Proof of the Riemann hypothesis

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Abstract

The Riemann hypothesis states that the real part of all non-trivial zeros of the Riemann zeta function is $\frac{1}{2}$ in the critical strip. In this paper we proved the hypothesis by using the properties of the Riemann zeta functional equation at $\zeta(s) = 0$ and also using the integral representation (Mellin transformation) of the Riemann zeta function in the critical strip.

1. Introduction

The Riemann hypothesis proposed by Bernhard Riemann in his 1859 paper asserts that the real part of all non-trivial zeros of the Riemann zeta function is $\frac{1}{2}$ on the critical line where $\zeta(s) = 0$. Proving the hypothesis could have a profound consequence in number theory and also help in understanding the distribution of prime numbers. In this proof, we show that indeed all the non-trivial zeros lie on the critical line in the critical strip of the complex plane.

2. Proof of the Riemann Hypothesis

The functional equation of the Riemann zeta function shows that;

$$\zeta(s) = 2^s \pi^{(s-1)} \sin\left(\frac{\pi s}{2}\right) \zeta(1-s)\Gamma(1-s).$$

For $\zeta(s) = 0$, whereby s is a non-trivial zero of the analytically continued Riemann zeta function. Take that s is a complex number $s = a + \boldsymbol{i}b$, for a and b being real numbers with $b \neq 0$ and for a in the region $0 < a < 1$;

$$\zeta(s) = \zeta(1-s) = 0.$$

From the Mellin transformation of the Riemann zeta function in the critical strip;

$$\zeta(s) = \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt.$$

When $\zeta(s) = 0$,

$$\int_0^\infty \frac{t^{s-1}}{e^t - 1} dt = 0. (equation\ 1)$$

Let's consider $\zeta(1-\bar{s}) = \zeta(\bar{s}) = 0$ for the conjugate non-trivial zero $\bar{s} = a - \boldsymbol{i}b$.

$$\zeta(1-\bar{s}) = \frac{1}{(1-2^{\bar{s}})\Gamma(1-\bar{s})}\int_0^\infty \frac{t^{-\bar{s}}}{e^t-1}\,dt$$

$$\int_0^\infty \frac{t^{-\bar{s}}}{e^t-1}\,dt = 0.\,(equation\ 2)$$

But $(equation\ 1) = (equation\ 2)$ since both equations are equal to zero. Implying that,

$$\int_0^\infty \frac{t^{s-1}}{e^t-1}\,dt = \int_0^\infty \frac{t^{-\bar{s}}}{e^t-1}\,dt.\,(equation\ 3)$$

We observe that the powers(constants) on both integrands must always be equal to each other since both integrals have the same limits and integrands. A proof of this claim can be illustrated using a simple integral shown in the lemma below.

3. Lemma

Consider two equal integrals with the same limits and same integrands with no constant multiplied to the integrand i.e. An equation in the form;

$$\int_c^d t^m\,dt = \int_c^d t^n\,dt.$$

The goal of the lemma is to show that if two integrals are equal to each other with the same limits and integrands with no constant multiplied to the integrand(as shown in the above), the power constants raised to the integrands of the integrals should also be equal.

Further more and this is very important: The form of the equation being given above clearly shows that the integrand is **not** multiplied to any other constant since the lemma wouldn't hold for particular integral equations despite having the same limits and integrands. An equation to highlight about here is shown below with two equal integrals with the same limits and integrands though the power raised to the integrands is not the same (for this case the powers are 1 and 2) and it leads to the same answer and equality holds but this is due to the constant multiplied to the variable "t".

$$\int_0^1 \left(\frac{3}{2}t\right)^{\mathbf{2}} dt = \int_0^1 \left(\frac{3}{2}t\right)^{\mathbf{1}} dt = 0.75$$

From such an observation, we can proceed to prove the lemma for the equality of the powers raised to the integrands of equal integrals with the same limits but with no constant multiplied to the variable.

4. Proof of the lemma

We have;

$$\int_c^d t^m\,dt = \int_c^d t^n\,dt.$$

Solving the integrals on both sides of the equation gives;

$$\left(\frac{d^{m+1}}{m+1}-\frac{c^{m+1}}{m+1}\right)=\left(\frac{d^{n+1}}{n+1}-\frac{c^{n+1}}{n+1}\right)$$

Setting m=n (substituting m with n) clearly gives;

$$\left(\frac{d^{n+1}}{n+1}-\frac{c^{n+1}}{n+1}\right)=\left(\frac{d^{n+1}}{n+1}-\frac{c^{n+1}}{n+1}\right).$$

Since the equation above is satisfied, the lemma is proven to be correct.

5. Continuation of the proof

From (*equation* 3), we can now claim that the power constants $s-1$ and $-\bar{s}$ must be equal for the equation to be satisfied and to follow the lemma we have proven.

Hence from

$$\int_0^\infty \frac{t^{s-1}}{e^t-1}dt=\int_0^\infty \frac{t^{-\bar{s}}}{e^t-1}dt\ (equation\ 3),$$

We deduct that;

$$s-1=-\bar{s}.$$

$$(a+\boldsymbol{i}b)-1=-(a-\boldsymbol{i}b)$$

$$a=\frac{1}{2}.$$

The real part of all the non-trivial zeros of the Riemann zeta function is $\frac{1}{2}$.

6. Testing the result $a=\frac{1}{2}$ to check for validity of (equation 3) we had acquired earlier.

This step can also act as a proof by contradiction and so we plug in $a=\frac{1}{2}$ *into the equation to see if it satisfies (equation 3) to give a correct mathematically standing statement.*

In (equation 3), we had;

$$\int_0^\infty \frac{t^{s-1}}{e^t-1}dt=\int_0^\infty \frac{t^{-\bar{s}}}{e^t-1}dt.$$

$$Where\ \bar{s}=a-\boldsymbol{i}b\ and\ s=a+\boldsymbol{i}b$$

$$\int_0^\infty \frac{t^{(a+\boldsymbol{i}b)-1}}{e^t-1}dt=\int_0^\infty \frac{t^{-(a-\boldsymbol{i}b)}}{e^t-1}dt.$$

Setting $a=\frac{1}{2}$, we have;

$$\int_0^{\infty} \frac{t^{\left(\frac{1}{2}+ib\right)-1}}{e^t-1}\,dt = \int_0^{\infty} \frac{t^{-\left(\frac{1}{2}-ib\right)}}{e^t-1}\,dt.$$

Which eventually gives;

$$\int_0^{\infty} \frac{t^{\left(-\frac{1}{2}+ib\right)}}{e^t-1}\,dt = \int_0^{\infty} \frac{t^{\left(-\frac{1}{2}+ib\right)}}{e^t-1}\,dt.\,(equation\ 4)$$

$$0 = 0.$$

$$(Since\ the\ integrals\ came\ from\ \zeta\,(\mathrm{s}) = \zeta\,(1-\bar{s}) = 0.)$$

From the result we have acquired in ($equation$ 4) and observing that the equation is well satisfied, it simply implies that the condition that the real part of all the non-trivial zeros of the Riemann zeta function to be $\frac{1}{2}$ is true and valid.

7. Conclusion

The Riemann hypothesis is correct since we have proven that the real part of all the non-trivial zeros of the Riemann zeta function must be $\frac{1}{2}$ on the critical line in the critical strip. From this proof of the Riemann hypothesis, we have used the Mellin transformation of the Riemann zeta function in the critical strip, utilized the properties of the zeta functional equation at $\zeta\,(\mathrm{s}) = 0$ and used a lemma that we later proved to show the relationship (equality) between the constant powers raised to integrands of two equal integrals with similar limits and integrands.

References

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