Proof of the Lehmer conjecture on Ramanujan's τ function

Minjia Shi, Lu Wang[*], Patrick Solé[†]

Abstract

A criterion for Lehmer's conjecture in terms of the spherical designs held in the shells of the lattice E_8 was derived by de La Harpe, Pache and Venkov circa 2005. We check that this criterion is satisfied by combining spherical designs, harmonic polynomials, weighted theta series, and Deligne's bound on the modulus of the τ function.

Keywords: Ramanujan's τ function, spherical designs, harmonic polynomials, modular forms, E_8 lattice, weighted theta series.

MSC (2020): 05E99, 11F11, 33C55

1 Introduction

The Lehmer conjecture (LC) on the non-vanishing of Ramanujan τ function is very important in number theory, historically and theoretically. Let $\Delta(q)$ denote the discriminant function, which, as a formal power series generates Ramanujan τ numbers

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n.$$

*Minjia Shi and Lu Wang are with the Key Laboratory of Intelligent Computing Signal Processing, Ministry of Education, School of Mathematical Sciences, Anhui University, Hefei 230601, China; State Key Laboratory of Integrated Service Networks, Xidian University, Xi'an, 710071, China. smjwcl.good@163.com, wanglu5666@163.com

† Patrick Solé is with I2M (Aix Marseille Univ, CNRS), Marseilles, France. sole@enst.fr

Lehmer conjectured in 1947 that $\forall n \geq 1$, $\tau(n) \neq 0$ [15]. This fact has been checked for $n \leq 10^{15}$ by Serre in 1985 [18], and for $n \leq 10^{23}$ in 2013 [8]. While Hardy was afraid that this function belonged to the "backwaters of mathematics " [11, Lect. X], it is now completely mainstream in modern mathematics. To wit its interpretation as the trace of a ℓ-adic representation [19], a conception which allowed Deligne to prove the third Ramanujan conjecture [11, 10.7.1] on that function at prime arguments [7], as a far reaching consequence of his proof of Weil conjectures [14]. This result in turn implies the estimate on the modulus of τ ([11, 10.7.2]) that we will need in this paper (Lemma 3).

In the present work, we prove the Lehmer conjecture by using a criterion based on spherical designs due to La Harpe, Pache and Venkov [12, 13, 17]. For backround material on spherical designs we recommend to the interested reader the survey [2]. Essentially, to check that this criterion is satisfied, we have to prove that the shells of the Gosset/Korkine/Zolotareff lattice (E_8 shortly) in dimension 8 never form a spherical design of strength 8. This is achieved by using a consequence of another criterion due to Venkov for a lattice to hold spherical designs in its shells. This Venkov criterion, when applied to E_8 tests the value of the average of the eighth powers of a given coordinate over a shell. This value is computed by projection of the homogeneous function x_1^8 on the spaces of harmonic polynomials of respective degree $0, 2, 4, 6, 8$. The technique of harmonic projection is known at least since Vilenkin's book [21], and has been used recently in mathematical physics [1]. The contribution of each degree (denoted by $c_n(j)$ in the text) is then evaluated by attaching to it a weighted theta series which turns out to have been determined by Pache [17]. In particular, degree 8 yields the discriminant function above as an harmonic modular form of weight 12. The violation of Venkov criterion is then achieved by having recourse to some estimates of arithmetic functions like the divisor functions $\sigma_k(n)$, and Ramanujan τ function.

The material is arranged as follows. The next section collects some background notions needed for the rest of the paper. Section 3 studies the Venkov criterion for E_8. Section 4 describes harmonic projection. Section 5 discusses weighted theta series. Section 6 proves Lehmer conjecture asymptotically. An effective proof is given in Section 7. Section 8 concludes this note.

2 Preliminaries

2.1 Spherical designs

Define the *unit sphere* $\Omega(n)$ of $\mathbb{R}^n$ as

$$\Omega(n) := \{x \in \mathbb{R}^n \mid (x, x) = 1\},$$

where $(.,.)$ denotes the standard euclidean inner product. Write $P(n,t)$ (resp. $H(n,t)$) the vector space of *homogeneous* (resp. homogeneous harmonic) polynomials in n variables of degree t. Here *harmonic* means being in the kernel of the Laplacian operator $\sum_{i=1}^{n} \frac{\partial^2}{\partial^2 x_i}$.

A *spherical design* of *strength* t (t-design shortly) is a finite subset X of $\Omega(n)$ such that the average of every element P in $P(n,j)$ over X for $1 \leq j \leq t$ equals its integral over $\Omega(n)$.

$$\frac{1}{|X|} \sum_{x \in X} P(x) = \int_{x \in \Omega(n)} P(x).$$

An alternative definition of a spherical design is

$$\forall 1 \leq j \leq t, \forall P \in H(n,j), \sum_{x \in X} P(x) = 0.$$

The *Venkov criterion* [16, Th. 3.2], [2, Th. 2.2] written here for $X = -X$ and t even, states that X is a t-design iff $\forall \alpha \in \mathbb{R}^n$, and all even $2 \leq j \leq t$, we have

$$\sum_{x \in X} (x, \alpha)^j = (\alpha, \alpha)^{j/2} \frac{j!!}{n(n+2)\dots(n+j-2)} |X|.$$

In this work, we will only use the case $\alpha = e_i$ with e_i the element of index i of the canonical basis of $\mathbb{R}^n$. The above equation becomes

$$\sum_{x \in X} x_i^j = \frac{j!!}{n(n+2)\dots(n+j-2)} |X|.$$

2.2 Lattices

A *lattice* L is an additive discrete subgroup of $\mathbb{R}^n$. The *norm* of a lattice is $\min\{(x,x) \mid 0 \neq x \in L\}$. For any integer m we define the *shell* L_m of order m of L as

$$L_m = \{x \in L \mid (x,x) = m\}.$$

There is a notion of duality for a lattice, that is

$$L^* = \{x \in \mathbb{R}^n \mid \forall y \in L,\, (x,y) = 0\}.$$

A lattice is *Type II* iff $L = L^*$ and the squared norm of each of its vectors is an even integer. A Type II lattice is *extremal* iff its norm is equal to $2(\lfloor n/24 \rfloor + 1)$. It is known that extremal lattices hold spherical designs in all their shells, their common strengths being $7, 3, 11$ depending on $n \equiv 8, 16, 24 \pmod{24}$ respectively [16, Th. 16.4, ch. 1]. The E_8 lattice is, up to equivalence, the only Type II lattice in dimension 8. It is defined explicitly in [5] as $E_8 = D_8 \cup (u + D_8)$, with $u = (1/2, \cdots, 1/2)$, and

$$D_8 = \{x \in \mathbb{Z}^8 \mid \sum_{i=1}^{8} x_i \equiv 0 \pmod{2}\}.$$

Given an harmonic polynomial P in n variables and L a lattice in $\mathbb{R}^n$, define the *weighted theta series* as the formal power series

$$\theta(L, P; q) = \sum_{x \in L} P(x) q^{(x,x)}.$$

Upon letting $q = \exp(\pi i z)$ with $\Im(z) > 0$, when L is Type II, these weighted theta series become modular forms of weight $n/2 + r$ where r is the degree of P [6, 9]. From the definitions, it is clear that, up to normalization, $(L)_m$ is a t-design iff the coefficient of q^m in $\theta(L, P; q)$ vanishes for all $P \in H(n, j)$ and all $1 \leq j \leq t$. (Cf. [17, Lemma 5]). By this simple observation, we deduce the de la Harpe, Pache, Venkov criterion (HPV criterion) on the vanishing of the τ function. Write, to simplify notation, $\sqrt{n} S_n = (E_8)_n$.

Proposition 1 *([17, Th. 32]) For all integers $n \geq 1$, $\tau(n) = 0$ iff S_{2n} is an 8-design.*

Proof. It is well-known that the shell S_{2n} is always a 7-design by extremality of E_8. By [17, Lemma 31] we know that

$$\theta(E_8, P; q) = c(P)\Delta(q^2)$$

for all $P \in H(8,8)$, and for some real $c(P)$. The result follows by taking coefficients of q^{2n} on both sides. ■

For general connections between spherical designs and extremal lattices see [3].

3 Venkov criterion

To contradict the HPV criterion, we need to check that S_n is not an 8-design. There is an equivalent definition of spherical designs due to Venkov [2, Th. 2.2.(6)], recalled in Section 2, that yields in this special case the following necessary condition.

For all $i \in [8]$, if S_n is a 8-design we should have

$$\sum_{u \in S_n} u_i^8 = |S_n| \frac{1.3.5.\ldots 15}{8.(8+2).\ldots(8+14)} = \frac{3 \times 13 |S_n|}{2^{15}},$$

and in particular

$$\frac{1}{|S_n|} \sum_{u \in S_n} u_i^8 = \frac{3 \times 13}{2^{15}}.$$

The LHS of the latter equation can be computed numerically for the first few values of n. Note that if $S(n)$ were an 8-design, then for all $i \in [8]$, we would have

$$\frac{1}{|S_n|} \sum_{u \in S_n} u_i^8 = \frac{3 \times 13}{2^{15}} \approx 0.001190.$$

The size of S_n can be obtained directly by

$$|S_n| = 240 \sigma_3 \left(\frac{n}{2} \right), \text{ where } \sigma_r(m) = \sum_{d | m} d^r.$$

By using Magma [4], for some small n, we get the set S_n and observe that it is not an 8-design in the following table. The value of $\frac{1}{|S(n)|} \sum_{u \in S_n} u_i^8$ is rounded to 6 decimal places. The entry for odd n is omitted since then S_n is empty. Table 1 suggests that the real value of $\sum_{u \in S_n} u_i^8$ is strictly larger than the value imposed by the 8-design property.

Table 1: Values of $\frac{1}{|S(n)|}\sum_{u\in S_n} u_i^8$ for some small n's.

n	2	4	6	8	10	12
$\lvert S_n \rvert$	240	2160	6720	17520	30240	60480
$\sum_{u\in\sqrt{n}S_n} u_i^8$	28.5	4356	67662	562848	2355255	9806832
$\frac{1}{\lvert S(n)\rvert}\sum_{u\in S_n} u_i^8$	0.007422	0.007878	0.007769	0.007843	0.007789	0.007820

4 Harmonic polynomials

For simplicity's sake, we let henceforth $i = 1$. Every homogeneous polynomial can be expressed on the basis of Harmonic polynomials by the explicit formulas of [1, 21].

Proposition 2 *The harmonic projection of x_1^8 is given by*

$$x_1^8 = \sum_{j=0}^{8} r^{8-j} h_j(x)$$

where $h_j(x)$ is a harmonic polynomial of degree j in 8 variables, $r^2 = (x, x)$, and $j = 0, 2, 4, 6, 8$.

Proof. This follows by [21, (3), p.443], or, alternatively by [1, (11)]. ■

Let $c_n(j) = \sum_{u\in S_n} h_j(u)$. We note, for future use, the relation

$$\sum_{u\in S_n} u_i^8 = \sum_{j=0}^{8} c_n(j), \tag{1}$$

which comes from Proposition 2, upon noting that $r = 1$ for u on the unit sphere.

5 Modular forms

To evaluate $c_n(j)$ we introduce the weighted theta series

$$T_j(q) := \theta(E_8, h_j) = \sum_{x\in E_8} h_j(x) q^{(x,x)}.$$

The connection with $c_n(j)$ is as follows.

Proposition 3 *For* $j \in \{0,2,4,6,8\}$, *we have* $T_j(q) = \sum_{n=0}^{\infty} c_n(j) n^{j/2} q^n$.

Proof. Write

$$\sum_{x \in E_8} h_j(x) q^{(x,x)} = \sum_{n=0}^{\infty} \sum_{x \in \sqrt{n} S_n} h_j(x) q^{(x,x)} = \sum_{n=0}^{\infty} q^n \sum_{x \in \sqrt{n} S_n} h_j(x).$$

We conclude by homogeneity of h_j. ∎

The evaluation of weighted theta series in [17] translates into evaluation of the $c_n(j)'$s.

Proposition 4 *For any integer* $n \geq 1$ *we have*

- $c_n(0) = \frac{|S_n|}{128}$,
- $c_n(j) = 0$, *for* $j = 2, 4, 6$
- $n^4 c_n(8) = 16 c_2(8)\, \tau(n/2) = -1.5\, \tau(n/2)$

Proof. The expression $h_0(x) = \frac{1}{128}$ can be obtained by applying formula (16) in [1]. Since h_8 is a homogeneous polynomial of degree 8, by using Magma [4], we have

$$16 c_2(8) = 16 \sum_{u \in S_2} h_8(u) = \sum_{u \in \sqrt{2} S_2} h_8(u) = -1.5\ .$$

Then the results follows immediately from [17, Lemma 31, (i)] by taking q-expansions of weighted theta series and using Proposition 3. ∎

6 Arithmetic functions

In this section, we derive an asymptotic version of Lehmer conjecture.

Theorem 1 *A sufficient condition for* S_n *to not be an* 8*-design is*

$$\frac{-1.5 \tau(n/2)}{n^4} + |S_n|/128 > \frac{3 \times 13}{2^{15}} |S_n|. \qquad (2)$$

Proof. By Equ.(1) and Proposition 4, we have

$$\sum_{x \in S_n} x_1^8 = c_n(0) + c_n(8) = \frac{-1.5\tau(n/2)}{n^4} + |S_n|/128.$$

The result follows then by Venkov criterion. ■

It should be noted that the values of $c_n(0)+c_n(8)$ are consistent with the data in Table 1 for $n \le 12$. The following estimates for divisor functions are well known [10, 20].

Lemma 1 *For fixed $k > 1$, we have*

$$\sigma_k(n) \ge n^k,$$

and

$$\sigma_k(n) \le \zeta(k) n^k,$$

where $\zeta(k)$ is the Riemann Zeta function given by $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$.

We know that $|S_n| = 240\sigma_3(n/2)$ [5]. Write $f(x) = O(g(x))$ if there is a constant $C > 0$, such that for all x large enough we have $|f(x)| \le C|g(x)|$. Thus, by the above estimates on $\sigma_k(n)$ we have

$$30n^3 \le |S_n| = O(n^3).$$

By a deep result of Deligne [5, Chap. 2, (55)], for any $\epsilon > 0$, we have $\tau(n) = O(n^{\frac{11}{2}+\epsilon})$, and so $\frac{-1.5\tau(n/2)}{n^4} = O(n^{3/2})$.

Thus, roughly speaking, for $n \to \infty$, the dominant term in the LHS of inequality (2) is the second one which is trivially larger than the RHS. We summarize the previous discussion as follows.

Theorem 2 *There exists an integer $n_0 > 0$, such that the inequality (2) holds for $n > n_0$.*

7 Effective bounds

In this section, we strive to make Theorem 1 effective. We begin by the crude, but explicit.

Lemma 2 *For $n \geq 1$, we have $\sigma_0(n) \leq 2\sqrt{n}$.*
Proof. By the definition of $\sigma_r(m)$, we have $\sigma_0(n) = \sum_{d|n} 1$, which is equivalent to the number of factors of n. If d divides n, so does n/d. For any factor d, one of d, n/d is smaller than or equal to $\sqrt{n}$. Then the result follows. ∎

We need a deep inequality which is called Ramanujan conjecture. The inequality was proved by Deligne in [14].

Lemma 3 *For $n \geq 1$, we have $|\tau(n)| \leq \sigma_0(n)n^{11/2}$.*

We are now in a position to state and prove the main result of this note.

Theorem 3 *For $n \geq 1$, we have $\tau(n) \neq 0$.*
Proof. We show that the bound of Theorem 1 holds for $n \geq N$, for some explicit N. Combining Lemmas 1, 2 and 3 gives a sufficient condition

$$\frac{-1.5 \times \left(\frac{n^6}{2^5}\right)}{n^4} + \frac{30n^3}{128} > \frac{3 \times 13}{2^{15}} \times 30\xi(3)n^3.$$

By using $\zeta(3) < 2$, we have a sufficient condition for Theorem 1, that is

$$-6n^2 + 30n^3 \geq \frac{3 \times 13 \times 30}{2^7} n^3,$$

which holds for $n \geq N$, with

$$N = \lceil \frac{3}{30 - 9.140625} \rceil = 1.$$

∎

8 Conclusion and open problems

In this note we have given a proof of Lehmer's conjecture on Ramanujan τ function based on spherical designs and estimates of some arithmetic functions. In particular, we have proved that the shells of the lattice E_8 never form 8-designs. The tool that have might been missed by the authors of [13] is probably the technique of harmonic projection [1, 21].

It seems likely that our techniques can be extended to other extremal lattices. For instance, it can be conjectured that the shells of the Leech lattice never form a 12-design (see the Remark in [17] after Theorem 32). This might require, however, estimates on arithmetic functions not yet available in the literature.

In another direction, the fundamental analogy between codes and lattices should provide arguments for the non-existence of high strength combinatorial designs in the supports of extremal codes.

Acknowledgement: The authors are indebted to Eiichi Bannai for helpful discussions.

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