Three-dimensional view about the Riemann hypothesis

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Abstract

We try interpreting the Riemann hypothesis as something three-dimensional.

1 Glossary

$a \in A$: a is a member of the set A.

$A \cong B$: A is isomorphic to B.

$\mathbb{C}$: the set of complex numbers .

$\det$: determinant .

DP: dot product .

HM: Hermitian matrix .

$\Im$: imaginary part .

$\imath$: imaginary unit .

I_n: $n \times n$ identity matrix .

LHS: left-hand side .

$\mathbb{N}$: $\{1, 2, 3, \ldots\}$.

NZ: nontrivial zero .

O: the origin $(0, 0)$ or $(0, 0, 0)$.

O_n: $n \times n$ null matrix .

$\mathbb{R}$: the set of real numbers .

$\Re$: real part .

RH: Riemann hypothesis .

RHS: right-hand side .

RZF: Riemann zeta function .

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SIM: singular matrix .

SU(n): special unitary group of degree n .

SYM: symmetric matrix .

tr: trace .

2 Introduction and 'dummy variable' $\jmath$

RZF and RH having been of some interest $[1, 2]$, we consider the RZF

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\,, \qquad s \in \mathbb{C} \qquad [3]. \tag{1}$$

The first few NZ's of (1) are

$$\tfrac{1}{2} + 14.13472\ldots\,\imath, \quad \tfrac{1}{2} + 21.02203\ldots\,\imath, \quad \tfrac{1}{2} + 25.01085\ldots\,\imath,$$

$$\tfrac{1}{2} + 30.42487\ldots\,\imath, \quad \tfrac{1}{2} + \ 32.93506\ldots\ \imath \quad [4, \mathbf{Figure\ 3.9}].$$

Talking of our idea, we try to 'higher-dimensionalise' RH somehow. To put this idea into practice, we employ the 'dummy variable' $\jmath$ [1] to rewrite *e.g.,* the first NZ as

$$\frac{1}{2} + 14.13472\ldots(\imath + \jmath). \tag{2}$$

3 'Decomposing' zero

It follows from (1) that

$$0 = \frac{1}{1^S} + \frac{1}{2^S} + \frac{1}{3^S} + \cdots\,, \tag{3}$$

where S is a certain NZ.

[1]By 'dummy', we mean stuff that exists but can be ignored by some measure(s). See footnote 3.

We now make the following interpretation.

Interpretation 3.1. (3) is a kind of 'decomposition' of 0 into an infinite series .

Next, let

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\imath & 0 \\ \imath & 0 & -\imath \\ 0 & \imath & 0 \end{pmatrix},\ J_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\ [5].$$

Remark 3.2. J_1 and J_3 are SYM's.

Remark 3.3. J_2 is a HM.

Remark 3.4. $\det(J_i) = 0$, where $i = 1, 2, 3$.

Remark 3.5. $\mathrm{tr}(J_i) = 0$, where $i = 1, 2, 3$.

We notice *e.g.*,

$$J_1 J_2 J_1 = O_3. \tag{4}$$

Along the lines of *Interpretation* 3.1, (4) is interpreted as

Interpretation 3.6. O_3 can be decomposed into the product of some SIM's, since $\det(J_1) = \det(J_2) = 0$ [2] .

Remark 3.7. Besides (4), $J_2 J_1 J_2 = O_3$, $J_1 J_3 J_1 = O_3$, etc. hold.

4 Some visualisations

Expanding (2), one gets

$$\tfrac{1}{2} + 14.13472\ldots\imath + 14.13472\ldots\jmath\ ^3,$$

which can be regarded as the DP of $(\frac{1}{2}, 14.13472\ldots, 14.13472\ldots)$ and $(1, \imath, \jmath)$, with $(1, \imath, \jmath)$ identified with (x, y, z). More generally, we consider

[2] See *Remark* 3.4.

[3] Replacing $\jmath$ by 0 leads to its 'disappearance', $\jmath$ being the 'dummy variable'. See footnote 1.

$$\tfrac{1}{2} + \alpha\imath \mapsto (\tfrac{1}{2}, \alpha, \alpha), \quad \alpha \in \mathbb{R},$$

as if NZ's were points in (conventional) three-dimensional space (3D space).

Example 4.1. $\frac{1}{2} + 14.13472\ldots\imath$, the first NZ of RZF, is regarded as the point $(\frac{1}{2}, 14.13472\ldots, 14.13472\ldots)$ in 3D space.

First, we visualise three NZ's in the complex plane :

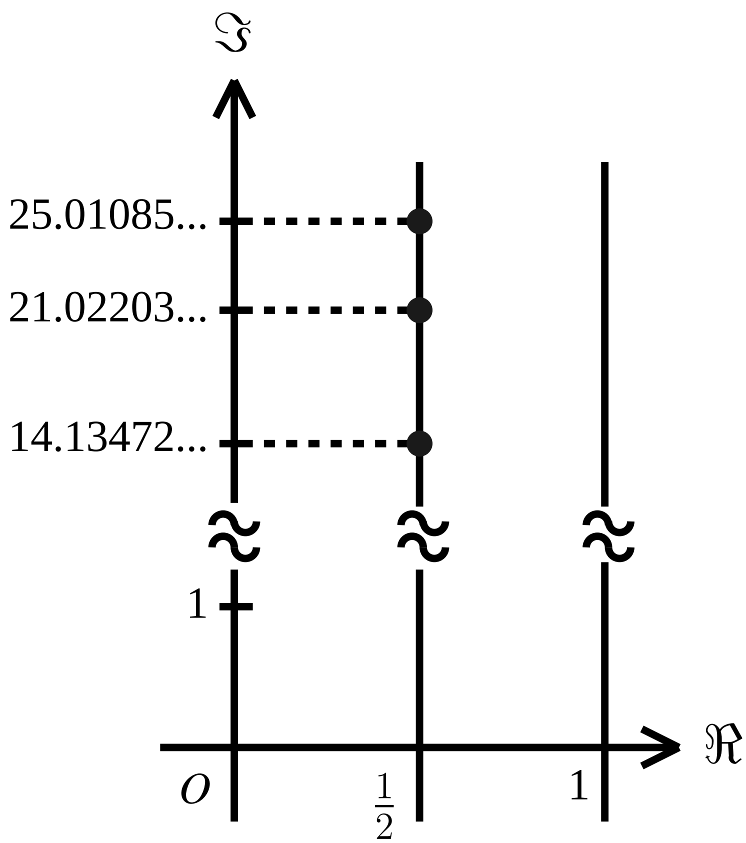

Fig. 1. Plotting NZ's in the complex plane . Black dots indicate the first three. Wavy lines in both this figure and Fig. 2 denote 'skipping' of some interval , *e.g.*, $[\,2, 10\,]$ on the $\Im$-axis. O stands for the origin $(0, 0)$.

We then go on to 'higher-dimensionalise' this along the lines with *Example* 4.1:

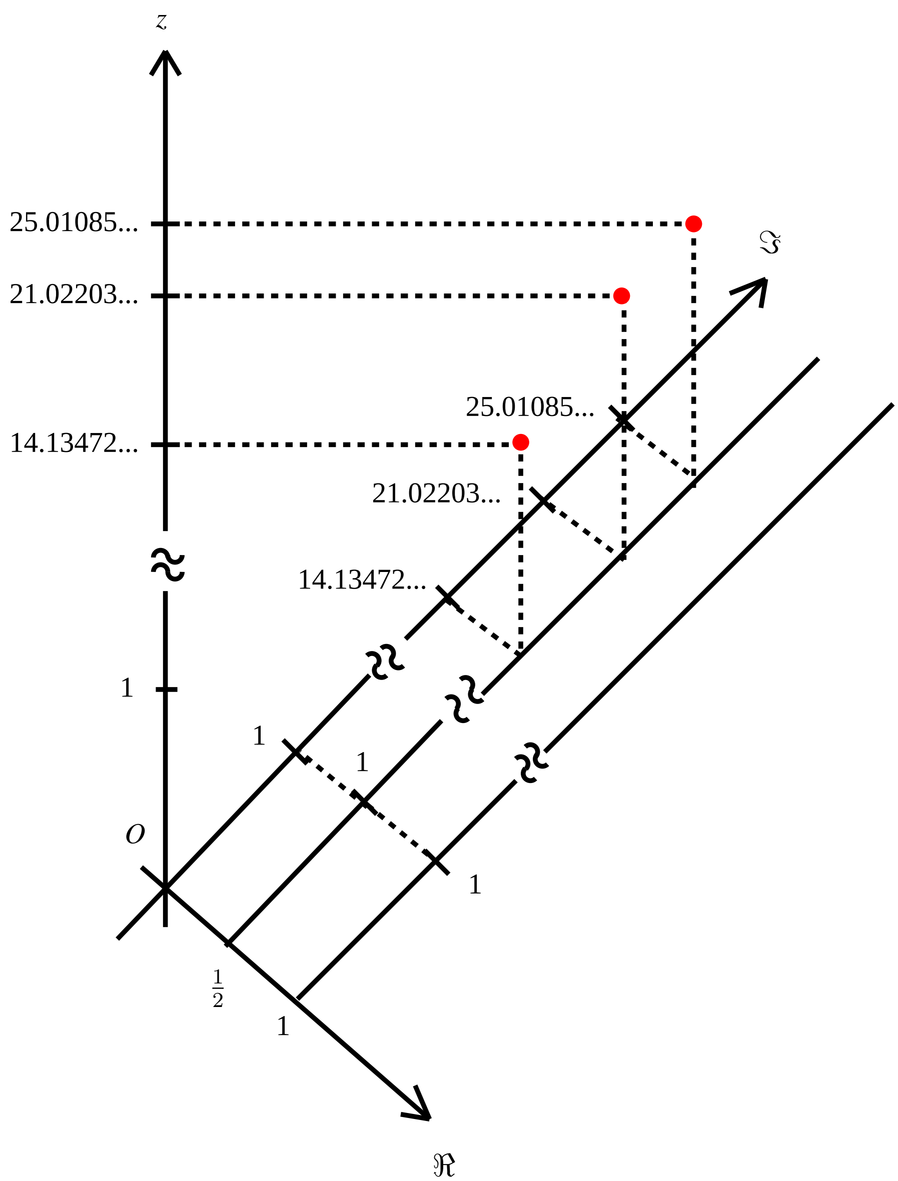

Fig. 2. ‘Higher-dimensionalisation’ of Fig. 1. Red dots correspond to black dots in Fig. 1.

N.B. Unlike Fig. 1, O in Fig. 2 stands for the origin $(0,0,0)$.

5 Discussion

Firstly, let $1, i, j, k$ be the basis elements of quaternions . Next, although the relation $iji = j$ holds, we refrain from drawing a parallel between (4) and it, since j in the RHS $\neq 0$. On the other hand, it follows from (4) that

$$A_1 \cdots A_i \cdot J_1 J_2 J_1 \cdot A_{i+1} \cdots A_n = A_1 \cdots A_i \cdot O_3 \cdot A_{i+1} \cdots A_n, \tag{5}$$

where A_i is a 3×3 matrix, and $n \in \mathbb{N}$. By the way, due to the Euler product formula for RZF , we have

$$\zeta(s) = \frac{1}{1 - \frac{1}{2^s}} \cdot \frac{1}{1 - \frac{1}{3^s}} \cdot \frac{1}{1 - \frac{1}{5^s}} \cdots \frac{1}{1 - \frac{1}{p^s}} \cdots , \tag{6}$$

where p is a prime . So by recalling (3) and letting n in the RHS of (5) tend to ∞, we draw some parallel between (5) and (6) [4] .

We also have

$$J_1 J_2 J_3 + J_3 J_2 J_1 = O_3. \tag{7}$$

Like *Interpretation* 3.6, this is interpreted as

Interpretation 5.1. O_3 can be decomposed into the sum of the products of some SIM's, since $\det(J_1) = \det(J_2) = \det(J_3) = 0$ [5] .

Remark 5.2. Besides (7), $J_1 J_3 J_2 + J_2 J_3 J_1 = O_3$, $J_2 J_1 J_3 + J_3 J_1 J_2 = O_3$, etc. hold.

Since J_1, J_2, J_3 are the 3×3 matrix representation of SU(2) [5], and SU$(2) \cong$ unit quaternions , we note the relation $ijk + kji = 0$. Identifying this relation with (7) seems to enable us to draw some parallel between quaternions and J_i, where $i = 1, 2, 3$. However, we also note that $i^2 = j^2 = k^2 = -1$, whereas $J_1^2, J_2^2, J_3^2 \neq -I_3$. Identifying $-I_3$ with -1, we acknowledge that i, j, k cannot always be identified with J_i.

As for Fig. 2, some might recall Miller index *e.g.*, (100) , and try to interpret the red dots as stuff in a certain plane. And regarding RH as basically two-dimensional, we propose the following.

Interpretation 5.3. Our introduction of 'dummy variable' $\jmath$ has virtually resulted in 'higher-dimensionalisation' of RH.

Eventually and needless to say, we presented neither proof of RH nor counterexample(s) to it. What is worse, quaternionic analogy is rather obscure in that i, j, k are not always identifiable with J_i. Nevertheless and finally, we wonder whether fully quaternionic formulation of RH, whose minimal framework we believe we have shown, would be feasible, if RH should be wrong.

[4] We believe we can draw a similar parallel, if we are allowed to regard $J_1 J_2 J_1$ in the LHS of (5) as 'one cluster'.

[5] See *Remark* 3.4.

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References

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[2] Connes, A., "An Essay on the Riemann Hypothesis," p225 in Nash, J. F. Jr. and Rassias, M. T. eds. "Open Problems in Mathematics," Springer 2016 .

[3] Borwein, P., Choi, S., Rooney, B., and Weirathmueller, A. eds. "The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike," Springer-Verlag 2008 p55.

[4] Eilers, S. and Johansen, R., "Introduction to Experimental Mathematics," Cambridge University Press 2017 p84.

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6 Appendix

6.1 Will a counterexample to RH be discovered soon?

To date, no counterexample seems to have been discovered , but. . ..

6.2 Can our 'higher-dimensionalisation' deal with such a (future) counterexample?

Time will tell.