Three Particle Amplitudes with a Massless Boson

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It seems possible to obtain massive three particle amplitudes with a massless boson including gravity from one formula for only massive particles similar to the case of massless three particle amplitudes. To achieve this one has to assume tree unitarity and non chiral interactions. The x-factor needed for amplitudes with two equal mass particles and one massless boson can be derived in this way. We give an interpretation of massive spinor brackets with negative exponents appearing in amplitudes with gravity and show that they lead to appropriate little group transformations.

1. Introduction

Amplitudes in particle physics are most easily calculated in the spinor helicity formalism, see for example the reviews in [1-4]. Three particle amplitudes are the building blocks from which higher point amplitudes can be calculated with gluing or by recursion. For amplitudes with only massless particles one uses the helicity spinors $\lambda_\alpha, \tilde{\lambda}_{\dot\alpha}$ with negative and positive helicity. For massive amplitudes one uses massive helicity spinors or spin spinors introduced in [5],[6],[7]. For massless particles the little group is U(1) and for massive particles the little group is SU(2). Massive particles are described by spinors $\lambda^I_\alpha, \tilde{\lambda}^I_{\dot\alpha}$ where $\alpha, \dot\alpha$ denote the $SL(2,\mathbb{C})$ indices and I, J the SU(2) spin indices. Amplitudes in this formalism have been investigated in [8-14].

It is well known [1-4] that there exists a formula comprising all massless three point amplitudes, which are determined by three point kinematics and momentum conservation. Massive three point amplitudes and their high energy limit were thoroughly investigated in [10],[11],[12] and classified in [13],[14]. One wonders immediately if a formula exists as well, comprising all massive three particle amplitudes. A suggestion was made in [14], which however had some limitations and excluded several amplitudes as discussed there. Here we propose that it seems possible to bypass some of these limitations and show how one can obtain many amplitudes from one formula for only massive amplitudes.

We confine the discussion to observed particles of spin 0, 1/2, 1, 2. In section 2 we discuss the notation and the general formulas for massless and massive amplitudes. In section 3 we investigate several massive amplitudes and how amplitudes with one massless boson can be derived from them under the assumptions of tree unitarity and vector-like interactions. Interestingly also amplitudes with gravitons can be obtained by this formula and the x-factor emerges. In section 4 we show that brackets with massive spinors and negative exponents can give the required little group transformations.

2. General Massless and Massive Three Particle Amplitudes

We investigate massive spinors introduced in [7] and take over the notation of [12-16]. Later in [17],[18] a slightly refined notation was introduced, replacing $|n] \to -|n]$ and $[n| \to -[n|$. This has the advantage of avoiding several minus signs and σ (=helicity category) in spinors and amplitudes, where only upper SU(2) indices appear. In order to make the paper self contained, we display this representation in appendix A.

We denote massive spinors $\lambda^I_\alpha = |i^I\rangle$ and $\tilde{\lambda}^{\dot\alpha I} = |i^I]$ together as $|i^I)_\sigma$ [15],[16]:

$$|\mathbf{i})_\sigma = |i^I)_\sigma = \begin{Bmatrix} |i^{\dot\alpha,I}] & \sigma=+ \\ |i^I_\alpha\rangle & \sigma=- \end{Bmatrix}, \quad (\mathbf{i}|_\sigma = (i^I|_\sigma = \begin{Bmatrix} [i^I_{\dot\alpha}| & \sigma=+ \\ \langle i^{\alpha,I}| & \sigma=- \end{Bmatrix} \tag{1}$$

The sign σ denotes the helicity category [13] of the massive spinor and corresponds to the helicity sign of the massless spinor remaining in the high energy limit. Contractions are only possible between spinors with the same sign σ

$$\left(\mathbf{i}\ \mathbf{j}\right)_\sigma = \begin{Bmatrix} \left[i^I \ \ j^J \right] \ \sigma = + \\ \left\langle i^I \ \ j^J \right\rangle \ \sigma = - \end{Bmatrix} \tag{2}$$

Massless spinors are denoted in the same way, where σ is the helicity sign, the formulas are obtained from above by unbolding and omitting the index I. The momentum of a massive particle with momentum $\mathbf{p_i}$ is given as

$$\mathbf{p_i} = \sigma \left| i^I \right)_{-\sigma} \left(i_I \right|_\sigma = \begin{Bmatrix} +\left| i^I \right\rangle \left[i_I \right| = \mathbf{p}_{i\alpha\dot{\alpha}}\,,\ \sigma = + \\ -\left| i^I \right] \left\langle i_I \right| = \bar{\mathbf{p}}_i^{\dot{\alpha}\alpha}\,,\ \sigma = - \end{Bmatrix} \tag{3}$$

The relations between massive spinors given in [12],[14],[17],[18] can be now written in a compact form where $\left(a, b = \alpha, \beta \text{ or } \dot{\beta}, \dot{\alpha}\right)$.

$$\begin{aligned} &\left(i^J \ \ i^K \right)_\sigma = \sigma m_i \epsilon^{JK}\,,\ \left(i_J \ \ i_K \right)_\sigma = -\sigma m_i \epsilon_{JK}\,,\ \left(i^J \ \ i_K \right)_\sigma = -\sigma m_i \delta^J_K\,,\ \left(i_J \ \ i^K \right)_\sigma = \sigma m_i \delta^K_J \\ &\left(i^J \ \ i_J \right)_\sigma = -\left(i_J \ \ i^J \right)_\sigma = -\sigma 2 m_i\,,\ \left| i^J \right)_\sigma \left(i_J \right|_\sigma = -\left| i_J \right)_\sigma \left(i^J \right|_\sigma = \sigma m_i \delta^b_a \\ &\mathbf{p}_i = \sigma \left| i^I \right)_{-\sigma} \left(i_I \right|_\sigma,\ \mathbf{p}_i \left| i^I \right)_{-\sigma} = m_i \left| i^I \right)_\sigma,\ \left(i^I \right|_{-\sigma} \mathbf{p}_i = -m_i \left(i^I \right|_\sigma,\ \varepsilon^{IJ} = -\varepsilon_{IJ} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{4}$$

We now introduce the following compact notation, mirror spinors are obtained by $\left|\ \ \right) \to \left(\ \ \right|$:

$$\begin{aligned} &\left| i^I \right)_\sigma = \left(\left| i \right)_\sigma \ \ \left| n_i \right)_\sigma \right),\ \left| i^I \right)_{-\sigma} = \left(\left| n_i \right)_{-\sigma} \ \ \left| i \right)_{-\sigma} \right) \\ &\left| i_I \right)_\sigma = \sigma\left(-\left| n_i \right)_\sigma \ \ \left| i \right)_\sigma \right),\ \left| i_I \right)_{-\sigma} = \sigma\left(-\left| i \right)_{-\sigma} \ \ \left| n_i \right)_{-\sigma} \right) \end{aligned} \tag{5}$$

The two-vectors in (5) must be understood in the following way: for $\sigma = +$ the two entries of the vector are in the right order, while for $\sigma = -$ the two entries must be swapped. Therefore we have for example $\left| i^I \right)_\sigma = \left(\left| i \right)_\sigma \ \ \left| n_i \right)_\sigma \right) = \left(\left| i \right] \ \ \left| n_i \right] \right)\big|_{\sigma=+}, \left(\left| n_i \right\rangle \ \ \left| i \right\rangle \right)\big|_{\sigma=-}$ and similarly for the mirror spinors. Massive spinors with index $-\sigma$ in (5), which are needed in amplitudes, are also obtained by swapping the two entries. For the helicity sign defined as $h_\sigma = \hat{\mathbf{p}} \cdot \vec{\sigma}$ one obtains that (conjugate) square i spinors have positive helicity sign and (conjugate) angle spinors negative helicity sign, which is reversed for n_i spinors. This suggests that the SU(2) indices I,J should run over $+,-$ [12]. One can check that all spin-spinors in appendix A (see also [17],[18]), are correctly described by (5). From the explicit representation one gets: $\left(i \ \ n_i \right)_\sigma = m_i$. Spinor contractions with only with upper indices appearing in amplitudes avoid any σ or minus signs and can now be written as:

$$\left(\mathbf{i}\ \mathbf{j}\right)_\sigma = \left(i^I \ \ j^J \right)_\sigma = \begin{pmatrix} \left(i \ \ j \right)_\sigma & \left(i \ \ n_j \right)_\sigma \\ \left(n_i \ \ j \right)_\sigma & \left(n_i \ \ n_j \right)_\sigma \end{pmatrix} = \begin{pmatrix} [i\ j] & [i\ n_j] \\ [n_i\ j] & [n_i\ n_j] \end{pmatrix}_{\sigma=+}, \begin{pmatrix} \langle n_i\ n_j \rangle & \langle n_i\ j \rangle \\ \langle i\ n_j \rangle & \langle i\ j \rangle \end{pmatrix}_{\sigma=-} \tag{6}$$

In the high energy (HE) limit we have $\left| i \right)_\sigma \sim \sqrt{E}$ and $\left| n_i \right)_\sigma \sim m_i / \sqrt{E}$.

From momentum conservation for the general three point amplitude is given by $\mathbf{p_i} + \mathbf{p}_j + \mathbf{p}_k = 0$ or more explicitly $\left| i \right)_{-\sigma} \left(i \right|_\sigma - \left| n_i \right)_{-\sigma} \left(n_i \right|_\sigma + \left| j \right)_{-\sigma} \left(j \right|_\sigma - \left| n_j \right)_{-\sigma} \left(n_j \right|_\sigma + \left| k \right)_{-\sigma} \left(k \right|_\sigma - \left| n_k \right)_{-\sigma} \left(n_k \right|_\sigma = 0$ one obtains in the HE limit:

$$\left(j\ i \right)_{-\sigma} \left(i\ k \right)_\sigma \approx 0,\ \left(n_j\ i \right)_{-\sigma} \left(i\ k \right)_\sigma \approx m_j \left(j\ k \right)_\sigma,\ \left(n_k\ i \right)_{-\sigma} \left(i\ j \right)_\sigma \approx m_k \left(k\ j \right)_\sigma \tag{7}$$

In three point amplitudes with two equal mass particles $m_i = m_j = m$ and one massless boson $m_k = 0$ one needs the so called x-factor [7] to write for example amplitudes of the form $\mathcal{A}_3 = \left(\mathbf{i}\ \mathbf{j}\right)_{-\sigma} x^\sigma$ (σ is the helicity sign of the massless boson k, note that we write $\sigma = \pm 1$ in terms). The x-factor and its HE limit are given by:

$$x^{\sigma} = \frac{1}{x^{-\sigma}} = \frac{(\varsigma_{-\sigma}\,\mathbf{p}_i\; k_{\sigma})}{m(\varsigma\; k)_{-\sigma}} \approx \frac{(j\; k)_{\sigma}(k\; i)_{\sigma}}{m(j\; i)_{\sigma}} = \frac{m(j\; i)_{-\sigma}}{(j\; k)_{-\sigma}(k\; i)_{-\sigma}} \tag{8}$$

It is well known that any massless three point amplitude with total helicity $h = h_1 + h_2 + h_3$ can be written as [1-4]

$$\mathcal{A}_3(h_1,h_2,h_3) = \tilde{g}(1\;2)_{\sigma}^{\sigma(h-2h_3)}(2\;3)_{\sigma}^{\sigma(h-2h_1)}(3\;1)_{\sigma}^{\sigma(h-2h_2)},\; [\tilde{g}] = 1-\sigma h \tag{9}$$

Note that all spinor products must have the same helicity sign. This can be easily derived from momentum conservation in the form $|i)_{\sigma}(i|_{-\sigma} + |j)_{\sigma}(j|_{-\sigma} + |k)_{\sigma}(k|_{-\sigma} = 0$. Multiplying from left with $(i|_{\sigma}$ and from right with $|k)_{-\sigma}$ one obtains $(i\;j)_{\sigma}(j\;k)_{-\sigma} = 0$. If we take for example $(j\;k)_{-\sigma} = 0$ then the massless three point amplitude must be given by products of the form $(i\;j)_{\sigma}$ with the same helicity sign. Furthermore we note that the mass dimension of the coupling can be obtained from $[\mathcal{A}_3] = [\tilde{g}] + \sigma(h-2h_3+h-2h_1+h-2h_2) = [\tilde{g}] + \sigma h \overset{!}{=} 1$ as $[\tilde{g}] = 1-\sigma h$ since every spinor product has mass dimension one. One can easily check that all massless three particle amplitudes are correctly described by (9), see for example [10].

Now we proceed to the massive three point amplitude by simply bolding the spinors and using that $s = \sigma h,\; s_j = \sigma h_j$, where $s = s_1 + s_2 + s_3$ is the total spin and s_j the spin of particle j [14].

$$\mathcal{A}_3(s_1,s_2,s_3) = \tilde{g}(\mathbf{1}\;\mathbf{2})^{(s-2s_3)}(\mathbf{2}\;\mathbf{3})^{(s-2s_1)}(\mathbf{3}\;\mathbf{1})^{(s-2s_2)},\; [\tilde{g}] = 1-s \tag{10}$$

The derivation of the mass dimension of $\tilde{g}$ goes as above. Note that we have omitted the subscript σ (denoting here the helicity category) in the spinor products from (9). The reason is that for massive particles one cannot conclude from momentum conservation that all σ are equal. Multiplying $|\mathbf{i})_{\sigma}(\mathbf{i}|_{-\sigma} + |\mathbf{j})_{\sigma}(\mathbf{j}|_{-\sigma} + |\mathbf{k})_{\sigma}(\mathbf{k}|_{-\sigma} = 0$ from left with $(\mathbf{i}|_{\sigma}$ and from right with $|\mathbf{k})_{-\sigma}$ one sees that $(\mathbf{i}\;\mathbf{j})_{\sigma}(\mathbf{j}\;\mathbf{k})_{-\sigma} \neq 0$ since $(\mathbf{i}\;\mathbf{i})_{\sigma} \neq 0$ and $(\mathbf{k}\;\mathbf{k})_{-\sigma} \neq 0$ from (4). On the other hand tree unitarity requires that not all brackets in (10) can have the same sign σ.

According [14] the round brackets are $(\mathbf{i}\;\mathbf{j}) = \{[\mathbf{i}\;\mathbf{j}], \langle\mathbf{i}\;\mathbf{j}\rangle\}$ and $(\mathbf{i}\;\mathbf{j})^2 = \{[\mathbf{i}\;\mathbf{j}][\mathbf{i}\;\mathbf{j}], [\mathbf{i}\;\mathbf{j}]\langle\mathbf{i}\;\mathbf{j}\rangle, \langle\mathbf{i}\;\mathbf{j}\rangle\langle\mathbf{i}\;\mathbf{j}\rangle\}$ and a tree of possible three point amplitudes was shown there. Furthermore it was imposed in [14] that the expression in (10) is only sensible when the exponents are not negative, meaning that the condition $s - 2\max(s_j) \geq 0$ should be satisfied. This however has the consequence that several amplitudes involving gravity like $\mathcal{A}_3\left(\tfrac{1}{2},\tfrac{1}{2},2\right)$ or others like $\mathcal{A}_3(0,0,2)$ cannot be obtained from (10). In the following we try to show that this condition may be too restrictive and therefore like (9) containing all massless amplitudes, (10) could describe many more massive three particle amplitudes under certain assumptions. This of course requires a valid interpretation of the term $(\mathbf{i}\;\mathbf{j})^{-1}$ in amplitudes.

3. Amplitudes with a Massless Boson

We consider the following massive particles: scalar $\varphi(s=0)$, fermion $f(s=1/2)$, vector $V(s=1)$, tensor $T(s=2)$. With the convention $s_1 \leq s_2 \leq s_3$ the particle 3 with highest spin is in many cases a boson with spin one or two. Now we investigate the limit $m_3 \approx 0$ which is nontrivial as discussed in [13] section 3.1. Another possibility to achieve the limit $m_3 \approx 0$ would be to consider a very small mass m_3. To obtain the high energy limit in contact with photon, gluon and graviton amplitudes we have to impose the following assumptions:

A) tree unitarity (only renormalisable interactions)

B) non chiral, vector-like interactions

Renormalisable interactions occur for $(\mathbf{2}\;\mathbf{3})_{-\sigma}(\mathbf{3}\;\mathbf{1})_{\sigma} \overset{HE}{\to} (2\;3)_{-\sigma}(3\;1)_{\sigma} = (2_{-\sigma}3\,1_{\sigma}) = -(2_{-\sigma}(1+2)\,1_{\sigma}) = 0$ without energy growth and non renormalisable ones with quadratic energy growth for $(\mathbf{2}\;\mathbf{3})_{\sigma}(\mathbf{3}\;\mathbf{1})_{\sigma} \approx (2\;3)_{\sigma}(3\;1)_{\sigma} \sim E^2$.

A non chiral interaction is given by $[\mathbf{2\ 3}]\langle\mathbf{3\ 1}\rangle+\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}]$ while $g_L[\mathbf{2\ 3}]\langle\mathbf{3\ 1}\rangle+g_R\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}]$ is chiral one.

We begin with the amplitude with $s_1 = s_2 = 1/2, s_3 = 1, s = 2$ and $[\tilde{g}] = -1$ satisfying the condition $s - 2\max(s_j) \geq 0$. Inserting the spin values from above in (10) gives $A_3\left(\frac{1}{2},\frac{1}{2},1\right) = \tilde{g}(\mathbf{2\ 3})(\mathbf{3\ 1})$.
From $(\mathbf{2\ 3})(\mathbf{3\ 1}) = \{[\mathbf{2\ 3}],\langle\mathbf{2\ 3}\rangle\}\cdot\{[\mathbf{3\ 1}],\langle\mathbf{3\ 1}\rangle\} = \{[\mathbf{2\ 3}][\mathbf{3\ 1}],[\mathbf{2\ 3}]\langle\mathbf{3\ 1}\rangle,\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}],\langle\mathbf{2\ 3}\rangle\langle\mathbf{3\ 1}\rangle\}$ one retains under assumption A) only the second and third term and under assumption B) only the combination $[\mathbf{2\ 3}]\langle\mathbf{3\ 1}\rangle+\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}]$, which is symmetric under the exchange $\langle\ \rangle \leftrightarrow [\ ]$. Under these assumptions one can derive the following equation (see appendix B), which will be used several times here:

$$(\mathbf{2\ 3})(\mathbf{3\ 1}) \approx [\mathbf{2\ 3}]\langle\mathbf{3\ 1}\rangle+\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}] \approx m_3(\mathbf{1\ 2})_{-\sigma}\, x^{\sigma} \tag{11}$$

Inserting this in the amplitude one gets:

$$\mathcal{A}_3\left(\tfrac{1}{2},\tfrac{1}{2},1\right) = \tilde{g}(\mathbf{2\ 3})(\mathbf{3\ 1}) \approx g(\mathbf{1\ 2})_{-\sigma}\, x^{\sigma} \tag{12}$$

Here $\tilde{g}$ is the coupling of a non renormalisable interaction $\tilde{g} = g/m_3$ and g is a dimensionless coupling constant. The amplitude in (12) describes the interaction of two equal mass fermions with a massless spin one boson (photon, gluon).

The amplitude with $s_1 = s_2 = 1/2, s_3 = 2, s = 3$ and $[\tilde{g}] = -2$ is excluded by the condition $s - 2\max(s_j) \geq 0$. From (10): $A_3\left(\frac{1}{2},\frac{1}{2},2\right) = \tilde{g}(\mathbf{1\ 2})^{-1}(\mathbf{2\ 3})^2\,(\mathbf{3\ 1})^2$. Now we interpret $(\mathbf{1\ 2})^{-1}$ as an term inverse to $(\mathbf{1\ 2}) = (\ 1^I\ \ 2^J\ )$. The main problem arising here is of course, how should one interpret this amplitude and what are the correct little group transformations. We discuss this more thoroughly in section 4. For the last two terms in the amplitude we have

$$(\mathbf{2\ 3})^2 = \{[\mathbf{2\ 3}],\langle\mathbf{2\ 3}\rangle\}\cdot\{[\mathbf{2\ 3}],\langle\mathbf{2\ 3}\rangle\} = \left\{[\mathbf{2\ 3}]^2,[\mathbf{2\ 3}]\langle\mathbf{2\ 3}\rangle,\langle\mathbf{2\ 3}\rangle^2\right\}$$

$$(\mathbf{3\ 1})^2 = \{[\mathbf{3\ 1}],\langle\mathbf{3\ 1}\rangle\}\cdot\{[\mathbf{3\ 1}],\langle\mathbf{3\ 1}\rangle\} = \left\{[\mathbf{3\ 1}]^2,[\mathbf{3\ 1}]\langle\mathbf{3\ 1}\rangle,\langle\mathbf{3\ 1}\rangle^2\right\}$$

Multiplying them would in general give nine terms, but from condition A) only the following three terms survive $\left\{[\mathbf{2\ 3}]^2\langle\mathbf{3\ 1}\rangle^2,[\mathbf{2\ 3}]\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}]\langle\mathbf{3\ 1}\rangle,\langle\mathbf{2\ 3}\rangle^2[\mathbf{3\ 1}]^2\right\}$. The only combination valid for a non chiral interaction symmetric under exchange of $\langle\ \rangle \leftrightarrow [\ ]$ is $\left([\mathbf{2\ 3}]\langle\mathbf{3\ 1}\rangle+\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}]\right)^2$. Therefore we obtain together with (11)

$$\mathcal{A}_3\left(\tfrac{1}{2},\tfrac{1}{2},2\right) \approx \tilde{g}(\mathbf{1\ 2})^{-1}\left([\mathbf{2\ 3}]\langle\mathbf{3\ 1}\rangle+\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}]\right)^2 \approx \tilde{g}(\mathbf{1\ 2})^{-1} m_3^2(\mathbf{1\ 2})_{-\sigma}^2\, x^{2\sigma}.$$

Now we make the bold assumption that $(\mathbf{1\ 2})^{-1}$ somehow cancels $(\mathbf{1\ 2})_{-\sigma}$. This seems justified by the final result, but is discussed more thoroughly in section 4. Then we obtain by putting $\tilde{g} = m/m_3^2 m_P$, where $m = m_1 = m_2$ and m_P is the Planck mass, the following result for the amplitude:

$$\mathcal{A}_3\left(\tfrac{1}{2},\tfrac{1}{2},2\right) = \tilde{g}(\mathbf{1\ 2})^{-1}(\mathbf{2\ 3})^2\,(\mathbf{3\ 1})^2 \approx \frac{m}{m_P}(\mathbf{1\ 2})_{-\sigma}\, x^{2\sigma} \tag{13}$$

which agrees with the amplitude for the interaction of two massive fermions with a massless graviton [10] .

The amplitude $s_1 = s_2 = 0, s_3 = 1, s = 1$, $[\tilde{g}] = 0$ violates $s - 2\max(s_j) \geq 0$. With $\tilde{g} = m_\varphi/m_3$ one gets from (10),(11)

$$\mathcal{A}_3(0,0,1) = \tilde{g}(\mathbf{1\ 2})^{-1}(\mathbf{2\ 3})(\mathbf{3\ 1}) \approx \tilde{g}\, m_3 x^{\sigma} = m_\varphi x^{\sigma} \tag{14}$$

The high energy limit for different scalars [20] is obtained by unbolding the spinors in (14) or with x^{σ} from (8).

The amplitude $s_1 = s_2 = 0, s_3 = 2, s = 2$, $[\tilde{g}] = -1$ violates the condition $s - 2\max(s_j) \geq 0$. With $\tilde{g} = m_\varphi^2/m_3^2\, m_P$ one gets

$$\mathcal{A}_3(0,0,2) = \tilde{g}(\mathbf{1\ 2})^{-2}(\mathbf{2\ 3})^2\,(\mathbf{3\ 1})^2 \approx \tilde{g}\, m_3^2 x^{2\sigma} = \frac{m_\varphi^2}{m_P} x^{2\sigma} \tag{15}$$

Again the high energy limit is obtained by unbolding the spinors or with x^{σ} from (8).

The amplitude $s_1 = s_2 = 1, s_3 = 2, s = 4$, $[\tilde{g}] = -3$ satisfies the condition $s - 2\max(s_j) \geq 0$. With $\tilde{g} = 1/m_3^2 m_P$ and (11)

$$\mathcal{A}_3(1,1,2) = \tilde{g}(\mathbf{2}\ \mathbf{3})^2(\mathbf{3}\ \mathbf{1})^2 \approx \tilde{g} m_3^2 (\mathbf{1}\ \mathbf{2})_{-\sigma}^2 x^{2\sigma} = \frac{1}{m_P}(\mathbf{1}\ \mathbf{2})_{-\sigma}^2 x^{2\sigma} \tag{16}$$

The last results agree with the corresponding amplitudes in [10].

For the amplitude $s_1 = s_2 = s_3 = 1, s = 3$ with $[\tilde{g}] = -2$ it was argued in [14], that the ansatz (10) also fails if $s - 2\max(s_j) > 0$. The reason is that there exists a relation between the possible spinor structures. Nevertheless we obtain from (10) and (11) in agreement with [10] the following amplitude, describing the interaction of two massive bosons and a massless one with $m = m_1 = m_2$ and $\tilde{g} = g/m_3 m$

$$\mathcal{A}_3(1,1,1) = \tilde{g}(\mathbf{1}\ \mathbf{2})(\mathbf{2}\ \mathbf{3})(\mathbf{3}\ \mathbf{1}) \approx \tilde{g} m_3 (\mathbf{1}\ \mathbf{2})_{-\sigma}^2 x^{\sigma} = \frac{g}{m}(\mathbf{1}\ \mathbf{2})_{-\sigma}^2 x^{\sigma} \tag{17}$$

Next we investigate the amplitude $s_1 = s_2 = s_3 = 2, s = 6$ with $[\tilde{g}] = -5$ and consider at first $m_3 \approx 0$ which gives $\mathcal{A}_3(2,2,2) = \tilde{g}(\mathbf{1}\ \mathbf{2})^2(\mathbf{2}\ \mathbf{3})^2(\mathbf{3}\ \mathbf{1})^2 \approx \tilde{g}(\mathbf{1}\ \mathbf{2})_{-\sigma}^2 m_3^2 (\mathbf{1}\ \mathbf{2})_{-\sigma}^2 x^{2\sigma}$. Now we can also take $m_{1,2} \approx 0$ by simply unbolding the brackets $(\mathbf{1}\ \mathbf{2})$ and using the high energy expression for x from (8) in the form $x^{\sigma} = 1/x^{-\sigma} \approx m(2\ 1)_{-\sigma}/(2\ 3)_{-\sigma}(3\ 1)_{-\sigma}$ where $m = m_1 = m_2$. We then finally find for all massless particles and by putting $\tilde{g} = 1/m_3^2 m^2 m_P$ in agreement with the three graviton amplitude [10]

$$\mathcal{A}_3(2,2,2) = \tilde{g}(\mathbf{1}\ \mathbf{2})^2(\mathbf{2}\ \mathbf{3})^2(\mathbf{3}\ \mathbf{1})^2 \approx \tilde{g} m_3^2 m^2 (1\ 2)_{-\sigma}^6/(2\ 3)_{-\sigma}^2(3\ 1)_{-\sigma}^2 = \frac{1}{m_P}(1\ 2)_{-\sigma}^6/(2\ 3)_{-\sigma}^2(3\ 1)_{-\sigma}^2 \tag{18}$$

A couple of other amplitudes also come out directly from the formula (10) without additional assumptions. The term $(\mathbf{i}\ \mathbf{j}) = \{[\mathbf{i}\ \mathbf{j}], \langle \mathbf{i}\ \mathbf{j}\rangle\}$ can give rise to several structures. Examples are the following amplitudes describing the coupling of a scalar to fermions, spin 1 bosons and a spin 2 boson.

$\mathcal{A}_3(0,\tfrac{1}{2},\tfrac{1}{2}) = \tilde{g}(\mathbf{2}\ \mathbf{3})$, $\mathcal{A}_3(0,1,1) = \tilde{g}(\mathbf{2}\ \mathbf{3})^2$, $\mathcal{A}_3(0,2,2) = \tilde{g}(\mathbf{2}\ \mathbf{3})^4$.

4. Interpretation of $(\mathbf{i}\ \mathbf{j})^{-1}$ in Amplitudes

First we investigate what is meant by the term $(\mathbf{i}\ \mathbf{j})^{-1} = (i^I\ j^J)^{-1}$. Of course the term $(i^I\ j^J)$ could be written as a matrix, but it is not a matrix. In the amplitude $(\mathbf{1}\ \mathbf{2})^a(\mathbf{2}\ \mathbf{3})^b(\mathbf{3}\ \mathbf{1})^c$ one must symmetrise over the implicit upper indices for spin $s \geq 1$, since a massive particle with spin s has $2s+1$ independent components and thereby can be described by a totally symmetric tensor $T^{I_1..I_{2s}}$, where the I_k run over 1,2 or $+,-$. The term $(1^I\ 2^J)^{-1}$ should cancel the term $(1^I\ 2^J)$. By inspecting the relations in (4) $|i^J)_\sigma(i_J|_\sigma = -|i_J)_\sigma(i^J|_\sigma = \sigma m_i \delta_a^b$, $(i^J\ i_K)_\sigma = -\sigma m_i \delta_K^J$ and their pendants for $\sigma \to -\sigma$ one sees that $(i^I\ j^J)_\sigma(j_J\ i_K)_\sigma = \sigma m_j (i^I\ i_K)_\sigma = \sigma m_j \cdot -\sigma m_i \delta_K^I = -m_i m_j \delta_K^I$ and therefore

$$(i^I\ j^J)_\sigma^{-1} = (j_J\ i_K)_\sigma / -m_i m_j \tag{19}$$

Now we consider the amplitude $A_3(\tfrac{1}{2},\tfrac{1}{2},2) = \tilde{g}(\mathbf{1}\ \mathbf{2})^{-1}(\mathbf{2}\ \mathbf{3})^2(\mathbf{3}\ \mathbf{1})^2$ describing two massive fermions interacting with a massive graviton. This amplitude should have the following little group transformations, where the symmetrisation over $\{K,L,M,N\}$ is indicated by the curly brackets:

$$A_3(\tfrac{1}{2},\tfrac{1}{2},2) \to W_{1I'}^{I} W_{2J'}^{J} W_{3K'}^{\{K} W_{3L'}^{L} W_{3M'}^{M} W_{3N'}^{N\}} A_3(\tfrac{1}{2},\tfrac{1}{2},2) \tag{20}$$

As discussed in section 3 the amplitude can be written as $(\mathbf{1}\ \mathbf{2})_{-\sigma}^{-1}\left((\mathbf{1}\ \mathbf{3})_{-\sigma}(\mathbf{3}\ \mathbf{2})_{\sigma}+(\mathbf{1}\ \mathbf{3})_{\sigma}(\mathbf{3}\ \mathbf{2})_{-\sigma}\right)^{2}$. Clearly the term $(\mathbf{1}\ \mathbf{2})_{-\sigma}^{-1}$ must cancel two of the implicit upper little group indices for the fermions $\mathbf{1},\mathbf{2}$ in order to satisfy (20). Using from (4) the relations $|i^{I})_{\sigma}(i_{I}|_{-\sigma}=-|i_{I})_{\sigma}(i^{I}|_{-\sigma}=-\sigma p_{i}$, $|i^{J})_{-\sigma}(i_{J}|_{-\sigma}=-|i_{J})_{-\sigma}(i^{J}|_{-\sigma}=-\sigma m_{i}\,\delta_{a}^{b}$ one obtains

$(2_J\ 1_I)_{-\sigma}\left((\mathbf{1}\ \mathbf{3})_{-\sigma}(\mathbf{3}\ \mathbf{2})_{\sigma}+(\mathbf{1}\ \mathbf{3})_{\sigma}(\mathbf{3}\ \mathbf{2})_{-\sigma}\right)=(\mathbf{3}\ \mathbf{2})_{\sigma}(2_J\ 1_I)_{-\sigma}(\mathbf{1}\ \mathbf{3})_{-\sigma}+(\mathbf{3}\ \mathbf{2})_{-\sigma}(2_J\ 1_I)_{-\sigma}(\mathbf{1}\ \mathbf{3})_{\sigma}=$

$-\sigma\cdot\sigma m_1(\mathbf{3}_{\sigma}\,p_2\,\mathbf{3}_{-\sigma})+-\sigma m_2\cdot-\sigma(\mathbf{3}_{-\sigma}\,p_1\,\mathbf{3}_{\sigma})=-m_1(\mathbf{3}_{\sigma}\,p_2\,\mathbf{3}_{-\sigma})+m_2(\mathbf{3}_{-\sigma}\,p_1\,\mathbf{3}_{\sigma})$

For $m=m_1=m_2$ and with $(\mathbf{3}_{\sigma}\,p_2\,\mathbf{3}_{-\sigma})=(\mathbf{3}_{-\sigma}\,p_2\,\mathbf{3}_{\sigma})$ (the implicit upper SU(2) indices are symmetrised) one gets for the amplitude (from (19) comes the minus sign)

$$A_3\left(\tfrac{1}{2},\tfrac{1}{2},2\right)=\tilde{g}(\mathbf{1}\ \mathbf{2})^{-1}(\mathbf{2}\ \mathbf{3})^{2}(\mathbf{3}\ \mathbf{1})^{2}=-\frac{\tilde{g}}{m}(\mathbf{3}_{-\sigma}|p_1-p_2|\mathbf{3}_{\sigma})\left((\mathbf{1}\ \mathbf{3})_{-\sigma}(\mathbf{3}\ \mathbf{2})_{\sigma}+(\mathbf{1}\ \mathbf{3})_{\sigma}(\mathbf{3}\ \mathbf{2})_{-\sigma}\right)\qquad(21)$$

Due to the symmetry under $\sigma\to-\sigma$ one can without restrictions assume $\sigma=+$. Furthermore in order to achieve the correct mass dimension of the amplitude we put again $\tilde{g}=m/m_P m_3^2$ as before (13) and write this as

$$A_3\left(\tfrac{1}{2},\tfrac{1}{2},2\right)=-\frac{1}{m_P\,m_3^2}\langle\mathbf{3}|p_1-p_2|\mathbf{3}]\left(\langle\mathbf{1}\ \mathbf{3}\rangle[\mathbf{3}\ \mathbf{2}]+[\mathbf{1}\,\mathbf{3}]\langle\mathbf{3}\ \mathbf{2}\rangle\right)\qquad(22)$$

This amplitude in (21),(22) has the little group transformations required in (20) and is anti-symmetric under exchange $1\leftrightarrow2$. We now investigate the high energy limit of (22) and at first the term $\langle\mathbf{3}|p_1-p_2|\mathbf{3}]$ with $p_i=|i\rangle[i|$.

From (5) and (A3) we see that $\langle\mathbf{3}|p_1-p_2|\mathbf{3}]=\begin{pmatrix}\langle n_3|p_1-p_2|3] & \langle n_3|p_1-p_2|n_3]\\ \langle3|p_1-p_2|3] & \langle3|p_1-p_2|n_3]\end{pmatrix}$.

The off-diagonal elements vanish in the HE limit and with (7) $\langle n_3\ 1\rangle[1\ 2]=m_3[3\ 2]$, $[n_3\ 1]\langle1\ 2\rangle=m_3\langle3\ 2\rangle$ as well as for $(1\leftrightarrow2)$ one obtains from

$\langle n_3|p_1|3]\approx\langle n_3\ 1\rangle[1\ 3]=\langle n_3\ 1\rangle[1\ 2][1\ 3]/[1\ 2]\approx m_3[3\ 2][1\ 3]/[1\ 2]$

$\langle n_3|p_2|3]\approx\langle n_3\ 2\rangle[2\ 3]=\langle n_3\ 2\rangle[2\ 1][2\ 3]/[2\ 1]\approx-m_3[3\ 1][2\ 3]/[1\ 2]$

and similarly from $\langle3|p_{1(2)}|n_3]=(-)m_3\langle2\ 3\rangle\langle3\ 1\rangle/\langle1\ 2\rangle$ together with (8)

$-\langle\mathbf{3}|p_1-p_2|\mathbf{3}]=-2m_3\begin{pmatrix}[2\ 3][3\ 1]/[1\ 2] & 0\\ 0 & \langle2\ 3\rangle\langle3\ 1\rangle/\langle1\ 2\rangle\end{pmatrix}=2m_3\dfrac{(2\ 3)_{\sigma}(3\ 1)_{\sigma}}{(2\ 1)_{\sigma}}\delta^{KL}=2m_3 m\,x^{\sigma}\delta^{KL}$.

The second term in (22) has as high energy limit see [10],[16]

$\langle\mathbf{1}\ \mathbf{3}\rangle[\mathbf{3}\ \mathbf{2}]+[\mathbf{1}\,\mathbf{3}]\langle\mathbf{3}\ \mathbf{2}\rangle\approx m_3(\mathbf{1}\ \mathbf{2})_{-\sigma}x^{\sigma}\approx m_3\begin{pmatrix}0 & -(3\ 1)_{\sigma}^{2}/(2\ 1)_{\sigma}\\ (2\ 3)_{\sigma}^{2}/(2\ 1)_{\sigma} & 0\end{pmatrix}$

Thereby the amplitude in (21) has as high energy limit

$A_3\left(\tfrac{1}{2},\tfrac{1}{2},2\right)\approx\dfrac{\tilde{g}}{m}\cdot2m_3(2\ 3)_{\sigma}(3\ 1)_{\sigma}/(2\ 1)_{\sigma}\,\delta^{KL}\cdot m_3\begin{pmatrix}0 & -(3\ 1)_{\sigma}^{2}/(2\ 1)_{\sigma}\\ (2\ 3)_{\sigma}^{2}/(2\ 1)_{\sigma} & 0\end{pmatrix}$

With $\tilde{g}=m/m_P m_3^2$ one obtains finally as high energy limit in agreement with [10],[16], where we have absorbed the factor 2 in the coupling (sign differences compared to [10] are due to another convention for $(i\ n_i)_{\sigma}$):

$$A_3\left(\tfrac{1}{2},\tfrac{1}{2},2\right)\approx\frac{1}{m_P}\begin{pmatrix}0 & -(3\ 1)_{\sigma}^{3}(2\ 3)_{\sigma}/(2\ 1)_{\sigma}^{2}\\ (2\ 3)_{\sigma}^{3}(3\ 1)_{\sigma}/(2\ 1)_{\sigma}^{2} & 0\end{pmatrix}\qquad(23)$$

The high energy limit together with the little group transformation given in (20) supports the view that (22) is the amplitude describing the interaction of fermions with a massive graviton. Directly from above we can obtain a formula for the x-factor expressed by massive spinors

$$x^{\sigma}\delta^{KL}=\frac{-1}{2m\,m_3}(\mathbf{3}_{-\sigma}|p_1-p_2|\mathbf{3}_{\sigma})\qquad(24)$$

5. Summary

We tried here to show that the application range of formula (10) first suggested in [14] might be greater than assumed there. With the assumptions of tree unitarity, vector-like interactions and a viable limit $m_3 \approx 0$ it seems possible to apply this formula for example to amplitudes with spins like $(0,0,1)$, $(\frac{1}{2},\frac{1}{2},1)$, $(1,1,1)$, $(0,0,2)$, $(\frac{1}{2},\frac{1}{2},2)$, $(1,1,2)$ which were partially excluded in [14]. We interpret $(\mathbf{1}\ \mathbf{2})^{-1}$ as the inverse term of $(\mathbf{1}\ \mathbf{2})_{-\sigma}$, which yields the correct final results for several amplitudes as shown in section 3. An advantage of the present ansatz is that one gains an additional explanation for three point amplitudes with one massless boson employing the x-factor and especially for amplitudes with gravitons. The x-factor according [7],[10] reproduces the high energy limits in the case of only massless particles. We also discuss the interpretation of $(\mathbf{i}\ \mathbf{j})_\sigma^{-1} = (i^I\ j^J)_\sigma^{-1} = (j_J\ i_K)_\sigma / -m_i m_j$ and show that this leads to the required little group transformations of the amplitude describing the interaction between fermions and a massive graviton.

Appendix A: Massive Spinor Representation

The explicit representation for massive spinors in [12-18] is based on the metric $(+---)$ and momentum

$$p^\mu = \begin{pmatrix} E & P\sin(\theta)\cos(\phi) & P\sin(\theta)\sin(\phi) & P\cos(\theta) \end{pmatrix} = \begin{pmatrix} E & Pc(s^* + s) & Pic(s^* - s) & P(cc - ss^*) \end{pmatrix} \tag{A1}$$

With the Pauli matrices we can write the momentum in bispinor form $p = p_\mu \sigma^\mu$ or $\overline{p} = p_\mu \overline{\sigma}^\mu$ using the equations $c = \cos(\theta/2)$, $s = \sin(\theta/2)\exp(i\phi)$, $s^* = \sin(\theta/2)\exp(-i\phi)$ with $cc + ss^* = 1$ which results in $(\sigma = \pm)$:

$$p = \begin{pmatrix} E - \sigma P(cc - ss^*) & -\sigma 2Pcs^* \\ -\sigma 2Pcs & E + \sigma P(cc - ss^*) \end{pmatrix} \tag{A2}$$

We write massive spinors in the 2-vector notation used in [19] see also [15],[16], which is better readable than enumerating all eight 2x2 matrices. We choose here the representation of [17],[18] which is identical to [16] with the exception that $|n] \to -|n]$ and $[n| \to -[n|$. This avoids minus signs and σ in spinors with upper indices appearing in amplitudes as can be seen in (16). Lowercase index spinors are obtained by $|i_I\rangle = \epsilon_{IJ} |i^J\rangle$ and mirror spinors by $|\ \rangle \to \langle\ |$ and $|\ ] \to [\ |$. One can obtain the expressions for $|i], |i\rangle, |n_i\rangle, |n_i]$ and its mirrors from the following equations.

$$|i^I] = (|i]\ \ |n_i]) = \left(\sqrt{E_i + P_i}\begin{pmatrix} c_i \\ s_i \end{pmatrix} \ \ \sqrt{E_i - P_i}\begin{pmatrix} -s_i^* \\ c_i \end{pmatrix} \right) \qquad |i^I\rangle = (|n_i\rangle\ \ |i\rangle) = \left(\sqrt{E_i - P_i}\begin{pmatrix} c_i \\ s_i \end{pmatrix} \ \ \sqrt{E_i + P_i}\begin{pmatrix} -s_i^* \\ c_i \end{pmatrix} \right) \tag{A3}$$

$$[i^I| = ([i|\ \ [n_i|) = \left(\sqrt{E_i + P_i}\begin{pmatrix} -s_i \\ c_i \end{pmatrix} \ \ -\sqrt{E_i - P_i}\begin{pmatrix} c_i \\ s_i^* \end{pmatrix} \right) \qquad \langle i^I| = (\langle n_i|\ \ \langle i|) = \left(-\sqrt{E_i - P_i}\begin{pmatrix} -s_i \\ c_i \end{pmatrix} \ \ \sqrt{E_i + P_i}\begin{pmatrix} c_i \\ s_i^* \end{pmatrix} \right)$$

$$|i_I] = (-|n_i]\ \ |i]) = \left(-\sqrt{E_i - P_i}\begin{pmatrix} -s_i^* \\ c_i \end{pmatrix} \ \ \sqrt{E_i + P_i}\begin{pmatrix} c_i \\ s_i \end{pmatrix} \right) \qquad |i_I\rangle = (-|i\rangle\ \ |n_i\rangle) = \left(-\sqrt{E_i + P_i}\begin{pmatrix} -s_i^* \\ c_i \end{pmatrix} \ \ \sqrt{E_i - P_i}\begin{pmatrix} c_i \\ s_i \end{pmatrix} \right)$$

$$[i_I| = (-[n_i|\ \ [i|) = \left(\sqrt{E_i - P_i}\begin{pmatrix} c_i \\ s_i^* \end{pmatrix} \ \ \sqrt{E_i + P_i}\begin{pmatrix} -s_i \\ c_i \end{pmatrix} \right) \qquad \langle i_I| = (-\langle i|\ \ \langle n_i|) = \left(-\sqrt{E_i + P_i}\begin{pmatrix} c_i \\ s_i^* \end{pmatrix} \ \ -\sqrt{E_i - P_i}\begin{pmatrix} -s_i \\ c_i \end{pmatrix} \right)$$

With $|i)_\sigma, |n_i)_\sigma$ defined in (6), the momentum (leading to (A2) for $\sigma = \pm$) is given by

$$\mathbf{p_i} = \sigma |i^I)_{-\sigma} (i_I|_\sigma = |i)_{-\sigma}(i|_\sigma - |n_i)_{-\sigma}(n_i|_\sigma \tag{A4}$$

One can check, that the relations in (4) are satisfied together with

$$(i\ n_i)_\sigma = m_i \tag{A5}$$

Appendix B: Massive Amplitudes and the x-Factor

First we note that $(\mathbf{2\ 3})(\mathbf{3\ 1})$ can be one of the following structures:

$(\mathbf{2\ 3})(\mathbf{3\ 1})=\{[\mathbf{2\ 3}],\langle\mathbf{2\ 3}\rangle\}\cdot\{[\mathbf{3\ 1}],\langle\mathbf{3\ 1}\rangle\}=\{[\mathbf{2\ 3}][\mathbf{3\ 1}],[\mathbf{2\ 3}]\langle\mathbf{3\ 1}\rangle,\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}],\langle\mathbf{2\ 3}\rangle\langle\mathbf{3\ 1}\rangle\}$

However assumption A) excludes the first and last possibility because their high energy behaviour is $\sim E^2$. For a chiral interaction we would obtain $g_L[\mathbf{2\ 3}]\langle\mathbf{3\ 1}\rangle+g_R\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}]$ [13]. From assumption B) we conclude that the only allowed term for a vector-like, non chiral interaction must be $[\mathbf{2\ 3}]\langle\mathbf{3\ 1}\rangle+\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}]$. With a massive polarisation defined by $\boldsymbol{\varepsilon}_3=\frac{\sqrt{2}}{m_3}|\,\mathbf{3}\rangle[\,\mathbf{3}\,|$ in [12],[13],[14] we can write this as

$$[\mathbf{2\ 3}]\langle\mathbf{3\ 1}\rangle+\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}]=\langle\mathbf{1\ 3}\rangle[\mathbf{3\ 2}]+\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}]=\frac{m_3}{\sqrt{2}}\left(\langle\mathbf{1}|\boldsymbol{\varepsilon}_3|\mathbf{2}]+\langle\mathbf{2}|\boldsymbol{\varepsilon}_3|\mathbf{1}]\right) \tag{B1}$$

With (A3) one derives for $\boldsymbol{\varepsilon}_3=\boldsymbol{\varepsilon}_3^{KL}$ with symmetrised indices K,L

$\boldsymbol{\varepsilon}_3^{KL}=\frac{\sqrt{2}}{m_3}|\,\mathbf{3}^K\rangle[\,\mathbf{3}^L\,|=\frac{\sqrt{2}}{m_3}\left\{\left(|\,n_3\rangle\ |\,3\rangle\right)_\sigma\circ\left([\,3|\ [\,n_3\,|\right)_\sigma\right\}=\frac{\sqrt{2}}{m_3}\left\{|\,n_3\rangle[\,3\,|,\frac{1}{2}\left(|\,3\rangle[\,3\,|+|\,n_3\rangle[\,n_3\,|\right),|\,3\rangle[\,n_3\,|\right\}.$

From $m_3=\langle 3\ n_3\rangle$ in the first term and $m_3=[\,3\ n_3\,]$ in the third term one obtains (B2) in the limit $m_3\approx 0$, $n_3\to\zeta$ by omitting the longitudinal component $\boldsymbol{\varepsilon}_3^{KL}\approx\{\varepsilon_3^+,\varepsilon_3^-\}$ with $\varepsilon_3^+=\sqrt{2}\,|\,\zeta\rangle[\,3\,|/\langle 3\ \zeta\rangle$ and $\varepsilon_3^-=\sqrt{2}\,|\,3\rangle[\,\zeta\,|/[\,3\ \zeta\,]$ since the longitudinal component proportional to p_3/m_3 becomes a pure gauge. Furthermore according [13] vector-like fermions do not interact with the longitudinal component and thereby $\boldsymbol{\varepsilon}_3^{KL}=\frac{\sqrt{2}}{m_3}|\,\mathbf{3}^K\rangle[\,\mathbf{3}^L\,|\approx\{\varepsilon_3^+,\varepsilon_3^-\}=\varepsilon_3^\sigma$.

$$[\mathbf{2\ 3}]\langle\mathbf{3\ 1}\rangle+\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}]\approx\frac{m_3}{\sqrt{2}}\left(\langle\mathbf{1}|\varepsilon_3^\sigma|\mathbf{2}]+\langle\mathbf{2}|\varepsilon_3^\sigma|\mathbf{1}]\right) \tag{B2}$$

Now with $\varepsilon_3^{+,-}$ the right side of (B2) give

$\frac{1}{\sqrt{2}}\left(\langle\mathbf{1}|\varepsilon_3^+|\mathbf{2}]+\langle\mathbf{2}|\varepsilon_3^+|\mathbf{1}]\right)=\frac{\langle\mathbf{1}\ \zeta\rangle[\,3\ \mathbf{2}\,]+\langle\mathbf{2}\ \zeta\rangle[\,3\ \mathbf{1}\,]}{\langle 3\ \zeta\rangle}$, $\frac{1}{\sqrt{2}}\left(\langle\mathbf{1}|\varepsilon_3^-|\mathbf{2}]+\langle\mathbf{2}|\varepsilon_3^-|\mathbf{1}]\right)=\frac{\langle\mathbf{1}\ 3\rangle[\,\zeta\ \mathbf{2}\,]+\langle\mathbf{2}\ 3\rangle[\,\zeta\ \mathbf{1}\,]}{[\,3\ \zeta\,]}$

From the Schouten identity $|\,\zeta\rangle\langle\mathbf{1\ 2}\rangle=-|\,\mathbf{1}\rangle\langle\mathbf{2}\ \zeta\rangle-|\,\mathbf{2}\rangle\langle\zeta\ \mathbf{1}\rangle$ multiplied from left with $[\,3\,|\mathbf{p}_1$ one derives $[\,3\,|\mathbf{p}_1|\,\zeta\rangle\langle\mathbf{1\ 2}\rangle=-[\,3\,|\mathbf{p}_1|\,\mathbf{1}\rangle\langle\mathbf{2}\ \zeta\rangle-[\,3\,|\mathbf{p}_1|\,\mathbf{2}\rangle\langle\zeta\ \mathbf{1}\rangle$. With $\mathbf{p}_1|\,\mathbf{1}\rangle=m|\,\mathbf{1}]$, $[\,3\,|\mathbf{p}_1|\,\mathbf{2}\rangle=-[\,3\,|\mathbf{p}_2|\,\mathbf{2}\rangle=-m[\,3\ \mathbf{2}\,]$ we have $[\,3\,|\mathbf{p}_1|\,\zeta\rangle\langle\mathbf{1\ 2}\rangle=-m[\,3\ \mathbf{1}\,]\langle\mathbf{2}\ \zeta\rangle+m[\,3\ \mathbf{2}\,]\langle\zeta\ \mathbf{1}\rangle=-m\left(\langle\mathbf{1}\ \zeta\rangle[\,3\ \mathbf{2}\,]+\langle\mathbf{2}\ \zeta\rangle[\,3\ \mathbf{1}\,]\right)$ and thereby

$\frac{\langle\mathbf{1}\ \zeta\rangle[\,3\ \mathbf{2}\,]+\langle\mathbf{2}\ \zeta\rangle[\,3\ \mathbf{1}\,]}{\langle 3\ \zeta\rangle}=\frac{[\,3\,|\mathbf{p}_1|\,\zeta\rangle\langle\mathbf{1\ 2}\rangle}{m\langle\zeta\ 3\rangle}=\langle\mathbf{1\ 2}\rangle x$. By a similar derivation ($m=m_1=m_2$)

$\frac{\langle\mathbf{1}\ 3\rangle[\,\zeta\ \mathbf{2}\,]+\langle\mathbf{2}\ 3\rangle[\,\zeta\ \mathbf{1}\,]}{[\,3\ \zeta\,]}=\frac{[\,\zeta\,|\mathbf{p}_1|\,3\rangle[\,\mathbf{1\ 2}\,]}{m[\,\zeta\ 3\,]}=[\,\mathbf{1\ 2}\,]x^{-1}$. Now we arrive finally at equation (11):

$$(\mathbf{2\ 3})(\mathbf{3\ 1})\approx[\mathbf{2\ 3}]\langle\mathbf{3\ 1}\rangle+\langle\mathbf{2\ 3}\rangle[\mathbf{3\ 1}]\approx m_3(\mathbf{1\ 2})_{-\sigma}x^\sigma \tag{B3}$$

Multiplying with $\tilde{g}=g/m_3$ gives then as in (12)

$$\tilde{g}(\mathbf{2\ 3})(\mathbf{3\ 1})\approx g(\mathbf{1\ 2})_{-\sigma}x^\sigma \tag{B4}$$

The symmetrised SU(2) indices K,L of the massive boson 3 collapsed to the two indices $\sigma=+,-$ of a massless boson.

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