A New Approach For The Proof of The *abc* Conjecture

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Abstract

In this paper, we assume that the explicit ***abc*** conjecture of Alan Baker and the conjecture $\boldsymbol{c < R^{1.63}}$ are true, we give a proof of the ***abc*** conjecture and we propose the constant $\boldsymbol{K(\epsilon)}$. Some numerical examples are given.

Keywords: prime numbers, the number the prime factors of the radical of the product ***abc***, the explicit ***abc*** conjecture of Alan Baker, the conjecture $\boldsymbol{c < R^{1.63}}$, the function exponential.

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To the memory of my Father who taught me arithmetic
To my wife ***Wahida****, my daughter* ***Sinda*** *and my son* ***Mohamed Mazen***
To Prof. ***A. Nitaj*** *for his work on the abc conjecture*

1 Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1. *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [2]. It concerned the best example given by E. Reyssat [2]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{4}$$

A conjecture was proposed that $c < rad^2(abc)$ [3]. In 2012, A. Nitaj [4] proposed the following conjecture:

Conjecture 2. *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < rad^{1.63}(abc) \tag{5}$$

$$abc < rad^{4.42}(abc) \tag{6}$$

In the following, we assume that the conjecture $c < rad^{1.63}(abc)$ is true. In 2004, Alan Baker [1], [5] proposed the explicit version of the *abc* conjecture namely:

Conjecture 3. *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} \tag{7}$$

with $R = rad(abc)$ and ω denote the number of distinct prime factors of abc.

A proof of the conjecture by the author is under review [6]. In the following, we assume also that the above conjecture is true, I will give an elementary proof of the *abc* conjecture by verifying the below inequality:

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} < ... < K(\epsilon)R^{1+\epsilon} \tag{8}$$

with a adequate choice of the constant $K(\epsilon)$. Let we denote $\alpha = \dfrac{6}{5}R\dfrac{(LogR)^\omega}{\omega!}$, we have remarked from some numerical examples (see below) that $c \ll \alpha - c$ when $\omega = 10$ and R not very large. With our choice, c will be very very small comparing to $K(\epsilon)R^{1+\epsilon}$.

2 The Proof of the *abc* conjecture

Proof. : Let $A=\dfrac{(Log(R^\epsilon))^\omega}{\omega!}$, and $\epsilon\in]0,0.63[$, we obtain:

$$R^\epsilon=e^{LogR^\epsilon}=1+Log(R^\epsilon)+\frac{(Log(R^\epsilon))^2}{2!}+\dots+A+\sum_{k=\omega+1}^{+\infty}\frac{(Log(R^\epsilon))^k}{k!}\Longrightarrow$$

$$A=R^\epsilon-1-\sum_{k=1,\neq\omega}^{+\infty}\frac{(Log(R^\epsilon))^k}{k!}\Longrightarrow$$

$$A=R^\epsilon\left(1-\frac{1}{R^\epsilon}\left[1+\sum_{k=1,\neq\omega}^{+\infty}\frac{(Log(R^\epsilon))^k}{k!}\right]\right)=R^\epsilon(1-B)>0,\ 0<B<1\Longrightarrow$$

$$A=\frac{(Log(R^\epsilon))^\omega}{\omega!}=R^\epsilon(1-B)>0 \tag{9}$$

We begin from the Baker's formula below :

$$c<\frac{6}{5}R\frac{(LogR)^\omega}{\omega!}=\frac{6}{5}R.\frac{1}{\epsilon^\omega}\frac{(\epsilon LogR)^\omega}{\omega!}=\frac{6}{5}\frac{R}{\epsilon^\omega}\frac{(Log(R^\epsilon))^\omega}{\omega!}$$

Using the term $\dfrac{(Log(R^\epsilon))^\omega}{\omega!}$ from (9), the equation above becomes :

$$c<\frac{6}{5}\frac{R}{\epsilon^\omega}R^\epsilon(1-B)<1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}}R^{1+\epsilon}\Longrightarrow \text{our choice of the constant } K(\epsilon)=1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}} \tag{10}$$

Now, is the following inequality true? :

$$\frac{6}{5}\frac{1}{\epsilon^\omega}(1-B)\overbrace{<}^{?}1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}} \tag{11}$$

Supposing that :

$$\frac{6}{5}\frac{1}{\epsilon^\omega}(1-B)>\frac{6}{5}e^{e^{\left(\frac{1}{\epsilon^4}\right)}}.\Longrightarrow 1>(1-B)>\epsilon^\omega.e^{e^{\left(\frac{1}{\epsilon^4}\right)}}$$

As $\omega\geq 4\Longrightarrow \omega=4\omega'+r,\ 0\leq r<3,\omega'\geq 1$, we write $\epsilon^\omega.e^{e^{(1/\epsilon)^4}}$ as:

$$\epsilon^\omega.e^{e^{(1/\epsilon)^4}}=\frac{e^{e^{(1/\epsilon)^4}}}{(1/(\epsilon^4))^{\omega'}}.\epsilon^r=\frac{e^{e^X}}{X^{\omega'}}.\epsilon^r$$

where $X = \frac{1}{\epsilon^4}$ and $1 \ll X$. Or we know that $X^{\omega'} \ll e^X \Longrightarrow X^{\omega'} \ll e^{e^X}$.

- If $\epsilon \in [0.1, 0.63[$, we obtain $\epsilon^r \geq 0.001$ and $e^X > 8.8e + 4342$, it follows that $\epsilon^\omega.e^{e^{\left(\frac{1}{\epsilon^4}\right)}} > 1$ and we obtain a contradiction and the inequality (11) is true.

- Now we consider $0 < \epsilon < 0.1$, when $\epsilon \longrightarrow 0^+, K(\epsilon) \longrightarrow +\infty$ and the inequality (11) becomes $+\infty \leq +\infty$ and the abc conjecture is true.

- For ϵ very small $\in]0, 0.10[$, e^{e^X} becomes very large, then $8.8e + 4342 \ll e^{e^X}$ and $1 \ll \frac{e^{e^X}}{X^{\omega'}}.\epsilon^r$, it follows a contradiction, then the equation (11) is true.

Finally, the choice of the constant $K(\epsilon) = 1.2e^{e^{\left(\frac{1}{\epsilon}\right)^4}}$ is acceptable for $\epsilon \in]0, 0.63[$. As we assume that the conjecture $c < R^{1+0.63}$ is true, we adopt $K(\epsilon) = 1.2$ for $\epsilon \geq 0.63$, and the *abc* conjecture is true for all $\epsilon > 0$.

The proof of the *abc* conjecture is finished.

Q.E.D

□

We give below some numerical examples.

3 Examples

3.1 Example 1. of Eric Reyssat

We give here the example of Eric Reyssat [1], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{12}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$.
$\omega = 4 \Longrightarrow \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} = 6\,437\,590.238 > 6\,436\,343$, $B = 0.86 < w = 4$.

$\epsilon = 0.5 \Longrightarrow \epsilon^\omega.e^{e^{\left(\frac{1}{\epsilon}\right)^4}} = 9.446e + 109 > 1 \Longrightarrow (1 - B) < 1$.

$\epsilon = 0.01 \Longrightarrow \epsilon^\omega = \epsilon^4 = 10^{-8} \ll e^{\left(\frac{1}{\epsilon}\right)^4}$ then $(1 - B) < 1$.

3.2 Example 2. of Nitaj

See [4]:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110$$

$\omega = 10 \Longrightarrow \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} = 7\,794\,478\,289\,809\,729\,132\,015,590 > 613\,474\,845\,886\,230\,468\,750$, $B = 0.9927 \ll (w = 10)$.

$\epsilon = 0.5 \Longrightarrow \epsilon^\omega = \epsilon^{10} = 0.009765625 \ll e^{1/(\epsilon^4)} \Longrightarrow (1 - B) < 1$.

$\epsilon = 0.001 \Longrightarrow \epsilon^\omega = \epsilon^{10} = 10^{-30}$, $1/(\epsilon^4) = 10^{12} \Longrightarrow \epsilon^{10}.e^{10^{12}} > 1 \Longrightarrow (1 - B) < 1$.

4 Conclusion

Assuming $c < R^{1.63}$ is true, and the explicit *abc* conjecture of Alan Baker true, we can announce the important theorem:

Theorem 4. *Assuming $c < R^{1.63}$ is true and the explicit abc conjecture of Alan Baker true then the abc conjecture is true:*
For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{13}$$

where K is a constant depending only of ϵ. For $\epsilon \in]0, 0.63[$, $K(\epsilon) = 1.2e^{e^{\left(\frac{1}{\epsilon}\right)^4}}$ and $K(\epsilon) = 1.2$ if $\epsilon \geq 0.63$.

Author contributions

This is the author contribution text.

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References

[1] M. Waldschmidt: (2013). *On the abc Conjecture and some of its consequences*, presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9, 2013.

[2] B. De Smit: https://www.math.leidenuniv.nl/ desmit/abc/. Accessed December 2020.

[3] P. Mihăilescu: (2014). *Around ABC*, European Mathematical Society Newsletter, **N° 93**, 29-34.

[4] A. Nitaj: (1996). *Aspects expérimentaux de la conjecture abc*. Séminaire de Théorie des Nombres de Paris (1993-1994), London Math. Soc. Lecture Note Ser., Vol **n°235**, 145-156, Cambridge Univ. Press.

[5] A. Baker: (2004). *Experiments on the abc-conjecture*, Publ. Math. Debrecen **65:3-4**, pp. 253-260.

[6] A. Ben Hadj Salem. (2025). The Explicit *abc* Conjecture of Alan Barker Is True. 5 pages. Submitted to the journal Quaestiones Mathematicae, April 2025.