juliacon

Differential geometric algebra with Leibniz and Grassmann

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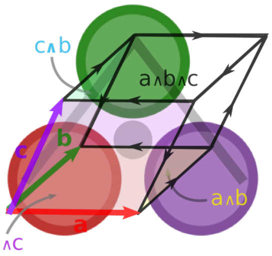


github.com/chakravala/Grassmann.jl

ABSTRACT

The *Grassmann.jl* package provides tools for computations based on multi-linear algebra and spin groups using the extended geometric algebra known as Leibniz-Grassmann-Clifford-Hestenes algebra. Combinatorial products include exterior, regressive, inner, and geometric; along with the Hodge star, adjoint, reversal, and boundary operators. The kernelized operations are built up from composite sparse tensor products and Hodge duality, with high dimensional support for up to 62 indices using staged caching and pre-compilation. Code generation enables concise yet highly extensible definitions. *DirectSum.jl* multivector parametric type polymorphism is based on tangent vector spaces and conformal projective geometry. Additionally, the universal interoperability between different sub-algebras is enabled by *AbstractTensors.jl*, on which the type system is built. !

—**DirectSum.jl**: Abstract tangent bundle vector space types (unions, intersections, sums, etc.)

—**AbstractTensors.jl**: Tensor algebra abstract type interoperability with vector bundle parameter

—**Grassmann.jl**: ⟨Leibniz+Grassmann-Clifford-Hestenes⟩ differential geometric algebra of multivector forms

—**Leibniz.jl**: Derivation operator algebras for tensor fields

—**Reduce.jl**: Symbolic parser generator for Julia expressions using REDUCE algebra term rewriter

Mathematical foundations and some of the nuances in the definitions specific to the *Grassmann.jl* implementation are concisely described, along with the accompanying support packages that provide an extensible platform for computing with geometric algebra at high dimensions. The design is based on the `TensorAlgebra` abstract type interoperability from *AbstractTensors.jl* with a `VectorBundle` type parameter from *DirectSum.jl*. Abstract vector space type operations happen at compile-time, resulting in a differential conformal geometric algebra of hyper-dual multivector forms.

The nature of the geometric algebra code generation enables one to easily extend the abstract product operations to any specific number field type (including differential operators with *Leibniz.jl* or symbolic coefficients with *Reduce.jl*), by making use of Julia's type system. Mixed tensor products with their coefficients are constructed from these operations to work with bivector elements of Lie groups [7][10].

1. Direct sum parametric type polymorphism

The *DirectSum.jl* package is a work in progress providing the necessary tools to work with an arbitrary `Manifold` specified by an encoding. Due to the parametric type system for the generating `VectorBundle`, the Julia compiler can fully pre-allocate and often cache values efficiently ahead of run-time. Although intended for use with the *Grassmann.jl* package, `DirectSum` can be used independently.

Definition 1 Vector bundle of submanifolds. *Let $M = T^\mu V \in \mathrm{Vect}_{\mathbb{K}}$ be a* `TensorBundle<:Manifold` *of rank n,*

$$T^\mu V = (n, \mathbb{P}, g, \nu, \mu), \quad \mathbb{P} \subseteq \langle v_\infty, v_\emptyset \rangle, \quad g : V \times V \to \mathbb{K}$$

The type `TensorBundle{n,ℙ,g,ν,μ}` *uses byte-encoded data available at pre-compilation, where $\mathbb{P}$ specifies the basis for up and down projection, g is a bilinear form that specifies the metric of the space, and μ is an integer specifying the order of the tangent bundle (i.e. multiplicity limit of Leibniz-Taylor monomials). Lastly, ν is the number of tangent variables.*

The dual space functor $(\cdot)' : \mathrm{Vect}_{\mathbb{K}}^{\mathrm{op}} \to \mathrm{Vect}_{\mathbb{K}}$ is an involution which toggles a dual vector space with inverted signature with property $V' = \mathrm{Hom}(V, \mathbb{K})$ and having `Basis` generators

$$\langle v_1, \dots, v_{n-\nu}, \partial_1, \dots, \partial_\nu \rangle = M \leftrightarrow M' = \langle w_1, \dots, w_{n-\nu}, \epsilon_1, \dots, \epsilon_\nu \rangle$$

where v_i, w_i are a basis for the vectors and covectors, while ∂_j, ϵ_j are a basis for differential operators and tensor fields. The direct sum operator `⊕` can be used to join spaces (alternatively `+`). The direct sum of a `VectorBundle` and its dual `V⊕V'` represents the full mother space `V*`. In addition to the direct-sum operation, several other operations exist, such as $\cup, \cap, \subseteq, \supseteq$ for set operations. Due to the design of the `VectorBundle` dispatch, these operations enable code optimizations at compile-time provided by the bit parameters.

$$\bigcup T^{\mu_i} V_i = \left(|\mathbb{P}| + \max\{n_i - |\mathbb{P}_i|\}_i, \bigcup \mathbb{P}_i, \cup g_i, \max\{\mu_i\}_i\right)$$

$$\bigoplus T^{\mu_i} V_i = \left(|\mathbb{P}| + \sum (n_i - |\mathbb{P}_i|), \bigcup \mathbb{P}_i, \oplus_i g_i, \max\{\mu_i\}_i\right)$$

These are roughly the formulas used for those operations.

Remark 1. *Although some type operations like $\bigcup$ and $\bigoplus$ are similar and sometimes result in equal values, the union and sum are entirely different operations in general.*

Calling manifolds with sets of indices constructs the subspace representations. Given `M(s::Int...)` one can encode `SubManifold{`$|s|, M, s$`}` with induced orthogonal space Z:

$$T^eV \subset T^\mu W \iff \exists Z \in \mathrm{Vect}_{\mathbb{K}}(T^e(V \oplus Z) = T^{e\leq\mu}W,\, V \perp Z),$$

such that computing unions of submanifolds is done by inspecting the parameter $s \in V \subseteq W$ and $s \notin Z$. Operations on `Manifold` types is automatically handled at compile time. The metric signature of the `SubManifold{V,1}` elements of a vector space `V` can be specified with the `V"..."` constructor by using `+` and `-` to specify whether the element of the corresponding index squares to `+1` or `-1`. For example, `S"+++"` constructs a positive definite 3-dimensional `TensorBundle`. It is also possible to specify an arbitrary `DiagonalForm` having numerical values for the basis with degenracy `D"1,1,1,0"`, although the $\pm$ format has a more compact representation. Further development will result in more metric types.
Declaring an additional plane at infinity is done by specifying it in the string constructor with `∞` at the first index (i.e. Riemann sphere `S"∞+++"`). The hyperbolic geometry can be declared by `∅` subsequently (i.e. Minkowski spacetime `S"∅+++"`). Additionally, the *null-basis* based on the projective split for confromal geometric algebra would be specified with `∞∅` initially (i.e. 5D CGA `S"∞∅+++"`). These two declared basis elements are interpreted in the type system.
The `tangent(V,`μ`,`ν`)` map can be used to specify μ and ν.

2. Tensor basis equivalence classes

The `AbstractTensors` package is intended for universal interoperability of the abstract `TensorAlgebra` type system. All `TensorAlgebra{V}` subtypes have type parameter `V`, used to store a `TensorBundle` value obtained from *DirectSum.jl*. By itself, this package does not impose any specifications or structure on the `TensorAlgebra{V}` subtypes and elements, aside from requiring V to be a `Manifold`. Hence all tensor types share a common underlying `Manifold` structure. The macro `@basis V` declares a local basis in Julia.

Definition 2. *Let $V \in Vect_{\mathbb{K}}$ be a* `TensorBundle` *with dual space V' and the basis elements $w_k : V \to \mathbb{K}$, then for all $x \in V, c \in \mathbb{K}$ the properties $(w_i + w_j)(x) = w_i(x) + w_j(x)$ and $(cw_k)(x) = cw_k(x)$ hold. An element of a mixed-symmetry* `TensorAlgebra{V}` *is a multilinear mapping that is formally constructed by taking the tensor products of linear and multilinear maps, $(\bigotimes_k \omega_k)(v_1, \dots, v_{\sum_k p_k}) = \prod_k \omega_k(v_1, \dots, v_{p_k})$.*

Definition 3 Mixed-symmetry basis. *Combining the linear basis generating elements with each other using the multilinear tensor product yields a graded (decomposable)* `SubManifold` *$\langle w_{p_1} \otimes \dots \otimes w_{p_k}\rangle_k$, where* `rank` *$k$ is determined by the number of w_i basis elements in its tensor product decomposition. The algebra partitions into symmetric and anti-symmetric tensor equivalence classes. For any pair,*

$$\underbrace{\omega \otimes \eta = -\eta \otimes \omega}_{\text{anti-symmetric}} \qquad \text{or} \qquad \underbrace{\omega \otimes \eta = \eta \otimes \omega.}_{\text{symmetric}}$$

Typically the k in a product $\left(\partial_{p_1} \otimes \dots \otimes \partial_{p_k}\right)^{(k)}$ is referred to as the `order` *of the element if it is fully symmetric, which is overall tracked separately from the* `grade` *such that $\partial_k \langle w_j\rangle_r = \langle \partial_k w_j\rangle_r$ and $(\partial_k)^{(r)}\omega_j = (\partial_k w_j)^{(r)}$. Hence, there is a partitioning into* `even` *grade components ω_+ and* `odd` *grade components ω_- such that $\omega_+ + \omega_- = \omega$.*

Remark 2. *Observe that the anti-symmetric property implies that $\omega \otimes \omega = 0$, while the symmetric property neither implies nor denies such a property. Grassmann remarked [6] in 1862 that the symmetric algebra of functions is by far more complicated than his anti-symmetric exterior algebra. The first part of the book focused on anti-symmetric exterior algebra, while the more complex symmetric function algebra of Leibniz was subject of the second multivariable part of the book. Elements ω_k in the space ΛV of anti-symmetric algebra are often studied as unit quantum state vectors in a unitary probability space, where $\sum_k \omega_k \neq \bigotimes_k \omega_k$ is entanglement.*

Definition 4. *The Grassmann anti-symmetric exterior basis is denoted by $v_{i_1\dots i_g} \in \Lambda_g V$ with its dual $w^{i_1\cdots i_g} \in \Lambda^g V$, while the Leibniz symmetric basis will be $\partial_{i_1}^{\mu_1} \dots \partial_{i_g}^{\mu_g} \in L_g V$ with $\epsilon_{i_1}^{\mu_1} \dots \epsilon_{i_g}^{\mu_g} \in L^g V$ dual elements. Let $\Lambda V = \bigoplus \Lambda^g V$.*

A higher-order tensor element is an oriented-multi-set X such that $w_X = \bigotimes_k w_{i_k}^{\otimes \mu_k}$ with $X = \big((i_1, \mu_1), \dots, (i_g, \mu_g)\big)$ and $|X| = \sum_k \mu_k$ is `grade`, `order`. Anti-symmetric indices $\Lambda X \subseteq \Lambda V$ have two orientations and higher multiplicities are degenerate, hence the only relevant multiplicity is $\mu_k \equiv 1$. The Leibniz-Taylor algebra [9] is a quotient polynomial ring $LV \cong R[x_1, \dots, x_n]/\{\prod_{k=1}^{\mu+1} x_{p_k}\}$, so that $\epsilon_k^{\mu+1} = 0$.
The Grassmann basis elements $v_k \in \Lambda_1 V$ and $w^k \in \Lambda^1 V$ are linearly independent vector and covector elements of V, while the Leibniz `Operator` elements $\partial_k \in L_1 V$ are partial tangent derivations and $\epsilon_k(x) \in L^1 V$ are dependent functions of the `tangent` manifold. Higher `grade` elements of ΛV correspond to `SubManifold` spaces, while higher `order` function elements of LV become homogenous polynomials and Taylor series.
Grassmann's exterior algebra doesn't invoke the properties of multi-sets, as it is related to the algebra of oriented sets; while the Leibniz symmetric algebra is that of unoriented multi-sets. Combined, the mixed-symmetry algebra yield a multi-linear propositional lattice. The formal sum of equal `grade` elements is an oriented `Chain` and with mixed `grade` it is a `MultiVector` simplicial complex. Thus, various standard operations on the oriented multi-sets are possible including $\cup, \cap, \oplus$ and the index operation $X \ominus Y = (X \cup Y)\backslash(X \cap Y)$, which is symmetric difference operation $\veebar$.
In order to work with a `TensorAlgebra{V}`, it is necessary for some computations to be cached. This is usually done automatically when accessed. Staging of precompilation and caching is designed so that a user can smoothly transition between very high dimensional and low dimensional algebras in a single session, with varying levels of extra caching and optimizations. The parametric type formalism in `Grassmann` is highly expressive and enables pre-allocation of geometric algebra computations involving specific sparse subalgebras, including the representation of rotational groups.
It is possible to reach `Simplex` elements with up to $n = 62$ vertices, requiring full alpha-numeric labeling with lowercase and capital letters. Full `MultiVector` allocations are only possible for $n \leq 22$, but sparse operations are also available at higher dimensions. While `Grassmann.Algebra{V}` is a container for the `TensorAlgebra` generators of V, the `Grassmann.Algebra` is only cached for $n \leq 8$. For the range of dimensions $8 < n \leq 22$, the `Grassmann.SparseAlgebra` type is used. To reach higher dimensions with $n > 22$, the `Grassmann.ExtendedAlgebra` type is used.

3. Geometric algebraic product structure

For the oriented sets of the Grassmann exterior algebra, the parity of $(-1)^\Pi$ is factored into transposition compositions when interchanging ordering of the tensor product argument permutations [1]. The symmetrical algebra does not need to track this parity, but has higher multiplicities in its indices. Symmetric differential function algebra of Leibniz trivializes the orientation into a single class of index multisets, while Grassmann's exterior algebra is partitioned into two oriented equivalence classes by anti-symmetry. Full tensor algebra can be sub-partitioned into equivalence classes in multiple ways based on the element symmetry, grade, and metric signature composite properties. Both symmetry classes can be characterized by the same geometric product, which is typically written as multiplication but explicitly denoted by $\ominus$ for clarity here.

Definition 5. *The geometric algebraic product is the Π oriented symmetric difference operator $\ominus$ (weighted by the bilinear form g) and multi-set sum $\oplus$ applied to multilinear tensor products $\otimes$ in a single operation:* $\omega_X \ominus \eta_Y =$

$$\underbrace{\overbrace{(-1)^{\Pi(X,Y)}}^{\textit{orient parity}} \overbrace{\det\left[g_{\Lambda(X\cap Y)}\right]}^{\textit{intersect metric}} (\overbrace{\bigotimes_{k\in\Lambda(X\ominus Y)} w^{i_k}}^{(X\cup Y)\setminus(X\cap Y)})}_{\Lambda^1\textit{-anti-symmetric, }\Lambda^g\textit{-mixed-symmetry}} \otimes \underbrace{(\overbrace{\bigotimes_{k\in L(X\oplus Y)} \epsilon_{i_k}^{\otimes\mu_k}}^{\textit{multi-set sum}})}_{L^g\textit{-symmetric}}$$

Remark 3. *The product symbol $\ominus$ will be used to denote explicitly usage of the geometric algebraic product, although the standard number product $*$ notation could also be used. The $\ominus$ choice helps emphasize that the geometric algebraic product is characterized by symmetric differencing of anti-symmetric indices.*

Definition 6 Null-basis of projective split. *Let $v_\pm^2 = \pm 1$ be a basis with $v_\infty = v_+ + v_-$ and $v_\emptyset = (v_- - v_+)/2$ An embedding space $\mathbb{R}^{p+1,q+1}$ carrying the action from the group $O(p+1,q+1)$ then has $v_\infty^2 = 0$, $v_\emptyset^2 = 0$, $v_\infty \cdot v_\emptyset = -1$, and $v_{\infty\emptyset}^2 = 1$ with Minkowski plane $v_{\infty\emptyset}$ having the Hestenes-Dirac-Clifford product properties*

$$v_{\infty\emptyset} \ominus v_\infty = -v_\infty, \qquad v_{\infty\emptyset} \ominus v_\emptyset = v_\emptyset,$$
$$v_\infty \ominus v_\emptyset = -1 + v_{\infty\emptyset}, \qquad v_\emptyset \ominus v_\infty = -1 - v_{\infty\emptyset}$$

Definition 7. *Symmetry properties of the tensor algebra can be characterized in terms of the geometric product by two averaging operations, which are the symmetrization $\odot$ and anti-symmetrization $\boxtimes$ operators:*

$$\bigodot_{k=1}^{j} \omega_k = \frac{1}{j!}\sum_{\sigma\in S_P} \bigominus_k \omega_{\sigma(k)}, \qquad \bigboxtimes_{k=1}^{j} \omega_k = \sum_{\sigma\in S_P} \frac{(-1)^{\Pi(\sigma)}}{j!} \bigominus_k \omega_{\sigma(k)}$$

These products satisfy various `MultiVector` properties, including the associative and distributive laws.

Definition 8 Exterior product. *Let $w_k \in \Lambda^{p_k}V$, then for all $\sigma \in S_{\sum p_k}$ define an equivalence relation $\sim$ such that*

$$\bigwedge_k \omega_k(v_1,\dots,v_{p_k}) \sim (-1)^{\Pi(\sigma)}(\bigotimes_k \omega_k)(v_{\sigma(1)},\dots,v_{\sigma(\sum p_k)})$$

if and only if $\bigominus_k \omega_k = \bigboxtimes_k \omega_k$ holds. It has become typical to use the $\wedge$ product symbol to denote products of such elements as $\bigwedge \Lambda V \equiv \bigotimes \Lambda V/\sim$ modulo anti-symmetrization.

Definition 9 Symmetric Leibniz differentials. *Let $\partial_k = \frac{\partial}{\partial x_k} \in L_g V$ be Leibnizian symmetric tensors, then there is an equivalence relation $\asymp$ which holds for each $\sigma \in S_p$*

$$(\partial_p \circ \dots \circ \partial_1)\omega \asymp (\bigotimes_k \partial_{\sigma(k)})\omega \iff \bigominus_k \partial_k = \bigodot_k \partial_k,$$

along with each derivation $\partial_k(\omega\eta) = \partial_k(\omega)\eta + \omega\partial_k(\eta)$.

Multiplication with an ϵ_i element is used help signify tensor fields so that differential operators are automatically applied in the `Basis` algebra as $\partial_j \ominus (\omega \otimes \epsilon_i) = \partial_j(\omega\epsilon_i) \neq (\partial_j \otimes \omega) \ominus \epsilon_i$.

```
julia> using Reduce, Grassmann; @mixedbasis tangent(ℝ^2,3,2);
julia> (∂1+∂12) * (:(x1^2*x2^2)*ϵ1 + :(sin(x1))*ϵ2)
0.0 + (2 * x1 * x2 ^ 2)∂₁ϵ¹ + (cos(x1))∂₁ϵ² + (4 * x1 * x2)∂₁₂ϵ¹
```

Since `VectorBundle` choices are fundamental to `TensorAlgebra` operations, the universal interoperability between `TensorAlgebra{V}` elements with different associated `VectorBundle` choices is naturally realized by applying the `union` morphism to type operations. For example,

$$\bigwedge : \Lambda^{p_1}V_1 \times \dots \times \Lambda^{p_g}V_g \to \Lambda^{\sum_k p_k} \bigcup_k V_k.$$

Definition 10 Reverse, involute, conjugate. *The `reverse` of $\langle\omega\rangle_r$ is defined as $\langle\tilde\omega\rangle_r = (-1)^{(r-1)r/2}\langle\omega\rangle_r$, while the `involute` is $\langle\omega\rangle_r^\times = (-1)^r\langle\omega\rangle_r$ and Clifford `conj` $\langle\omega\rangle_r^\ddagger$ is the composition of `involute` and `reverse`.*

Definition 11 Reversed product. *Define the index reversed product $*$ which yields a Hilbert space structure:*

$$\omega * \eta = \tilde\omega \ominus \eta, \qquad or \qquad \omega *' \eta = \omega \ominus \tilde\eta,$$

$$|\omega|^2 = \omega * \omega, \qquad |\omega| = \sqrt{\omega * \omega}, \qquad ||\omega|| = \textit{Euclidean } |\omega|.$$

Remark 4. *Observe that $*$ and $*'$ could both be exchanged in `abs`, `abs2`, and `norm`; however, these are different products. The scalar product $\circledast$ is the `scalar` part, so $\eta \circledast \omega = \langle \eta * \omega\rangle$. In general $\sqrt{\omega} = e^{(\log\omega)/2}$ is valid for invertible ω.*

Definition 12 Inverse. $\omega^{-1} = \omega * (\omega * \omega)^{-1} = \tilde\omega/|\omega|^2$, *with $\eta/\omega = \eta \ominus \omega^{-1}$ and $\eta\backslash\omega = \eta^{-1} \ominus \omega$.*

Definition 13 Sandwich product. *This product can be defined as $\eta \oslash \omega = \omega\backslash\eta \ominus \omega^\times$. Alternatively, the reversed definition is $\eta \circledS \omega = \eta^\times \ominus \omega/\eta$ or in Julia `η>>>ω`, which is often found in literature.*

Remark 5. *Observe that it is overall more simple and consistent to use $\{*, \oslash\}$ operations instead of $\{*', \circledS\}$.*

The `real` part $\Re\omega = (\omega + \tilde\omega)/2$ is defined by $|\Re\omega|^2 = (\Re\omega)^{\ominus 2}$ and the `imag` part $\Im\omega = (\omega - \tilde\omega)/2$ by $|\Im\omega|^2 = -(\Im\omega)^{\ominus 2}$, such that $\omega = \Re\omega + \Im\omega$ has real and imaginary partitioned by

$$\langle\tilde\omega\rangle_r / \left|\langle\omega\rangle_r\right| = \sqrt{\langle\tilde\omega\rangle_r^2 / \left|\langle\omega\rangle_r\right|^2} = \sqrt{\langle\omega\rangle_r * \langle\omega\rangle_r^{-1}}$$
$$= \sqrt{\langle\tilde\omega\rangle_r / \langle\omega\rangle_r} = \sqrt{(-1)^{(r-1)r/2}} \in \left\{1, \sqrt{-1}\right\},$$

which is a unique partitioning completely independent of the metric space and manifold of the algebra [8].

$$\omega * \omega = |\omega|^2 = |\Re\omega + \Im\omega|^2 = |\Re\omega|^2 + |\Im\omega|^2 + 2\Re(\Re\omega * \Im\omega)$$

The `radial` and `angular` components in a multivector exponential are partitioned by the parity of their metric.

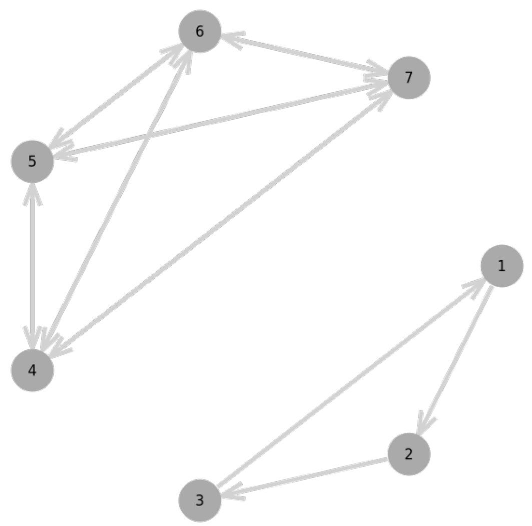
4. Leibniz operators and Grassmann's Hodge-DeRahm theory

A universal unit volume element can be specified in terms of `LinearAlgebra.UniformScaling`, which is independent of V and has its interpretation only instantiated by the context of the `TensorAlgebra{V}` element being operated on. Universal interoperability of `LinearAlgebra.UniformScaling` as the pseudoscalar element which takes on the `TensorBundle` form of any other `TensorAlgebra` element is handled globally. This enables the usage of `I` from `LinearAlgebra` as a universal pseudoscalar element defined at every point x of a `Manifold`, which is mathematically denoted by $I = I(x)$ and specified by the $g(x)$ bilinear tensor field of TM.

Definition 14 Poincare-Hodge dual complement. *Let* $\star\langle\omega\rangle_p = \langle\omega\rangle_p * I = \langle\tilde\omega\rangle_p \ominus I$, *then* $\star : \Lambda^p V \to \Lambda^{n-p} V$.

Remark 6. *While* $\star\omega$ *is* `complementright` *of* ω, *the* `complementleft` *would be* $I*'\omega$ *and* $!\omega$ *denotes the non-metric variant the complement. The* $\star$ *symbol was added to the Julia language as unary operator on Julia's v1.2 release.*

```
using Grassmann, Compose
x = Grassmann.Algebra(ℝ^7).v123
Grassmann.graph(x+!x)
draw(PDF("simplex.pdf",16cm,16cm),x+!x)
```



Triangle with its tetrahedron complement $v_{123} + \star v_{123}$ in $\mathbb{R}^7$

John Browne has discussed Grassmann duality principle in book [3], stating that every theorem (involving either of the exterior and regressive products) can be translated into its dual theorem by replacing the $\wedge$ and $\vee$ operations and applying *Poincare duality* (homology). First applying this Grassmann duality principle to the $\wedge$ product alone, let $\{\omega_k\}_k \in \Lambda^{p_k} V, P = \sum_k p_k$, then it is possible to obtain the co-product $\bigvee : \Lambda^{p_1} V_1 \times \cdots \times \Lambda^{p_g} V_g \to \Lambda^{P-(g-1)\#V} \bigcup_k V_k$. Grassmann's original notation implicitly combined $\wedge, \vee, \star$.

The join $\wedge$ product is analogous to union $\cup$, the meet $\vee$ product is analogous to intersection $\cap$, and the orthogonal complement $\star \mapsto^\perp$ is negation. Together, $(\wedge, \vee, \star)$ yield an orthocomplementary propositional lattice (quantum logic):

$$(\star \bigvee_k \omega_k)(v_1, \dots, v_P) = (\bigwedge_k \star\omega_k)(v_1, \dots, v_P) \quad \textit{DeMorgan's Law},$$

where DeMorgan's law is used to derive tensor contractions.

Definition 15. *Skew left* $\lrcorner$ *and right* $\llcorner$ *contractions are symmetrically defined* $\langle\omega\rangle_r \cdot \langle\eta\rangle_s = \begin{cases} \omega \llcorner \eta = \omega \vee \star\eta & r \geq s \\ \omega \lrcorner \eta = \eta \vee \star\omega & r \leq s \end{cases}$. *Note for* ω, η *of equal grade,* $\omega \circledast \eta = \omega \odot \eta = \omega \cdot \eta = \omega \llcorner \eta = \omega \lrcorner \eta$ *are all symmetric operations. In Julia,* $\lrcorner$ *is* $<$ *and* $\llcorner$ *is* $>$.

Definition 16. *Let* $\nabla = \sum_k \partial_k v_k$ *be a vector field and* $\epsilon = \sum_k \epsilon_k(x) w_k \in \Omega^1 V$ *be unit sums of the mixed-symmetry basis. Elements of* $\Omega^p V$ *are known as differential p-forms and both* ∇ *and* ϵ *are tensor fields dependent on* $x \in W$. *Another notation for a differential form is* $dx_k = \epsilon_k(x) w_k$, *such that* $\epsilon_k = dx_k / w_k$ *and* $\partial_k \omega(x) = \omega'(x)$.

Remark 7. *The space* W *does not have to equal* $V \in \mathit{Vect}_{\mathbb{K}}$ *above, as* $\Omega^p V$ *could have coefficients from* $\mathbb{K} = LW$.

Definition 17. *Define differential* $d : \Omega^p V \to \Omega^{p+1} V$ *and co-differential* $\delta : \Omega^p V \to \Omega^{p-1} V$ *such that [2]*

$$\star d\omega = \star(\nabla \wedge \omega) = \nabla \times \omega, \quad \omega \cdot \nabla = \omega \vee \star\nabla = \partial\omega = -\delta\omega.$$

These two maps have the special properties $d \circ d = 0$ *and* $\partial \circ \partial = 0$ *for any form* ω *and vector field* ∇. *In topology there is boundary operator* ∂ *defined by* $\partial\epsilon = \epsilon \cdot \nabla = \sum_k \partial_k \epsilon_k$ *and is commonly discussed in terms the limit* $\epsilon(x) \cdot \nabla\omega(x) = \lim_{h\to 0} \frac{\omega(x+h\epsilon)-\omega(x)}{h}$, *which is the directional derivative [11].*

Example 1 Vorticity curl of vector-field. $\star d(dx_1 + dx_2 + dx_3) = (\partial_2 - \partial_3) dx_1 + (\partial_3 - \partial_1) dx_2 + (\partial_1 - \partial_2) dx_3$.

Example 2 Boundary of 3-simplex. *Faces of simplex (oriented):* $\partial(w_{1234}) = -\partial_4 w_{123} + \partial_3 w_{124} - \partial_2 w_{134} + \partial_1 w_{234}$.

Theorem 18 Integration by parts & Stokes. *Let* $\nabla \in \Omega_1 V$ *be a Leibnizian vector field operator, then* $d, -\partial$ *are Hilbert adjoint Hodge-DeRahm operators with* $\langle * \rangle$

$$\int_M d\omega \wedge \star\eta + \int_M \omega \wedge \star\partial\eta = 0, \qquad \langle d\omega * \eta \rangle = \langle \omega * -\partial\eta \rangle.$$

Proof. $\partial\omega = \omega \cdot \nabla = \star^{-1}(\star\omega \wedge \star^2 \nabla) = (-1)^n (-1)^{nk} \star d \star \omega$. Then substitute this into $\int_M \omega \wedge (-1)^{mk+m+1} \star \star d \star \eta = (-1)^{km+m+1} (-1)^{(m-k+1)(k-1)} \int_M \omega \wedge d \star \eta$, apply the identity $(-1)^{km+m+1} (-1)^{(m-k+1)(k-1)} = (-1)^k$ and $(-1)^k \int_M \omega \wedge d \star \eta = \int_M d(\omega \wedge \star\eta) - (-1)^{k-1} \omega \wedge d \star \eta = \int_M d\omega \wedge \star\eta$. Stokes identity can be proved by relying on a variant of the *common factor theorem* by Browne [3]. □

Theorem 19 Clifford-Dirac-Laplacian. *The Dirac operator [5] is* $(\nabla^2)^{\frac{1}{2}} \omega = \pm\nabla \ominus \omega = \pm\nabla \wedge \omega \pm \nabla \cdot \omega = \pm d\omega \pm \partial\omega$.

$$\nabla^2 \omega = \nabla \wedge (\omega \cdot \nabla) + (\nabla \wedge \omega) \cdot \nabla) = \mp(\mp\omega \ominus \nabla) \ominus \nabla).$$

Elements ω *are harmonic if* $\nabla\omega = 0$ *and both closed* $d\omega = 0$ *and coclosed* $\delta\omega = 0$, *such that* $\mathcal{H}^p M = \{\nabla\omega = 0 : \omega \in \Omega^p M\}$. *Hodge [12]:* $\Omega^p M = \mathcal{H}^p M \oplus im(d\Omega^{p-1} M) \oplus im(\partial\Omega^{p+1} M)$. *Note:* $\nabla\omega = -\omega\nabla, \nabla^2\omega = \omega\nabla^2$ *for higher-order tensor fields!*

For the null-basis, complement operations are different:

$$\star v_\infty = \star(v_+ + v_-) = (v_- + v_+)v_{1\dots n} = v_{\infty 1\dots n}$$
$$\star 2v_\emptyset = \star(v_- - v_+) = (v_+ - v_-)v_{1\dots n} = -2v_{\emptyset 1\dots n}$$

The Hodge complement satisfies $\langle \omega * \omega\rangle I = \omega \wedge \star\omega$. This property is naturally a result of using the geometric product in the definition. An additional metric independent version of the complement operation is available with the ! operator,

$$!v_\infty = !(v_+ + v_-) = (v_- - v_+)v_{1\dots n} = 2v_{\emptyset 1\dots n}$$
$$!2v_\emptyset = !(v_- - v_+) = (v_+ + v_-)v_{1\dots n} = -v_{\infty 1\dots n}$$

For that variation of complement, $||\omega||^2 I = \omega \wedge !\omega$ holds.

EXAMPLE 3. $S"\infty\emptyset{+}{+}{+}"(\nabla\text{^}2) \mapsto -2\partial_{\infty\emptyset} + \partial_1^2 + \partial_2^2 + \partial_3^2$

Let $\nabla \in \Lambda^1 V$, then $\omega = (\nabla\backslash\nabla)\ominus\omega = \nabla\backslash(d\omega + \partial\omega)$ where $\nabla \parallel \partial\omega$ and $\nabla \perp d\omega$. Let's reflect across the hyperplane $\star\nabla$,

$$\begin{aligned}\nabla\backslash(d\omega - \partial\omega) &= \nabla\backslash(d\omega - \partial\omega)\ominus(\nabla\backslash\nabla)\\ &= -\nabla^2\backslash(d\omega+\partial\omega)\ominus\nabla = -\nabla\backslash\omega\ominus\nabla.\end{aligned}$$

Hence, reflection by hyperplane $\star\nabla$ has isometry $\omega\oslash\nabla$ which for $\nabla = v_j$ is the map $\mathbb{R}^n \to \mathbb{R}_1 \times \cdots \times \overline{\mathbb{R}}_j \cdots \times \mathbb{R}_n$.

THEOREM 20 CARTAN-DIEUDONNE. *Every isometry of $V \to V$ is the composite of at most k reflections across non-singular hyperplanes. Hence there exist vectors ∇_j such that*

$$(((\omega \oslash \nabla_1)\oslash\nabla_2)\oslash\cdots)\oslash\nabla_k = \omega\oslash(\nabla_1\ominus\nabla_2\ominus\cdots\ominus\nabla_k)$$

for any isometry element of the orthogonal group $O(p,q)$.

Note that elements under transformations of this group preserve inner product relations. The even grade operators make up the rotational group, where each bivector isometry is a composition of two reflections [1] [4].
Consider the differential equation $\partial_i\epsilon_j = \epsilon_j \oslash \omega$ with the solution $\epsilon_j(x) = \epsilon_j(0)\oslash e^{x_i\omega}$ where $\theta = 2x_i$ is the parameter of the Lie group. Then for a normalized ω,

$$e^{\theta\omega} = \sum_k \frac{(\theta\omega)^{\ominus k}}{k!} = \begin{cases}\cosh\theta + \omega\sinh\theta, & \text{if } \omega^2 = 1,\\ \cos\theta + \omega\sin\theta, & \text{if } \omega^2 = -1,\\ 1+\theta\omega, & \text{if } \omega^2 = 0.\end{cases}$$

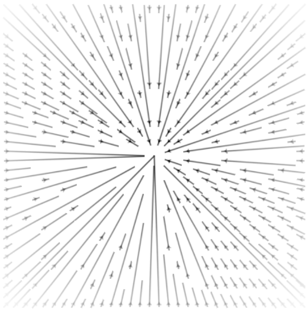
Note that $\nabla\oslash e^{\theta\omega/2} = \nabla\ominus e^{\theta\omega}$ is a double covering when using the complex numbers in the Euclidean plane.

THEOREM 21 LEIBNIZ-TAYLOR SERIES. $\partial_X = \bigominus_k \partial_k^{\mu_k}$ *is defined so that* $|X| = \sum_k \mu_k$*, then* $e^{\partial\epsilon}\omega(x)$ *is*

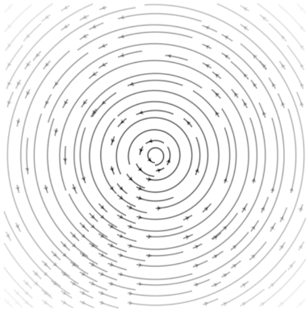
$$e^{\partial\epsilon}\omega(x) = \sum_{j=0}^{\mu}\frac{(\partial\epsilon)^{\ominus j}}{j!}\omega(x) = \sum_{j=0}^{\mu}\sum_{|X|=j}\bigominus_k \frac{(\partial_k\epsilon_k(x))^{\mu_k}}{\mu_k!}\omega(x).$$

The multivariate *product rule* is encoded into the geometric algebraic product when using mixed-symmetry.

```
using Grassmann, Makie
basis"2" # Euclidean
streamplot(vectorfield(exp(π*v12/2)),-1.5..1.5,-1.5..1.5)
streamplot(vectorfield(exp((π/2)*v12/2)),-1.5..1.5,-1.5..1.5)
streamplot(vectorfield(exp((π/4)*v12/2)),-1.5..1.5,-1.5..1.5)
streamplot(vectorfield(v1*exp((π/4)*v12/2)),-1.5..1.5,-1.5..1.5)
@basis S"+-" # Hyperbolic
streamplot(vectorfield(exp((π/8)*v12/2)),-1.5..1.5,-1.5..1.5)
streamplot(vectorfield(v1*exp((π/4)*v12/2)),-1.5..1.5,-1.5..1.5)
```

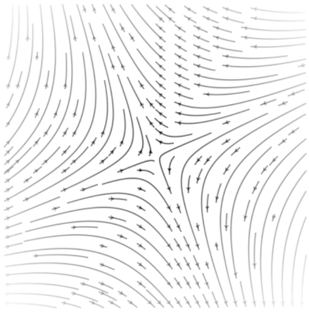


(a) $x \oslash e^{\pi v_{12}/2}$, $\mathbb{R}^2$

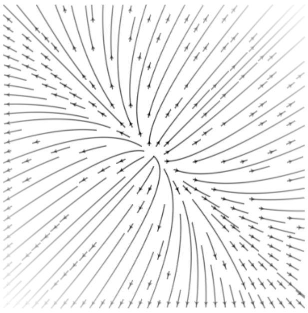


(b) $x \oslash e^{\frac{\pi}{2} v_{12}/2}$, $\mathbb{R}^2$

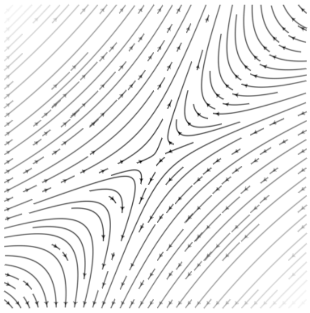
(c) $x \oslash e^{\frac{\pi}{4} v_{12}/2}$, $\mathbb{R}^2$



(d) $x \oslash (v_1 \ominus e^{\frac{\pi}{4} v_{12}/2})$, $\overline{\mathbb{R}}\oplus\mathbb{R}$



(e) $x \oslash e^{\frac{\pi}{8} v_{12}/2}$, $\mathbb{R}\oplus\mathbb{R}'$



(f) $x \oslash (v_1 \ominus e^{\frac{\pi}{4} v_{12}/2})$, $\overline{\mathbb{R}}\oplus\mathbb{R}'$

As a result of Grassmann's exterior & interior products, the Hodge-DeRahm chain complex from cohomology theory is

$$0 \underset{\partial}{\overset{d}{\rightleftarrows}} \Omega^0(M) \underset{\partial}{\overset{d}{\rightleftarrows}} \Omega^1(M) \underset{\partial}{\overset{d}{\leftrightarrows}} \cdots \underset{\partial}{\overset{d}{\rightleftarrows}} \Omega^n(M) \underset{\partial}{\overset{d}{\rightleftarrows}} 0,$$

having dimensional equivalence brought by the Grassmann-Poincare-Hodge complement duality,

$$\mathcal{H}^{n-p}M \cong \frac{\ker(d\Omega^{n-p}M)}{\operatorname{im}(d\Omega^{n-p+1}M)}, \quad \dim\mathcal{H}^pM = \dim\frac{\ker(\partial\Omega^pM)}{\operatorname{im}(\partial\Omega^{p+1}M)}$$

The rank of the grade p boundary incidence operator is

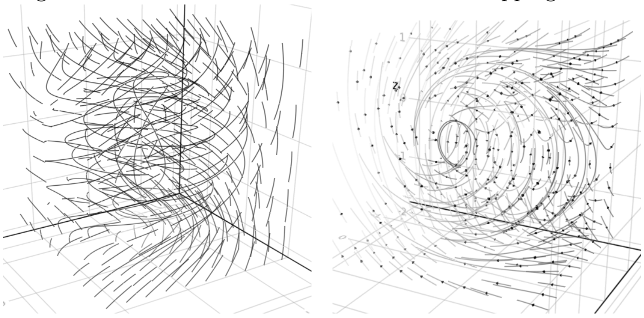
$$\operatorname{rank}\left\langle\partial\langle M\rangle_{p+1}\right\rangle_p = \min\left\{\dim\left\langle\partial\langle M\rangle_{p+1}\right\rangle_p, \dim\langle M\rangle_{p+1}\right\}$$

Invariant topological information can be computed using the rank of homology groups, where $b_p(M) = \dim\mathcal{H}^pM$ are

$$b_p(M) = \dim\langle M\rangle_{p+1} - \operatorname{rank}\left\langle\partial\langle M\rangle_{p+1}\right\rangle_p - \operatorname{rank}\left\langle\partial\langle M\rangle_{p+2}\right\rangle_{p+1}$$

Betti numbers with Euler characteristic $\chi(M) = \sum_p(-1)^p b_p$.

Fig. 2: Variations of sub-manifold vector field mappings in $\mathbb{R}^4$.



(a) $x \oslash e^{\frac{\pi}{4}(v_{12}+v_{\infty 3})}$ (b) $V_{\infty 12} \to V_{123}$

Let's obtain the full `skeleton` of a simplical complex $\Delta(\omega) = \mathcal{P}(\omega) \backslash \Lambda^0(V)$ from the power set $\mathcal{P}(\omega)$ of all vertices with each `subcomplex` $\Delta(\partial(\omega))$ contained in the edge graph:

$$\Delta(\omega) = \sum_{g=1}^{n} \sum_{k=1}^{\binom{n}{g}} \left(\operatorname{abs} \langle \omega \rangle_{g,k} + \Delta \left(\operatorname{abs} \partial \langle \omega \rangle_{g,k} \right) \right).$$

Example 4 Topology. *Compute the value $\chi(\Delta(\omega)) = 1$ and $\chi(\Delta(\partial(\omega))) = ?$ for any* `Simplex` *ω. As an exercise, also compute the corresponding* `betti` *numbers..*

In Fig. 4, different possible discrete bivector topologies in a projective Riemann sphere setting are examined. The figures are based on the product topology of two rotation bivectors. When the Euclidean $\mathbb{R}^4$ basis is combined with projective geometric algebra, resulting one parameter Lie groups can be visualized as a fibration of a torus in $\mathbb{R}^3$. When the fourth v_∞ basis direction is rotated into the Minkowski plane, the double rotation becomes a helix with translational single rotation. In examples (b)∼(c) the bivector is modulated.

5. Conclusion

Grassmann.jl and its accompanying support packages provide an extensible platform for fully generalized computing with geometric algebra at high dimensions. All of the types and operations in this paper are implemented using only a few thousand lines of code with Julia's type polymorphism code generation, with the mixed-symmetry interaction of `Leibniz` and `Grassmann` available for research. Thus, computations involving fully general rotational algebras and Lie bivector groups are possible with a full trigonometric suite. Conformal geometric algebra is possible with the Minkowski plane $v_{\infty\emptyset}$, based on the null-basis. In general, multivalued quantum logic is enabled by the $\wedge, \vee, \star$ Grassmann lattice. Mixed-symmetry algebra with *Leibniz.jl* and *Grassmann.jl*, having the geometric algebraic product chain rule, yields automatic differentiation and Hodge-DeRahm co/homology as unveiled by Grassmann. Most importantly, the Dirac-Clifford product yields generalized Hodge-Laplacian and the Betti numbers with Euler characteristic `χ`.

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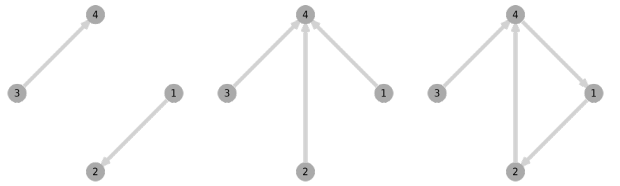
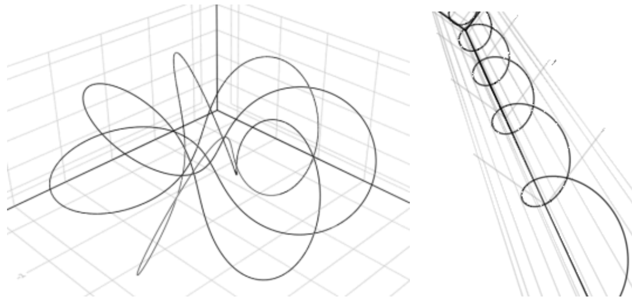
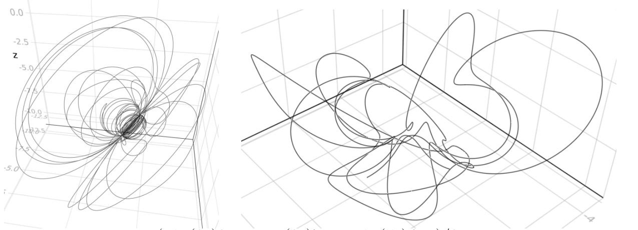
Fig. 3: Different variations of bivectors in $\mathbb{R}^4$ discrete topology.



(a) $v_{12} + v_{34}$ (b) $v_{14} + v_{24} + v_{34}$ (c) $\partial v_{124} + v_{34}$



(a) Doubly periodic flow (independent) (a) Period translation



(b)∼(c) $\downarrow (e^{tv_\infty(sin(3t)3v_1+cos(2t)7v_2-sin(5t)4v_3)/2} \oslash \uparrow (v_1 + v_2 - v_3))$

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