TOTAL}$ values got by direct mass and NFW total mass. Test I

In the framework of DMbQG theory was got the formulas $M_{200-TOTAL} = \frac{a^{12/5}}{G \cdot (10 \cdot H)^{2/5}}$ (3.4) and $R_{200-TOTAL} = \left[\frac{a^2}{100 \cdot H^2}\right]^{2/5}$ (3.5) so using the parameter a^2 for M31 = $2.235 \cdot 10^{21}$ m$^{5/2}$s^{-2} , see table 1, it is possible to calculate rightly both values.

In the table 16 are summarized both concepts got by the two different methods and also it is shown its relative difference, which are very low.

Table 16	By parameter a^2	By NFW-total	Relative diff.%
$R_{200\text{-}TOTAL}$	232.15	238.6 kpc	2.7
$M_{200\text{-}TOTAL}$	$1.4245 \cdot 10^{12}$ M_Θ	$1.546 \cdot 10^{12}$ M_Θ	7.8

In the table 17 are checked the mean density of the $R_{200\text{-}TOTAL}$ radius sphere got by the two different methods, and the matching versus $200\rho_C$ is almost perfect for both.

Table 17 M31	$R_{200\text{-TOTAL}}$	$M_{200\text{-TOTAL}}$ $M_\odot$	Mean dens/ $200\rho_C$
NFW-total	238.6 kpc	$1.546\cdot10^{12}$	0.999595
By parameter a^2	232.15	$1.4245\cdot10^{12}$	0.999959

6.2 Calculus of parameter a² using $M_{200\text{-TOTAL}}$ got by NFW . Test II

As in the epigraph 4 was calculated $M_{200\text{-TOTAL}}$ using the NFW method, then it is possible to use such result to calculate the parameter a^2 by the formula:

$$a^2 = (G \cdot M_{200})^{5/6} \cdot (10 \cdot H)^{1/3} \quad (3.3)$$

Then this result may be compared with the parameter a^2 got in [1] Abarca, M.2024 in the framework of DMbQG theory that for M31 is $2.235\cdot10^{21}$ $m^{5/2}s^{-2}$

In the table 18 is compared both results of parameter a^2 with an excellent result.

Table 18	$M_{200\text{-TOTAL}}$ $M_\odot$	Parameter a^2	Relative diff. %
NFW-total	$1.546\cdot10^{12}$	$2.393\cdot10^{21}$	6.6
a^2 as reference		$2.235\cdot10^{21}$	

6.3 Calculus of parameter a² using $M_{200\text{-TOTAL}}$ and $R_{200\text{-TOTAL}}$ got by NFW. Test III

In this test the parameter a^2 is got by the formula $a^2 = \dfrac{G \cdot M_{200}}{\sqrt{R_{200}}}$ (3.2),

using $M_{200\text{-TOTAL}}$ and $R_{200\text{-TOTAL}}$ got by the NFW-total method.

The result of parameter a^2 in this test is the same that in the test II because the formula (3.3) is mathematically equivalent to (3.2)

Table 19	$R_{200\text{-TOTAL}}$	$M_{200\text{-TOTAL}}$ $M_\odot$	Formula (3.2)	Relative diff. %
NFW-total	238.6 kpc	$1.546\cdot10^{12}$	$2.393\cdot10^{21}$	6.6
a^2 as reference			$2.235\cdot10^{21}$	

6.4 Comparison of Direct mass formula with the NFW-total mass formula into the halo region up to $R_{200\text{-TOTAL}}$. Test IV

As it was shown in the epigraph 5.4 to compare both formulas of the masses is enough to compare the called dimensionless functions of masses.

The table 20 is right to define the NFW function mass and his dimensionless function whose formula is: $F\,M_{TOTAL}^{NFW}(<r) = \left[f_{BA} + f(x)\right]$ (5.5) being $x = r/R_0$

Table 20 NFW-Sofue	R_0 34.6 kpc	$R_{200\text{-TOTAL}}$ 238.6 kpc	$K_{NFW} = 1.16\cdot10^{12}$ $M_\odot$	$f_{BA} = 0.14$

By other side, the dimensionless direct function mass formula is:

$FM_{TOTAL}^{DIRECT}(<r) = f_T \cdot \sqrt{x}$ (5.4) being x = r/R$_0$ being $f_T = K_T / K_{NFW}$ and $K_T = \frac{a^2 \cdot \sqrt{R_0}}{G}$

Being $a^2 = 2.235 \cdot 10^{21}$ and $R_0 = 34.6$ kpc then $K_T = 5.5 \cdot 10^{11} M_\odot$ and $f_T = 0.474$

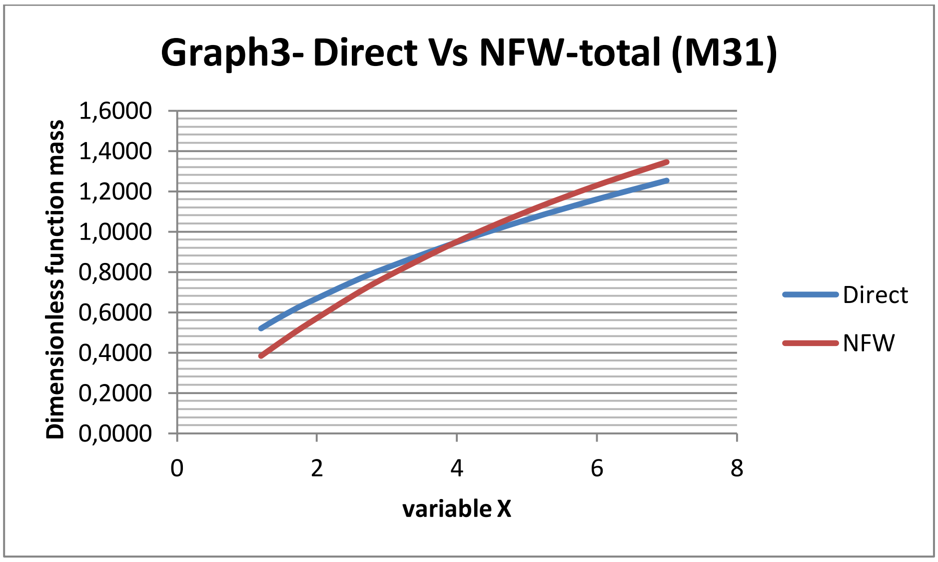
With these parameters the dimesionless function of direct mass is defined.

As the radius dominion is from 40 kpc up to 240 kpc the variable x ranges from 1.2 up to 6.9

In the table 21 are tabulated both functions into its dominion and its relative difference, that for variable x bigger than 2, the relative difference is under 15%

Table 21 Radius kpc	Variable X	Direct mass	NFW-total	Relat. Diff%
41,52	1,2	0,51924	0,38300	26,238
48,44	1,4	0,56084	0,43214	22,949
55,36	1,6	0,59957	0,48013	19,921
62,28	1,8	0,63594	0,52676	17,168
69,2	2	0,67034	0,57195	14,678
103,8	3	0,82099	0,77629	5,444
138,4	4	0,94800	0,94944	-0,152
173	5	1,05990	1,09843	-3,635
207,6	6	1,16106	1,22877	-5,832
242,2	7	1,25409	1,34444	-7,205

In the graph 3 it is shown how close are both functions.

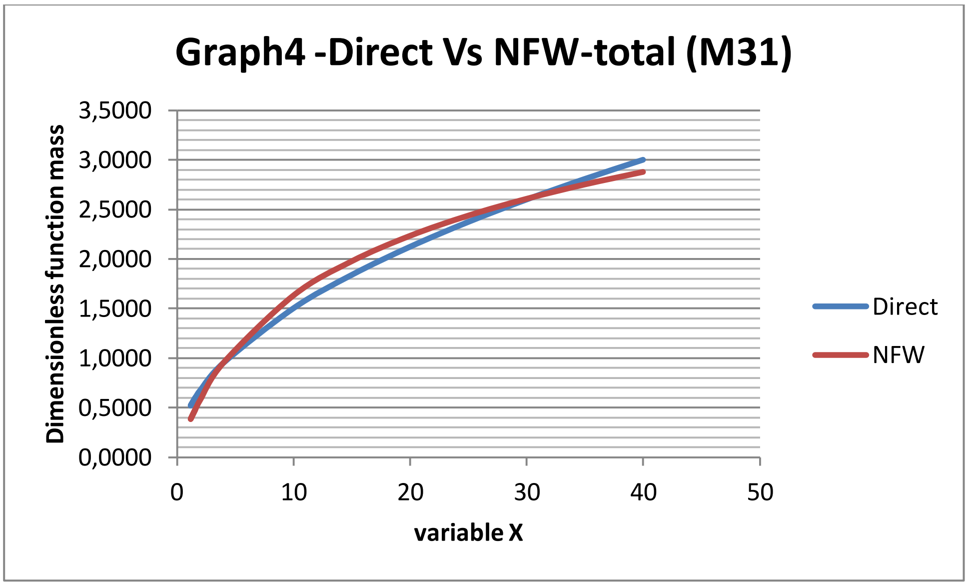


In the table 22 both functions are tabulated from 40 kpc up to 1380 kpc to show how in this dominion so wide, the relative difference remain negligible.

Table 22 Radius kpc	Variable X	Direct mass	NFW-total	Relat. Diff %
41,52	1,2	0,5192	0,3830	26,24
51,9	1,5	0,5805	0,4563	21,40
69,2	2	0,6703	0,5719	14,68
86,5	2,5	0,7495	0,6785	9,47
138,4	4	0,9480	0,9494	-0,15
207,6	6	1,1611	1,2288	-5,83
276,8	8	1,3407	1,4483	-8,03
415,2	12	1,6420	1,7819	-8,52
588,2	17	1,9544	2,0859	-6,73
761,2	22	2,2233	2,3190	-4,31
899,6	26	2,4169	2,4729	-2,31
1038	30	2,5962	2,6062	-0,39
1211	35	2,8042	2,7513	1,89
1384	40	2,9978	2,8780	4,00

In the graph 4 is shown how close are both function into a dominion so wide.

According the DMbQG theory the DM grows with the square root of radius without limit. In [1] Abarca, M.(2024), was demonstrated that the halo for the Local Group is about 2 Mpc.



7. Concluding remarks

The main goal of this paper is to demonstrate the Direct mass and the NFW-total mass functions are equivalents into the halo up to the galactic virial radius.

As it was pointed at the introduction, in the chapter 4 it was developed a method to integrate the baryonic mass into the NFW mass formula because the Direct mass is a function for the total mass, and as the reader knows the NFW is a DM density profile only. The new NFW-total mass function developed in the chapter 4 has been used to be compared with the Direct mass through four tests. These tests has been tested for the MW in the chapter 5 and tested for the M31 in the chapter 6.

All the calculus and the results of the four tests are shown conveniently and it is clear that the relative differences of the calculus made by the two different functions of masses are below the error measures published by the authors.

Namely the relative differences in the test IV using the [4] Sofue data for MW are below 10% into the whole halo dominion and the same test IV using the [9] Karukes data for MW are below 4% into the radius from 40 kpc up to 200 kpc.

The same test IV made to M31 gives a successful result as well because the relative differences between both formula of masses is below 15% from 65 kpc up to 1.3 Mpc where its relative difference is 4% only.

The good matching between the two function mass formulas is shown in the graph 4 into a radius dominion that ranges from 40 kpc up to 1380 kpc, almost twice the distance MW-M31.

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