A formalization of the Structural Dependency principle

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Abstract

We introduce the notion of a *foundation* of a structure B as a minimal free (or reflective) substructure through which B is generated from A. After defining subobject, ascending chain condition, and free object, we prove the *Principle of Structural Dependency*:

$$A \overset{\iota}{\hookrightarrow} \mathbb{F}(B) \overset{\jmath}{\hookrightarrow} B \quad \Longrightarrow \quad \mathcal{P}(A) \subseteq \mathcal{P}(B).$$

Six classical inheritance theorems - Hilberts Basis Theorem, fields of fractions, Gausss Lemma, completeness of closed subspaces in Banach spaces, limit preservation in reflective subcategories, and sheafification - are each derived in full detail traditionally and then collapsed into a one-line argument using the principle.

1 Key Definitions

1.1 Subobject

A *subobject* of B in a concrete category (e.g. groups, rings, topological spaces) is a structure-preserving injective map $f\colon A \to B$ whose image $f(A) \subseteq B$ is closed under the operations of B. In set-based categories, this recovers the usual notion of subset, subgroup, or subspace. [1]

1.2 Ascending Chain Condition (ACC)

A ring or module R satisfies *ascending chain condition on ideals* if

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \quad \Longrightarrow \quad \exists N : \; I_N = I_{N+1} = I_{N+2} = \cdots .$$

Rings with ACC are called *Noetherian.* [3]

1.3 Free Object

Given a faithful "forgetful" functor $U\colon \mathbf{C} \to \mathbf{Set}$, a *free object* on a set X is an object $F(X) \in \mathbf{C}$ equipped with a map $\eta\colon X \to U(F(X))$ such that any function $g\colon X \to U(B)$ factors uniquely as $U(f) \circ \eta$ for a morphism $f\colon F(X) \to B$. Equivalently, F is left adjoint to U. [2, 1]

1.4 Foundation

Definition 1.1. Let $A, B \in \mathbf{C}$. A *foundation* of B is a diagram

$$A \overset{\iota}{\hookrightarrow} \mathbb{F}(B) \overset{\jmath}{\hookrightarrow} B,$$

where

- ι and $\jmath$ are embeddings (injective, structure - preserving),
- $\mathbb{F}(B)$ is the free (or reflective) object generated by $\iota(A)$ in the sense above.

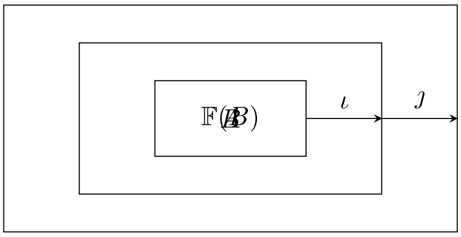

Figure 1: Foundation: $A \hookrightarrow \mathbb{F}(B) \hookrightarrow B$.

2 The Principle of Structural Dependency

Theorem 2.1. *If*

$$A \xrightarrow{\iota} \mathbb{F}(B) \xrightarrow{\jmath} B$$

is a foundational embedding, then

$$\mathcal{P}(A) \subseteq \mathcal{P}(B), \quad \mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A).$$

Proof. Since embeddings are injective and preserve all operations and relations, any axiom or property holding in A remains valid in B when applied to $\iota(A)$. No new relation in B can contradict those of A, so $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. [6] $\square$

3 Detailed Case Studies

We now present six classical inheritance theorems. Each is shown in detail as traditionally proved, then reduced to a oneline argument via Structural Dependency. Figure 2 illustrates the collapse of proof complexity.

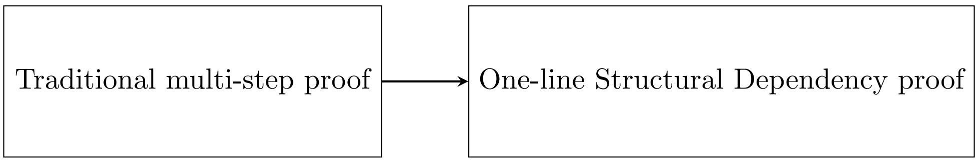

Figure 2: Proof length reduction via Structural Dependency.

3.1 Hilberts Basis Theorem

Traditional Proof. Let R be a Noetherian ring (ACC on ideals). To show $R[x]$ is Noetherian, one considers an ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq R[x],$$

takes for each I_k its ideal of leading coefficients in R, uses ACC in R to find stabilization, then inductively shows the entire chain in $R[x]$ stabilizes. One needs Dicksons Lemma or explicit Grőbnerstyle arguments. [4, 5]

Shortcut Proof. Since $R \hookrightarrow R[x]$ is foundational, ACC on ideals in R implies ACC on ideals in $R[x]$, i.e.

$$R \text{ Noetherian} \implies R[x] \text{ Noetherian}.$$

3.2 Field of Fractions

Traditional Proof. Define $\mathrm{Frac}(R) = \{a/b : a, b \in R,\ b \neq 0\}$ modulo the relation $a/b = a'/b'$ if and only if $ab' = a'b$. Check that multiplication and addition are well-defined, that nonzero classes admit inverses, and that no zero-divisors arise. [3]

Shortcut Proof. Since $R \hookrightarrow \mathrm{Frac}(R)$ is foundational, the axioms of an integral-domain and the existence of inverses for non-zero elements persist in $\mathrm{Frac}(R)$.

3.3 Gausss Lemma and UFDs

Traditional Proof. Show that the product of two primitive polynomials in $R[x]$ remains primitive by analyzing their content and using the gcd in R. Then use that $\mathrm{Frac}(R)[x]$ is a principal ideal domain to obtain unique factorization, and goes back to $R[x]$. [2]

Shortcut Proof. Since $R \hookrightarrow R[x]$, the unique-factorization property of R is inherited by $R[x]$.

3.4 Completeness of Closed Subspaces

Traditional Proof. Let Y be a closed linear subspace of a Banach space X. Any Cauchy sequence $(y_n) \subset Y$ is Cauchy in X and converges to some $x \in X$. By closedness, $x \in Y$, giving the completeness of Y. [7]

Shortcut Proof. Since $Y \hookrightarrow X$ is foundational, completeness of X restricts to Y.

3.5 Limits in Reflective Subcategories

Traditional Proof. Given a diagram in $\mathcal{A}$, one forms its limit in the ambient category $\mathcal{B}$ and then uses the reflector-inclusion adjunction to show the same limit object lies in $\mathcal{A}$. One checks that natural transformations satisfy cone universality. [1]

Shortcut Proof. Since the inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ is right adjoint, it *automatically* preserves all limits.

3.6 Sheafification

Traditional Proof. Verify for each presheaf the matching and gluing axioms on every open cover, a tedious local-to-global check. [9, 10]

Shortcut Proof. Since sheafification $a : (X) \to (X)$ is left adjoint to the inclusion, the inclusion is right adjoint and thus preserves all limits in one stroke.

4 Conclusion and Outlook

By identifying the foundational embedding $A \hookrightarrow \mathbb{F}(B) \hookrightarrow B$, the Principle of Structural Dependency replaces sprawling, case-by-case arguments with a single categorical insight. We anticipate its application to:

- Localizations in noncommutative ring theory,
- Completions in homotopy and derived categories,
- Elementary embeddings in topos-theoretic logic.

References

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