Exact Relativistic Corrections to the Quantization of a Classical Spinning Particle with Constant Electric and Magnetic fields

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Abstract

This paper is an extension of a paper by the author where a quantization of classical spinning particle equations is carried out using the Euler angles of the particle. Relativistic corrections are found and compared to the Foldy-Wouthuysen transformation of the Dirac equation. We only consider constant linear electric and magnetic fields, and find agreement to all orders of $1/c^2$.

I.Introduction

This paper is an extension of a previous paper by the author [1], referred to as the initial paper in this article. The initial paper is based on the equations of motion for a relativistic spinning charge in an external electric and magnetic field. These equations can be found in Jackson [2]. Only linear constant external fields are considered. Using the Euler angles and particle position as degrees of freedom, a Lagrangian and Hamiltonian are found for these equations and then the system is quantized using the method of Bopp and Haag [3].

In the initial paper we found that we could quantize the system up to third order in $1/c^2$, and found agreement with the Foldy-Wouthuysen transformation [4] of the Dirac equation. More on the Dirac equation can be found in Sakauri [5]. In this extension of the initial paper we find a solution to all orders of $1/c^2$ which agrees with the Foldy-Wouthuysen transformation as found in Chiou and Chen [6].

II.Review of initial paper

From Jackson [2] we have the equations of motion for a spinning charge in a general inertial frame

$$m\frac{d\boldsymbol{\beta}}{dt} = \frac{q}{\gamma c}(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B} - \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})) \tag{1}$$

$$\frac{d\mathbf{s}}{dt} = \frac{q}{mc}\mathbf{s}\times\{\left(\frac{g}{2} - 1 + \frac{1}{\gamma}\right)\mathbf{B} - (\frac{g}{2} - 1)\frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta} - (\frac{g}{2} - \frac{\gamma}{\gamma+1})\boldsymbol{\beta}\times\mathbf{E}\} \tag{2}$$

where $\mathbf{s}$ is the spin angular momentum in the particle's rest frame and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields in a general frame. In these expressions $\boldsymbol{\beta} = \mathbf{v}/c$ and $\gamma = (1 - \beta^2)^{-1/2}$ where $\mathbf{v}$ is the velocity of the particle in the general frame and c is the speed of light. q is the charge of the particle, m is its rest mass, and t represents the time in a general inertial frame. g is the gyromagnetic factor which based on the Dirac equation is taken to be 2. We use a bold symbol to indicate a vector.

Now set $\mathbf{s} = I\gamma\boldsymbol{\omega}$ where I is the moment of inertia for the particle and $\boldsymbol{\omega}$ is its angular velocity. A factor of γ is included because in this paper $\boldsymbol{\omega}$ represents the derivatives of the Euler angles with respect to the time t in a general frame, not the time in the rest frame of the particle.

Using the Euler angles and particle position as the degrees of freedom, we have from the initial paper the Hamiltonian H and conjugate momentum $\mathbf{p}_{\mathbf{v}}$ and $\mathbf{p}_{\boldsymbol{\omega}}$

$$H = \left(m_0c^2 + \frac{1}{2}I\gamma^2\omega^2\right)\gamma + \frac{Iq}{mc}\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\cdot(\boldsymbol{\omega}\times\mathbf{E}) + q\Phi \tag{3}$$

$$\mathbf{p_v} = \left(m_0c^2 + \frac{1}{2}I\gamma^2\omega^2\right)\frac{1}{c}\gamma\boldsymbol{\beta} + \frac{q}{c}\mathbf{A} + \frac{Iq}{mc^2}\left(\frac{\gamma}{\gamma+1}\boldsymbol{\omega}\times\mathbf{E} + \frac{\gamma^3}{(\gamma+1)^2}(\boldsymbol{\beta}\cdot(\boldsymbol{\omega}\times\mathbf{E}))\boldsymbol{\beta}\right) \quad (4)$$

$$\mathbf{p_\omega} = I\gamma\boldsymbol{\omega} + \frac{Iq}{mc}\left(\mathbf{B} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}\times\mathbf{E}\right) \quad (5)$$

where $m_0 = m - \frac{1}{2c^2}I\gamma^2\omega^2$ and can be considered the non-rotating rest mass. $\mathbf{A}$ is the vector potential and Φ is the scalar potential.

To quantize the system we want to express the Hamiltonian H in terms of $\mathbf{p_v}$ and $\mathbf{p_\omega}$. From the initial paper we were able to do this exactly for only the case of a non-zero **B** field where we obtained

$$H = \{m'^2c^4 + c^2\pi^2 - \frac{2q}{m}m'c\mathbf{p_\omega}\cdot\mathbf{B}\}^{\frac{1}{2}}$$

$$= m'c^2\left(1 + \frac{\pi^2}{m'^2c^2}\right)^{\frac{1}{2}} - \frac{q}{mc}\left(1 + \frac{\pi^2}{m'^2c^2}\right)^{-\frac{1}{2}}\mathbf{p_\omega}\cdot\mathbf{B} \quad (6)$$

where $\boldsymbol{\pi} = \mathbf{p_v} - \frac{q}{c}\mathbf{A}$ and $m' = m_0 + \frac{1}{2Ic^2}p_{\boldsymbol{\omega}}{}^2$. We have only kept linear fields.

When we include a non-zero **E** field we were able to find a solution up to third order in $1/c^2$ which is given by

$$H = m'c^2\left(1 + \frac{1}{2}\frac{\pi^2}{m'^2c^2} - \frac{1}{8}\frac{\pi^4}{m'^4c^4} + \frac{1}{16}\frac{\pi^6}{m'^6c^6}\right) + \frac{q}{mc}\left(-1 + \frac{1}{2}\frac{\pi^2}{m'^2c^2} - \frac{3}{8}\frac{\pi^4}{m'^4c^4} + \frac{5}{16}\frac{\pi^6}{m'^6c^6}\right)\mathbf{p_\omega}\cdot\mathbf{B}$$

$$- \frac{1}{2}\frac{q}{mm'c^2}\left(1 - \frac{3}{4}\frac{\pi^2}{m'^2c^2} + \frac{5}{8}\frac{\pi^4}{m'^4c^4}\right)\boldsymbol{\pi}\cdot(\mathbf{p_\omega}\times\mathbf{E}) + q\Phi$$

$$= m'c^2\left(1+\frac{1}{2}\frac{\pi^2}{m'^2c^2}-\frac{1}{8}\frac{\pi^4}{m'^4c^4}+\frac{1}{16}\frac{\pi^6}{m'^6c^6}\right)-\frac{q}{mc}\left(1+\frac{\pi^2}{m'^2c^2}f(\frac{\pi^2}{m'^2c^2})\right)\mathbf{p_\omega}\cdot\mathbf{B}$$

$$+\frac{q}{mm'c^2}f(\frac{\pi^2}{m'^2c^2})\boldsymbol{\pi}\cdot(\mathbf{p_\omega}\times\mathbf{E})+q\Phi \tag{7}$$

where

$$f\left(\frac{\pi^2}{m'^2c^2}\right)=-\frac{1}{2}+\frac{3}{8}\frac{\pi^2}{m'^2c^2}-\frac{5}{16}\frac{\pi^4}{m'^4c^4} \tag{8}$$

By comparing equation (6) to equation (7) we find

$$f\left(\frac{\pi^2}{m'^2c^2}\right)=(\frac{\pi^2}{m'^2c^2})^{-1}((1+\frac{\pi^2}{m'^2c^2})^{-\frac{1}{2}}-1) \tag{9}$$

to second order in $1/c^2$.

III.Exact Solution and Quantization

To extend the solution given in equation (7), assume that equation (9) is valid to all orders in $1/c^2$ so that equation (7) becomes

$$H = m'c^2(1+\frac{\pi^2}{m'^2c^2})^{\frac{1}{2}}-\frac{q}{mc}(1+\frac{\pi^2}{m'^2c^2})^{-\frac{1}{2}}\mathbf{p_\omega}\cdot\mathbf{B}$$

$$+\frac{q}{mm'c^2}(\frac{\pi^2}{m'^2c^2})^{-1}((1+\frac{\pi^2}{m'^2c^2})^{-\frac{1}{2}}-1)\,\boldsymbol{\pi}\cdot(\mathbf{p_\omega}\times\mathbf{E})+q\Phi \tag{10}$$

Then using the relation

$$x\left\{\frac{1}{1+(1+x)^{1/2}}-\frac{1}{(1+x)^{1/2}}\right\} = \frac{1}{(1+x)^{1/2}}-1 \tag{11}$$

for some number x, set $x = \frac{\pi^2}{m'^2c^2}$ so that we can express equation (10) for H as

$$H = m'c^2(1+\frac{\pi^2}{m'^2c^2})^{\frac{1}{2}} - \frac{q}{mc}(1+\frac{\pi^2}{m'^2c^2})^{-\frac{1}{2}}\mathbf{p}_{\boldsymbol{\omega}}\cdot\mathbf{B}$$

$$+\frac{q}{mm'c^2}\{\frac{1}{1+(1+\frac{\pi^2}{m'^2c^2})^{1/2}}-\frac{1}{(1+\frac{\pi^2}{m'^2c^2})^{1/2}}\}\,\boldsymbol{\pi}\cdot(\mathbf{p}_{\boldsymbol{\omega}}\times\mathbf{E})+q\Phi \tag{12}$$

To test our assumption of the validity of equation (12) we can use equations (4) and (5) for $\mathbf{p}_{\mathbf{v}}$ and $\mathbf{p}_{\boldsymbol{\omega}}$ in equation (12) to see if we obtain equation (3) for H. Using equation (5) we can write

$$m_0c^2+\frac{1}{2}I\gamma^2\omega^2 = m'c^2-\frac{Iq}{mc}\gamma\boldsymbol{\omega}\cdot(\mathbf{B}-\frac{\gamma}{\gamma+1}\boldsymbol{\beta}\times\mathbf{E}) \tag{13}$$

Then using equation (13) in equation (4) along with $\boldsymbol{\pi} = \mathbf{p}_{\mathbf{v}}-\frac{q}{c}\mathbf{A}$ we find

$$\boldsymbol{\pi} = m'c\gamma\boldsymbol{\beta}+\frac{Iq}{mc^2}\{-\gamma^2(\boldsymbol{\omega}\cdot\mathbf{B})\boldsymbol{\beta}+\frac{\gamma}{\gamma+1}\boldsymbol{\omega}\times\mathbf{E}-\frac{\gamma^4}{(\gamma+1)^2}\big(\boldsymbol{\beta}\cdot(\boldsymbol{\omega}\times\mathbf{E})\big)\boldsymbol{\beta}\} \tag{14}$$

and that

$$(1+\frac{\pi^2}{m'^2c^2})^{\frac{1}{2}} = \gamma+\frac{qI}{mm'c^3}\{-\gamma^2\beta^2\boldsymbol{\omega}\cdot\mathbf{B}+\frac{\gamma+\gamma^2-\gamma^3}{1+\gamma}\ \boldsymbol{\beta}\cdot(\boldsymbol{\omega}\times\mathbf{E})\} \tag{15}$$

Then using equations (14) and (15) in equation (12) we obtain

$$H = m'c^2\gamma + \frac{q}{mc}\{I\left\{-\gamma^2\beta^2\boldsymbol{\omega}\cdot\mathbf{B} + \frac{\gamma+\gamma^2-\gamma^3}{1+\gamma}\ \boldsymbol{\beta}\cdot(\boldsymbol{\omega}\times\mathbf{E})\right\} - \frac{1}{\gamma}\mathbf{p}_{\boldsymbol{\omega}}\cdot\mathbf{B}$$

$$- \frac{1}{1+\gamma}\boldsymbol{\beta}\cdot(\mathbf{p}_{\boldsymbol{\omega}}\times\mathbf{E})\} + q\Phi \qquad (16)$$

Then using equations (13) and (5) in equation (16) we obtain equation (3). In these calculations we have used the relation $\gamma^2 - 1 = \gamma^2\beta^2$ and only kept linear field terms. Since we can obtain equation (3) in this way it shows that equation (12) is a valid solution.

Following the initial paper we can quantize the solution for H in equation (12) by replacing m' by $m = m_0 + \frac{3\hbar^2}{8Ic^2}$, $\boldsymbol{\pi}$ by $\hat{\boldsymbol{\pi}} = -i\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}$, and $\mathbf{p}_{\boldsymbol{\omega}}$ by $\frac{1}{2}\hbar\boldsymbol{\sigma}$ where $\boldsymbol{\sigma}$ are the Pauli spin matrices in vector form and a hat represents an operator. Thus equation (12) becomes the corresponding quantum equation

$$i\hbar\frac{\partial\Psi}{\partial t} = [mc^2\hat{\gamma}_{\boldsymbol{\pi}} - \frac{q\hbar}{2mc}\frac{1}{\hat{\gamma}_{\boldsymbol{\pi}}}\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{q\hbar}{2m^2c^2}\{\frac{1}{1+\hat{\gamma}_{\boldsymbol{\pi}}} - \frac{1}{\hat{\gamma}_{\boldsymbol{\pi}}}\}\ \hat{\boldsymbol{\pi}}\cdot(\boldsymbol{\sigma}\times\mathbf{E}) + q\Phi]\Psi \qquad (17)$$

where $\hat{\gamma}_{\boldsymbol{\pi}} = (1 + \frac{\hat{\pi}^2}{m^2c^2})^{\frac{1}{2}}$. This agrees with the Foldy-Wouthuysen expansion of the Dirac equation given by Chiou and Chen [6]

Conclusion

As in the initial paper we have only considered the spin 1/2 case and compared our results to the Foldy-Wouthuysen transformation of the Dirac equation. In principle higher order spins could be considered and compared to non-relativistic expansions of higher order relativistic spin equations. It would be interesting if this method could also be extended to non-constant fields.

The fact that we get agreement to all orders of the Foldy-Wouthuysen transformation as given by Chiou and Chen makes it more probable that the electron can be considered as actually spinning, or at least that its spin is as real as its position. From this calculation it would appear that the Dirac equation is equivalent to the canonical quantization of a classical spinning charge, at least for constant fields of low intensity. The fact that we ignore non-linear field terms is an indication that this result only agrees with the Foldy-Wouthuysen transformation for low intensity fields.

References

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