SOME NEW IDENTITIES FOR PRODUCTS OF FIBONACCI-LIKE SEQUENCES

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Abstract. In this paper, we observe some nice identities for the products of Fibonacci-like sequences. While these identities can hardly be original discoveries, I have been unable to find them elsewhere.

Fibonacci Numbers

The Fibonacci sequence (OEIS A000045) begins as follows:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34...$$

In this sequence, starting with 0 and 1, each term is found by adding the previous two numbers. This sequence is defined by the linear recurrence equation, where $F_0 = 0$ and $F_1 = F_2 = 1$:

$$F_n = F_{(n-1)} + F_{(n-2)}.$$

Fibonacci Identity 1

We observe that the product of 5 consecutive Fibonacci numbers is equal to the middle term to the fifth power minus that number, such that:

$$\prod_{i=(n-2)}^{(n+2)} F_i = F_n^5 - F_n.$$

For example, if $n = 4$,

$$(F_{(2)}F_{(3)}F_{(4)}F_{(5)}F_{(6)}) = (1 \times 2 \times 3 \times 5 \times 8) = 3^5 - 3,$$

or if $n = 7$,

$$(F_{(5)}F_{(6)}F_{(7)}F_{(8)}F_{(9)}) = (5 \times 8 \times 13 \times 21 \times 34) = 13^5 - 13.$$

Fibonacci Identity 2

The product of 3 consecutive Fibonacci numbers is equal to the cube of the middle term plus/minus that number (depending on whether n is odd (-) or even (+)) such that:

$$\prod_{i=(n-1)}^{(n+1)} F_i = F_n^3 \pm F_n.$$

For example, if $n = 5$,

$$(F_{(4)}F_{(5)}F_{(6)}) = (3 \times 5 \times 8) = 5^3 - 5$$

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or if $n = 6$,

$$(F_{(5)}F_{(6)}F_{(7)}) = (5 \times 8 \times 13) = 8^5 + 8.$$

Pell Numbers

There are similar identities for the Pell sequence (OEIS A000129), which begins:

$$0, 1, 2, 5, 12, 29, 70, 169, ...$$

This sequence is defined by the linear recurrence equation, where $P_0 = 0$ and $P_1 = 1$:

$$P_n = 2P_{(n-1)} + P_{(n-2)}.$$

Pell Identity 1

We observe that the product of 4 consecutive Pell numbers is equal to one less than a perfect square, such that:

$$\prod_{i=(n-1)}^{(n+2)} P_i = [P_{(n)}P_{(n+1)} - 1]^2 - 1.$$

For example, if $n = 4$,

$$(P_{(3)}P_{(4)}P_{(5)}P_{(6)}) = (2 \times 5 \times 12 \times 29) = 59^2 - 1$$

or if $n = 6$,

$$(P_{(5)}P_{(6)}P_{(7)}P_{(8)}) = (12 \times 29 \times 70 \times 169) = 2029^2 - 1.$$

Pell Identity 2

The product of 3 consecutive Pell numbers is equal to the middle term to the power of 3 plus/minus that term (depending on whether n is odd (+) or even (-)):

$$\prod_{i=(n-1)}^{(n+1)} P_i = P_{(n)}^2 \pm P_{(n)}.$$

For example, if $n = 4$,

$$(P_{(3)}P_{(4)}P_{(5)}) = (2 \times 5 \times 12) = 5^3 - 5$$

or if $n = 5$,

$$(P_{(4)}P_{(5)}P_{(6)}) = (5 \times 12 \times 29) = 12^3 + 12.$$

"Bronze Fibonacci Numbers"

The so-called 'Bronze Fibonacci' sequence (OEIS A006190) begins as follows:

$$0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970, 42837, ...$$

This sequence is defined by the linear recurrence equation, where $B_0 = 0$ and $B_1 = 1$:

$$B_n = 3B_{(n-1)} + B_{(n-2)}.$$

Bronze Fibonacci Identity 1

We observe that the product of 3 consecutive Bronze Fibonacci numbers is equal to the middle term to the fifth power minus that number (depending on whether n is odd (-) or even (+)), such that:

$$\prod_{i=(n-1)}^{(n+1)} B_i = B_n^3 \pm B_n.$$

For example, if $n = 3$,

$$(B_{(2)}B_{(3)}B_{(4)}) = (3 \times 10 \times 13) = 10^3 - 10,$$

or if $n = 4$,

$$(B_{(3)}B_{(4)}B_{(5)}) = (10 \times 33 \times 109) = 33^3 + 33.$$

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